CHAPTER-11 DETERMINANTS & MATRICES

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Definition of determinants and its properties
- Solving system of linear equations by Cramer's rule.
- Matrices and its Classification
- Different types of operation on Matrices
- Solving system of linear equations by inverse matrix method
- Business problem solution

11.1 INTRODUCTION

A scalar arrangement of an expression or number of elements into rows and columns, enclosed within $| \ |$ is called a determinant. The determinant is a single number or scalar and is found only for square matrices. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A, $|A| = ad - bc$.

$ad - bc$ is also called the *value of the determinant.*

Therefore, The value of the determinant of order two can be obtained by the product of the diagonal elements minus the product of anti-diagonal elements.

11.2 DETERMINANTS

A scalar quantity obtained by expanding the elements of a square matrix, say $A = |a_{ij}|$ with respect to the elements of any row or column is called a determinant of matrix A and is denoted by $|A|$ or, det A.

Determinant of second order: Consider the square matrix of order 2 as:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The product of the elements in the principal diagonal is $a_{11}.a_{22}$ and the product of the elements in the off diagonal is: $a_{12}.a_{21}$. The difference $(a_{11} \times a_{22}) - (a_{12} \times a_{21})$ is called the determinant of A and is denoted by $|A|$ or, det A. expressed as:

$$|A| = \det A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}.a_{22} - a_{12}.a_{21}$$

Example: $\begin{vmatrix} 2 & -4 \\ 6 & -8 \end{vmatrix} = (2)(-8) - (-4)(6) = -16 + 24 = 8$

Remarks:

1. The determinant of a square matrix is associated with some value whereas a matrix is essentially an arrangement of numbers and as such has no value.
2. The change in the position of elements off a matrix will give a different matrix.

Determinant of order three: Consider the square matrix of order 3 as:

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & a_3 \end{bmatrix}$$

Then its determinant is written as:

$$|A| = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & a_3 \end{bmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

The value of this determinant is obtained expanding it with the help of the elements of the elements of any row or column.

11.3 RULE FOR EXPANSION OF DETERMINANT

Multiply each element of the first row (or column) by the determinant obtained after removing the row and column to which it belongs, the signs of the products being taken alternately positive and negative.

In other words, to expand $|A|$ with respect to elements of a row or a column, we multiply each element a_{ij} in ith row with $(-1)^{i+j}$ times the determinant of the sub-matrix obtained by deleting the row and column passing through the element.

Example: Expanding the following determinant with respect to the elements of the first row, i.e. 2,3, and 4. Thus, we get

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 7 \\ 4 & 5 & 6 \end{bmatrix} = 2\begin{vmatrix} 0 & 7 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 1 & 7 \\ 4 & 6 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 4 & 5 \end{vmatrix} = 2(0-35) - 3(6-28) + 4(5-0) = 16.$$

Minors and Co-factors

Minor: The determinant of the square sub-matrix $|M_{ij}|$ of order $(n-1)$ obtained by deleting its i^{th} row and j^{th} column is called the minor of the element a_{ij} of matrix A of order n.

If $A=\begin{bmatrix}7 & 8 & 9\\ 4 & 5 & 6\\ 1 & 2 & 3\end{bmatrix}$, then $M_{23}=\begin{bmatrix}7 & 8\\ 1 & 2\end{bmatrix}$ is the sub-matrix obtained by deleting 2^{nd} row and 3^{rd} column of A. Thus, the determinant of any such sub-matrix is called the minor of element a_{23} in A. i.e. minor of the element a_{23} in A is:

$$|M_{23}|=\begin{bmatrix}7 & 8\\ 1 & 2\end{bmatrix}=14-8=6.$$

Similarly, the minor of the element a_{22} (i.e.5) is $|M_{22}|=\begin{bmatrix}7 & 9\\ 1 & 3\end{bmatrix}=21-9=12.$

Co-factors: The co-factor a_{ij} of an element a_{ij} in the square matrix A is the minor of that element with proper sign depending on the row and column in which the element is located. Thus co-factor of an element a_{ij} belonging to ith row and jth column of $|A|$, denoted by a_{ij} is given by $A_{ij}=(-1)^{i+j}|M_{ij}|$.

Hence, the co-factor of an element a_{ij} of a square matrix, $A=|a_{ij}|$ is defined as:

$$A_{ij}=\begin{cases}-|M_{ij}| & if\ \ i+j\ is\ odd\\ |M_{ij}| & if\ \ i+j\ is\ even\end{cases}$$

In the previous example, we see that the minor of the element a_{23} in A is $|M_{23}|=\begin{vmatrix}7 & 8\\ 1 & 2\end{vmatrix}=6$

Therefore co-factor of a_{23} in $|A|$ is $A_{23}=(-1)^{2+3}|M_{23}|=(-1)^5 6=-6$

Similarly, co-factor of a_{22} in A is $A_{22}=(-1)^{2+2}|M_{22}|=(-1)^4 12=12$

Now using the concept of minor and co-factor, we can write the expansion of the determinant $|A|$ as:

$$\det A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & c_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$$

$$=a_{11}(-1)^{1+1}|M_{11}|+a_{12}(-1)^{1+2}|M_{12}|+a_{13}(-1)^{1+3}|M^{13}|$$

$$=a_{11}|M_{11}|-a_{12}|M_{12}|+a_{13}|M_{13}|$$

$$= a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example: $$\begin{vmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 7(-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix} + 8(-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix} + 9(-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix}$$

$$= 7(15-12) - 8(12-6) + 9(8-5) = 21 - 48 + 27 = 0.$$

Remarks:

1. The value of the determinant is equal to the sum of the products of the elements of any row or column and their corresponding co-facto Tk.
2. The sum of the products of the elements of a row or a column with corresponding co-factors of any other parallel row (or column) is zero.

11.4 PROPERTIES OF DETERMINANTS

Following are the number of useful properties of determinant of order three. However, all these properties hold for determinant of any order.

1. The value of the determinant remains unchanged, if the rows and the columns of the determinant are interchanged. That is, $|A| = |A^T|$. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & a_3 \end{vmatrix} \text{ then } |A^T| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}. \text{ Hence } |A| = |A^T|.$$

2. If any two adjacent rows (or columns) of the determinant are interchanged, its value remain unchanged, but its sign is changed. That is, $|A| = -|A|$. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ then } \begin{vmatrix} a_2 & b_2 & c_3 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = -|A|.$$

3. If two rows (or columns) of a determinant are identical, then the value of the determinant is zero. That is

$$|A| = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0.$$

4. If every element in any row (or column) of the determinant is multiplied by same scalar k. then the value of the determinant gets multiplied by the same scalar i.e., value of new determinant is k times the value of original determinant. That is,

$$\begin{vmatrix} ka_1 & a_1 & b_1 \\ ka_2 & a_2 & b_2 \\ ka_3 & a_3 & b_3 \end{vmatrix} = k\begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = k.0 = (\text{by proposition 3}).$$

5. If the elements of any row (or column) are the sum (or difference) of two terms, then the determinant can be expressed as the sum (or difference) of two determinants of the same order, That is,

$$\begin{vmatrix} a_1+k_1 & a_2 & a_3 \\ b_1+k_2 & b_2 & b_3 \\ c_1+k_3 & c^2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & a_2 & a_3 \\ k_2 & b_2 & b_3 \\ k_3 & c_2 & c_3 \end{vmatrix}$$

6. If each element of any row (or column) of a determinant are added (or subtracted) equimultiples of the corresponding element of one or more rows (or columns), the value of the determinant remains same. That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\lambda a_2-ma_3 & a_2 & a_3 \\ b_1+\lambda b_2-mb_3 & b_2 & b_3 \\ c_1+\lambda c_2-mc_3 & c_2 & c_3 \end{vmatrix}$$

7. If $A=[a_{ij}]$ is a diagonal matrix of order n(>2), then $|A| = a_{11}.a_{22}.a_{33}.....a_{nn}$

8. If A and B are square matrices of the same order then $|AB|=|A||B|$.

Remark: In the application of the above property it should be remember that the elements of the row (or column) which is going to be replaced should remain as such. Otherwise, divide the whole determinant by the same quantity, Which is used to multiple the elements of that row (or column).

11.5 CRAMER'S RULE

One of the applications of the determinants is to solve a system of linear equations in which number of variables are equal to the number of equations and the co-efficient matrix of the system of equations is non-singular. This method enables us to determine solution directly without computing the inverse of the matrix.

Let us consider the following linear equations:

$$\begin{aligned} a_1x+b_1y&=c_1 \\ a_2x+b_2y&=c_2 \end{aligned} \quad \text{or,} \quad \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Let D be the determinant of the co-efficient of the variables x and y such that

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Further, let $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ be the determinant obtained from D by replacing the first column by the elements c_1, c_2 and by replacing the second column by the elements c_1, c_2 respectively. Thus the values of x and y can be expressed in the form of determinant as:

$$\frac{x}{D_x} = \frac{y}{D_y} = \frac{1}{D}, \text{ provided } D \neq 0.$$

$$\therefore x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}, if \ D \neq 0.$$

The Cramer's rule for solving a system of linear equations can also be extended for equations involving more that two variables by using following general formula:

Cramer's formula: $x_j = \frac{D_j}{D}, j = 1,2,\ldots,n.$

Where the determinant D_i can be obtained from D (determinant of the coefficients of unknowns) by replacing ith column of D by the column of right hand side constant terms.

ILLUSTRATIONS (Determinants)

Illustration-01: Show that $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + fgh - af^2 - bg^2 - ch^2.$

Solution : Expanding the determinant in terms of the elements of first row, we get

$$|A| = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$$

$$= a(bc - f^2) - h(hc - gf) + g(hf - gb)$$

$$= abc - af^2 - ch^2 + fgh + ghf - g^2b$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2$$

Illustration-02: Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution : Applying elementary column operations as indicated follows :

$$\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=\begin{vmatrix}1 & 0 & 0\\ a & b-a & c-a\\ a^2 & b^2-a^2 & c^2-a^2\end{vmatrix}\begin{matrix}C_2'\to C_2-C_1\\ C_3'\to C_3-C_1\end{matrix}$$

Expanding with respect to first row, we have

$$=\begin{vmatrix}b-a & c-a\\ b^2-a^2 & c^2-a^2\end{vmatrix}=\begin{vmatrix}b-a & c-a\\ (b-a)(b+a) & (c+a)(c+a)\end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix}1 & 1\\ b+a & c+a\end{vmatrix}$$

$$=(b-a)(c-a)\{(c+a)-(b+a)\}$$

$$=(b-a)(c-a)(c-b)=(a-b)(b-c)(c-a).$$

Illustration-03: Prove that $\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}=(a+b+c)^3$

Solution: Applying elementary row operations, we have

$$\begin{vmatrix}a+b+c & a+b+c & a+b+c\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix},R_1'\to R_1+R_2+R_3$$

Taking $(a+b+c)$ common from first row, we get

$$(a+b+c)\begin{vmatrix}1 & 1 & 1\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix}1 & 0 & 0\\ 2b & -b-c-a & 0\\ 2c & 0 & -c-a-b\end{vmatrix},\begin{matrix}C_2'\to C_2-C_1\\ C_3'\to C_3-C_1\end{matrix}$$

Expanding with respect to first row, we get

$$(a+b+c)[(-b-c-a)(-c-a-b)]=(a+b+c)^3$$

Illustration-04: Show that $\begin{vmatrix}-a^2 & ab & ac\\ ba & -b^2 & bc\\ ac & bc & c^2\end{vmatrix}=4a^2b^2c^2$

Solution: Taking out a, b, c common from first, second and third row respectively and then from columns, and applying elementary column operations a as indicated follows:

$$|A| = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \quad \because C_2' \to C_2 + C_1,\ C_3' \to C_3 + C_1$$

$$= a^2b^2c^2\begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix}$$

$= 4a^2b^2c^2$ (Expanding by first row)

Illustration-05: Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$.

Solution: Applying elementary column operations, we have

$$\begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}, C_1' \to C_1 + C_2 + C_3$$

$$= (2a+2b+2c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}, \begin{matrix} R_2' \to R_2 - R_1 \\ R_3' \to R_3 - R_1 \end{matrix}$$

Expanding with respect to the elements of first column, we get

$$= 2(a+b+c)\begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}$$

$= 2(a+b+c)[(b+c+a)(c+a+b)] = 2(a+b+c)^3$.

Illustration-06: Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

Solution: Taking out a, b, c as common factor from R_1, R_2 and R_3 respectively, we get

$$abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying row operation, $R_1' \to R_1 + R_2 + R_3$, we get

$$abc\begin{vmatrix} 1+\frac{1}{b}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}; \begin{matrix} C_2' \to C_2 - C_1 \\ C_3' \to C_3 - C_1 \end{matrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \text{ (Expanding with respect to first row)}$$

Illustration-07: Show that $x = 2$ is the root of the equation

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$$ and solve it completely.

Solution: Applying elementary row operations, we have

$$\begin{vmatrix} x-2 & -6+3x & -1-x+3 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0; R_1' \to R_1 - R_2$$

$$\text{or, } \begin{vmatrix} x-2 & 3(x-2) & -(x-2) \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0 \Rightarrow (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$$

$$\text{or, } (x-2)\begin{vmatrix} 1 & 0 & 0 \\ 2 & -3(x+2) & x-1 \\ -3 & 2x+9 & x-1 \end{vmatrix}; \begin{matrix} C_2' \to C_2 - 3C_1 \\ C_3' \to C_3 + C_1 \end{matrix}$$

$$\text{or, } (x-2)\begin{vmatrix} -3(x+2) & x-1 \\ 2x+9 & x-1 \end{vmatrix} = 0 \text{ (Expanding with respect to first row)}$$

$$\text{or, } (x-2)(x-1)\{-3(x+2)-(2x+9)\} = 0$$

$$\text{or, } -5(x-2)(x-1)(x+3) = 0$$

i.e. $x = 2, x = 1, x = -3$ is the required solution.

Illustration-08: If x, y, z are all different and if

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0, \text{ prove that } xyz = -1.$$

Solution: The given determinant may be written as:

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

Taking out x, y, z common from R_1, R_2 and R_3 of the second determinant on R.H.S., we get

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Applying row operations as indicated below:

$$= (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}; \begin{matrix} R_2' \to R_2 - R_1 \\ R_3' \to R_3 - R_1 \end{matrix}$$

$$= (1+xyz)\{(x-y)(y-z)(z-x)\}$$

Now $\Delta = 0 \Rightarrow (1+xyz)(x-y)(y-z)(z-x) = 0$

As x, y, z are all different, therefore $x - y \neq -, z - x \neq 0$. Hence, $1 + xyz = 0$ or, $xyz = -1$.

Illustration-09: Solve the following system of equations using Cramer's rule
$x+y=11,\ 2x-6y=0.$

Solution: The determinant of the coefficients of x and y is:

$$D=\begin{vmatrix}1 & 1\\ 2 & -6\end{vmatrix}=-6-2=-8(\neq o).$$

To find D_x, we change first column of D by R.H.S. terms 11, 0. Then

$$D_x=\begin{vmatrix}11 & 1\\ 0 & -6\end{vmatrix}=-66.$$

Similarly to find D_y, we change first second column of D by R.H.S. terms 11, 0.

Then $$D_y=\begin{vmatrix}1 & 11\\ 2 & 0\end{vmatrix}=-22.$$

Hence, $$x=\frac{D_x}{D}=\frac{-66}{-8}=\frac{33}{4};\ y=\frac{D_y}{D}=\frac{-22}{-8}=\frac{11}{4}.$$

Illustration-10: Solve the following system of equations using Cramer's method.
$x+y+z=6;\quad x-y+z=2;\quad 2x+y-z=1$

Solution: The determinant of the coefficients of x, y and z is given by

$$D=\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 2 & 1 & -1\end{vmatrix}=(1-1)-1(-1-2)+1(1+2)=6(\neq 0).$$

The solution is given by $\frac{x}{D_x}=\frac{y}{d_y}=\frac{z}{D_z}=\frac{1}{D}$

Where $$D_x=\begin{vmatrix}6 & 1 & 1\\ 2 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=6\begin{vmatrix}-1 & 1\\ 1 & -1\end{vmatrix}-\begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix}+\begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix}=6$$

$$D_y=\begin{vmatrix}1 & 6 & 1\\ 1 & 2 & 1\\ 2 & 1 & -1\end{vmatrix}=1\begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix}-6\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}+\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}=12$$

$$D_z=\begin{vmatrix}1 & 1 & 6\\ 1 & -1 & 2\\ 2 & 1 & 1\end{vmatrix}=1\begin{vmatrix}-1 & 2\\ 1 & 1\end{vmatrix}-1\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}+6\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix}=18$$

Thus, we have $\frac{x}{6}=\frac{y}{12}=\frac{z}{18}=\frac{1}{6}$ or $x=1, y=2, z=3.$

EXERCISE-I (Determinants)

1. Find the value of the determinant

(a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$, where $(a+b+c)=0$.

(b) $\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ (d) $\begin{vmatrix} 1 & 18 & 72 \\ 2 & 40 & 148 \\ 2 & 45 & 150 \end{vmatrix}$

2. If $a+b+c=0$, then prove that

(a) $\begin{vmatrix} a-b & c & b \\ c & b-c & a \\ b & a & c-a \end{vmatrix}=0$ (b) $\begin{vmatrix} a & b & c \\ 0 & a & b \\ b & 0 & a \end{vmatrix}=c(2ab-c^2)$

3. Show that

(a) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=(a-b)(b-c)(c-a)$ (b) $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}=0$

(c) $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}=2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

(d) $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$

4. Prove that

(a) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & xz & xy \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$

(b) $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\gamma)$

(c) $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} = a^3+b^3+c^3-3\,abc.$

(d) $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

5. (a) Solve that equation $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0.$

(b) If $x \neq y \neq z$ and $\begin{vmatrix} x & y & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then prove that $xyz = -1.$

(c) Prove that, $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

6. Find the value of the determinant and express its square as a third order determinant:

$$\begin{vmatrix} a & b & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$$

and hence show that

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ab-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$$

7. Prove that: $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

8. Prove the following:

(a) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

(b) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx).$

(c) $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$

9. Prove that

(a) $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

(b) $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

(c) $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

10. Solve the following system of linear equations by using Crame's rule:

(a) $x-y+2z=9;\quad 2x-5y+3z=18;\quad 6x+7y+10z=35$

(b) $3x+3y-z=11;\quad 2x-y+2z=9;\quad 4x+3y+2z=25$

(c) $x+y-z=-3;\quad 2x+3y+z=2;\quad 8y+3z=1$

(d) $x+y+z+1=0;\quad x+2y+3z+4=0;\quad x+3y+4z+6=0$

(e) $2x+y+z=7;\quad 3x-y-z=-2;\quad x+2y-3z=-4$

(f) $2x-3y+z=7;\quad 2x+y-z=1;\quad 4x+3z=-11.$

MATRICES

A matrix is an ordered system of numbers arranged in formation of rows and columns, describing various aspects of a phenomenon inter-related to each other. For example, it is used in the study of dominance within a group by sociologists, in the study of births and survivals, marriage and decent, class structure and mobility by demographers, etc.

11.6 DEFINITION AND NOTATIONS OF MATRIX

Definition: A rectangular arrangement of numbers (real or complex) in m rows and n columns is called matrix of order (or size) m by n, denoted by $m \times n$. A general form of the matrix of order $m \times n$ is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \rightarrow \text{Row} \end{matrix}$$

$$\uparrow$$

Column

The number $a_{ij}\,(i = 1,2,\ldots, m \text{ and } j = 1,2,\ldots, n)$ in this array (rectangular or square) of mn numbers is called an element of the matrix A. In compact form, the matrix of order $m \times n$ is also written as :

$$A = (a_{ij})_{m\times n}, i = 1,2,\ldots, m; j = 1,2,\ldots, n.$$

In this double subscript notation, the first subscript indicates the row and the second subscript indicates the column in which the element lies. For example, a_{ij} is an element of matrix appearing in i^{th} row and j^{th} column.

Remarks: A matrix is not a number. It has got no numerical value. For example, 5 is simply a number, but in the notation of matrix [5] is a matrix of order 1×1.

Notation: A matrix is some times represented by pairs of parentheses ( ) or a pair of double bars $\| \ \|$ or a pair of brackets [ ]. Matrices are generally denoted by capital letter of English alphabet such as

$A, B, C, X, Y,$ etc., and their elements by corresponding small letters a, b, c etc. For example, the following matrices

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 4 & -2 & 2 \end{bmatrix} \text{ or } \begin{pmatrix} 1 & 3 & 6 \\ 4 & -2 & 2 \end{pmatrix} \text{ or } \begin{Vmatrix} 1 & 3 & 6 \\ 4 & -2 & 2 \end{Vmatrix}, B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

have two rows and three columns. Therefore, A and B are matrices of order 2×3.

11.7 TYPES OF MATRICES

Row and column matrices: A matrix $(a_{ij})_{1\times n}$ in which there is only one row is called a row matrix. A matrix $(a_{ij})_{m\times 1}$ in which there is only one column is called a column matrix.

A row matrix is also called row vector and a column matrix is also called column vector.

Example: (i) The matrix $A=[3 \quad 6 \quad 2]$ is a 1×3 matrix or row matrix (or row vector).

(ii) The matrix $A=\begin{bmatrix}2\\1\end{bmatrix}$ is a 2×1 matrix or column matrix (or column vector).

Square matrix: An $m\times n$ matrix A is said to be a square matrix if $m=n$, i.e., number of rows is equal to the number of columns.

Thus, the matrix $A=(a_{ij})_{n\times n}, i=1,2,...,n; j=1,2,...,n.$ is a $n\times n$ square matrix.

For example, the following matrices:

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}_{2\times 2}, \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}_{3\times 3}, \begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}_{n\times n}$$ are square matrices.

In a square matrix of order n, elements a_{ij} for which $i=j$, i.e. the elements $a_{11}, a_{22},..., a_{nn}$ are known as diagonal elements, and the line along which the above elements lie is called the principal diagonal or the diagonal of the matrix. For example, the matrix

$$\begin{bmatrix}3 & 4 & 5\\ 6 & 2 & 3\\ -2 & 5 & 1\end{bmatrix}_{3\times 3}$$

is the square matrix of order 3, in which the diagonal elements are 3,2, and 1.

Null Matrix: A matrix in which every element is zero is called a zero (or null) matrix and is denoted by 0.

For example: $A=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ is a null matrix.

Equal Matrices: Two matrices are said to be equal if and only if

(a) they are of the same order, i.e. they have the same number of rows and columns and

(b) each eiement of one is equal to the corresponding element of the other.

Given two matrices A and B.

$$A=\begin{bmatrix}a & b\\ d & e\end{bmatrix}, B=\begin{bmatrix}1 & -1\\ 3 & 2\end{bmatrix}$$

The matrix A is of order 2×2 and B is also of the order 2×2, so A and B are equal if

$a = 1, b = -1$

$d = 3, e = 2$

Diagonal matrix: A square matrix $(a_{ij})_{n\times n}$ is called a diagonal matrix if each of its non-diagonal element is zero i.e. if $a_{ij} = 0$ when $i \neq j$. Thus, the matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

is a diagonal matrix of order 3, and it can be written as diag. $[a_{11} \quad a_{22} \cdots a_{33}]$

Scalar matrix: A diagonal matrix whose diagonal elements are all equal is called a scalar matrix. i.e. a square matrix where diagonal elements are all equal and the remaining elements are zero is a scalar matrix. Thus, the matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$ is a scalar matrix of order 3.

Unit (or Identity) matrix: A square matrix where diagonal elements are unity (or one) and remaining elements are zero is called a unit (or identity) matrix

A square matrix $A = (a_{ij})_{n\times n}$ is a unit matrix if (i) $a_{ij} = 1$ when $i = j$, and

(ii) $a_{ij} = 0$ when $i \neq j$.

Such a matrix is denoted by I_n. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$ are unit matrices of order two and three,

respectively.

Symmetric matrix: A matrix is said to be symmetric, if it is

(i) a square matrix and

(ii) $a_{ij} = a_{ji}$, i.e. (i,j)th element is the same as the (j,i)th element.

In other words, a symmetric matrix does not change if we interchange rows and columns. For example, the following two matrices

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix}, \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

are symmetric matrices because $a_{12} = a_{21}, a_{13} = a_{31}$ and so on.

Skew symmetric matrix: A square matrix $A=(a_{ij})_{n\times n}$ is said to be skew symmetric, if $a_{ij}=-a_{ji}$, i.e. (i,j)th element is the negative of (j,i)th element.

If we put $j=i$, we have $a_{ii}=-a_{ii}=0$ which implies that the elements of the diagonal of skew symmetric are all zero. For example, matrices

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

are skew symmetric matrices because $a_{12}=-a_{21}, a_{11}=a_{22}=a_{33}=0, a_{13}=-a_{31}.$

Sub Matrix: A matrix which is obtained from a given matrix by deleting any number of rows and any number of columns is called a sub matrix of the given matrix.

For example, if we delete first row and first column from the matrix $\begin{bmatrix} 5 & 3 & 2 \\ 1 & 1 & 2 \\ 7 & 3 & 4 \end{bmatrix}$

of order 3×3 ,then the matrix so obtained $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is the sub-matrix of the given matrix.

Upper and lower triangular matrices: Upper triangular matrix is a square matrix in which all the elements below the principal diagonal are zero, i.e., $a_{ij}=0$ for all $i>j$.

Lower triangular matrix is a square matrix in which all the elements above the principal diagonal are zero, i.e. $a_{ij}=0$ for all $i<j$.

For example, the following two matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n\times n} \text{ and } \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

are upper and lower triangular matrices, respectively.

A triangular matrix, $A=(a_{ij})_{n\times n}$ is called a strictly triangular if $a_{ij}=0$ for all $i=1,2,\ldots,n$.

11.8 ALGEBRA OF MATRICES

In the algebra of matrices, we shall define mathematical operations on matrices, which enable us to combine matrices so as to produce new matrix. The main operations are:

(i) Addition and subtraction of matrices

(ii) Scalar multiplication i.e., multiplication of a matrix by a scalar.

(iii) Multiplication of matrices.

Addition and Subtraction of Matrices

Addition: The sum of two matrices of the same order is obtained by adding their corresponding elements and their sum is said to be defined as the matrix of the same order.

If $A=\lfloor a_{ij}\rfloor$ and $B=\lfloor b_{ij}\rfloor$ are two matrices of the same order $m\times n$, then their sum is another matrix $C=\lfloor c_{ij}\rfloor_{m\times n}$, where $c_{ij}=a_{ij}+b_{ij}$, i.e. each element of C is the sum of the corresponding elements of A and B and is denoted by $A+B$.

Subtraction: Let $A=\lfloor a_{ij}\rfloor$ and $B=\lfloor b_{ij}\rfloor$ be two matrices of same order $m\times n$. Then $C=A-B=A+(-B)=\lfloor a_{ij}\rfloor_{m\times n}+\lfloor -b_{ij}\rfloor_{m\times n}=\lfloor a_{ij}-b_{ij}\rfloor_{m\times n}$ i.e. the difference $A-B$ is obtained by subtracting each element of B from the corresponding element of A.

Example: If $A=\begin{bmatrix}3 & 2 & 0\\ 2 & 5 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & -1 & 0\\ 2 & 0 & 3\end{bmatrix}$ are two matrices of same order. then

$$A+B=\begin{bmatrix}3+1 & 2-1 & 0+0\\ 2+2 & 5+0 & -1+3\end{bmatrix}=\begin{bmatrix}4 & 1 & 0\\ 4 & 5 & 2\end{bmatrix}$$

and $A-B=\begin{bmatrix}3-1 & 2-(-1) & 0-0\\ 2-2 & 5-0 & -1-3\end{bmatrix}=\begin{bmatrix}2 & 3 & 0\\ 0 & 5 & -4\end{bmatrix}$ are two matrices, of order

Properties of Matrix Addition:

Let $A=[a_{ij}]$, $B=[b_{ij}]$ *and* $C=[c_{ij}]$ be three matrices of order $m\times n$ each. Then

(i) Matrix addition is commutative: $A+B=B+A$

(ii) Matrix addition is associative: $(A+B)+C=A+(B+C)$.

(iii) Distributive law for scalar multiplication: $\alpha(A+B)=\alpha A+\alpha B$ where α is a scalar.

(iv) Existence of additive identity: For any matrix, say $A=\lfloor a_{ij}\rfloor$ of order $m\times n$, there is a null (or zero) matrix 0 of the same order, such that $A+0=0+A=A$

This is because $a_{ij}+0=0+a_{ij}=a_{ij}$ for each (i,j).

(v) Additive inverse: Any matrix A of order $m\times n$ is called the additive inverse of a matrix $B(=-A)$ of the same order if $A+(-A)=0=(-A)+A$, where 0 is the additive identity.

(vi) Cancellation law: If A, B, and C are matrices of the same order, then

$A+C=B+C$ implies $A=B$.

For example: Let $A=\begin{bmatrix}2 & 3 & 5\\ 6 & 7 & 8\end{bmatrix}, B=\begin{bmatrix}7 & 6 & 3\\ 1 & 4 & 5\end{bmatrix}, C=\begin{bmatrix}6 & -2 & 7\\ 6 & 0 & 9\end{bmatrix}$ and $0=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

(a) Addition is commutative $A+B=\begin{bmatrix}9 & 9 & 8\\ 7 & 11 & 13\end{bmatrix}, B+A=\begin{bmatrix}9 & 9 & 8\\ 7 & 11 & 13\end{bmatrix}$

(b) Addition is associative

$$A+(B+C)=\begin{bmatrix}2 & 3 & 5\\ 6 & 7 & 8\end{bmatrix}+\begin{bmatrix}13 & 4 & 10\\ 9 & 4 & 14\end{bmatrix}=\begin{bmatrix}15 & 7 & 15\\ 15 & 11 & 22\end{bmatrix}$$

$$(A+B)+C=\begin{bmatrix}9 & 9 & 8\\ 7 & 11 & 13\end{bmatrix}+\begin{bmatrix}6 & -2 & 7\\ 8 & 0 & 9\end{bmatrix}=\begin{bmatrix}15 & 7 & 15\\ 15 & 11 & 22\end{bmatrix}$$

(c) Additive identity

$$A+0=\begin{bmatrix}2 & 3 & 5\\ 6 & 7 & 8\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}2 & 3 & 5\\ 6 & 7 & 8\end{bmatrix}=A$$

(d) Additive inverse

$$A+(-A)=\begin{bmatrix}2 & 3 & 5\\ 6 & 7 & 8\end{bmatrix}+\begin{bmatrix}-2 & -3 & -5\\ -6 & -7 & -8\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0$$

Scalar Multiplication of a Matrix

Definition: If $A=[a_{ij}]$ is an $m\times n$ matrix and k is a real number (also called a scalar), then kA is defined as the matrix each element of which is k times the corresponding element of the matrix A, i.e. $kA=[ka_{ij}]_{m\times n}$,

For example, if $A=\begin{bmatrix}2 & 3 & 5\\ 6 & 7 & 8\end{bmatrix}$, then $2A=\begin{bmatrix}4 & 6 & 10\\ 12 & 14 & 16\end{bmatrix}$.

Properties of scalar multiplication: If A, B and O are three matrices of same order, then for any scalar k, k_1 and k_2, we have the following properties of scalar multiplication:

(i) $k(A+B)=kA+kB$ (ii) $(k_1+k_2)A=k_1A+k_2A$

(iii) $k_1(k_2A)=(k_1k_2)A$ (iv) $0A=0$

(v) $k0=0$ (vi) $(-k)A=-(kA)=k(-A)$

For example If $A=\begin{bmatrix}0 & 2 & 3\\ 2 & 1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}7 & 6 & 3\\ 1 & 4 & 5\end{bmatrix}$, find the value of $2A+3B$.

Solution: $2A=2\begin{bmatrix}0 & 2 & 3\\ 2 & 1 & 4\end{bmatrix}=\begin{bmatrix}0 & 4 & 6\\ 4 & 2 & 8\end{bmatrix}$, $3B=3\begin{bmatrix}7 & 6 & 3\\ 1 & 4 & 5\end{bmatrix}=\begin{bmatrix}21 & 18 & 9\\ 3 & 12 & 15\end{bmatrix}$

$$\therefore 2A+3B=\begin{bmatrix}0 & 4 & 6\\ 4 & 2 & 8\end{bmatrix}+\begin{bmatrix}21 & 18 & 9\\ 3 & 12 & 15\end{bmatrix}=\begin{bmatrix}21 & 22 & 15\\ 7 & 14 & 23\end{bmatrix}.$$

11.9 MULTPLICATION OF MATRICES

If A and B are two matrices such that the number of columns in A is equal to the number of rows in B, i.e. if $A=[a_{ij}]_{m\times n}$ and $B=[b_{jk}]_{n\times p}$, then the product of A and B denoted by AB is defined as $C=[c_{ik}]m\times p$, For example, consider the matrices

$$A=\begin{bmatrix}2 & 3 & 1\\ 3 & 1 & 0\end{bmatrix}_{2\times 3}, B=\begin{bmatrix}4\\ 6\\ 7\end{bmatrix}_{3\times 1}$$

Here the number of columns in $A(=3)$ = the number of rows in B and thus, we can evaluate AB.

Let $AB = [c_{ij}]$, where $[c_{ij}]$ is 2×1 matrix. Now to write c_{ij} we take the elements of the first row of A, i.e., 2, 3, 1 and the elements of first column of B i.e. 4,6,7, and form the product and finally add them i.e.,

$$c_{11} = 2\times4+3\times6+1\times7 = 33.$$

Similarly, $$c_{21} = 3\times4+1\times6+0\times7 = 18.$$

Hence, $$AB = \begin{bmatrix}2 & 3 & 1\\ 3 & 1 & 0\end{bmatrix}\begin{bmatrix}4\\6\\7\end{bmatrix} = \begin{bmatrix}8+18+7\\12+6+0\end{bmatrix} = \begin{bmatrix}33\\18\end{bmatrix}_{2\times1}$$

REMARKS: The product AB can be calculated if and only if the number of columns in A are equal to the number of rows in B. The two matrices A and B satisfying this condition are called comfortable to multiplication.

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$ are two matrices and their rows and columns be denoted by $R_1, R_2 \ldots, R_m$ and $C_1, C_2 \ldots, C_P$ respectively. Then, we have

$$AB = \begin{bmatrix}R_1\\R_2\\\vdots\\R_m\end{bmatrix}_{m\times n} [C_1C_2\ldots C_P]_{n\times p} = \begin{bmatrix}R_1C_1 & R_1C_2 & \cdots & R_1C_P\\ R_2C_1 & R_2C_2 & \cdots & R_2C_P\\ \vdots & \vdots & & \vdots\\ R_mC_1 & R_mC_2 & \cdots & R_mC_P\end{bmatrix}_{m\times p}$$

POST-MULTIPLICATION AND PRE-MULTIPLICATION OF MATRICES

In the matrix product AB, the matrix A is pre-multiplied (also called pre-multiplier) with matrix B (also called post multiplier).

The products AB and BA may or may not exists and may be equal or different. However, if AB exists, then it is not necessary that the product BA will also exist, i.e., in general $AB \neq BA$.

For example: If $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B = \begin{bmatrix}2 & 3\\ 4 & 7\end{bmatrix}$, then compute AB.

Solution:
$$AB = \begin{matrix}R_1\\R_2\end{matrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}2^c & 3^c\\ 4 & 7\end{bmatrix} = \begin{bmatrix}R_1C_1 & R_1C_2\\ R_2C_1 & R_2C_2\end{bmatrix} = \begin{bmatrix}1\times2+2\times4 & 1\times3+2\times7\\ 3\times2+4\times4 & 3\times3+4\times7\end{bmatrix}$$

$$= \begin{bmatrix}2+8 & 3+14\\ 6+16 & 9+18\end{bmatrix} = \begin{bmatrix}10 & 17\\ 22 & 37\end{bmatrix}.$$

PROPERTIES OF MATRIX MULTIPLICATION

(i) Multiplication is associative: Let $A=[a_{ij}]_{m\times n}, B=[b_{jk}]_{n\times p}, C=[c_{kt}]_{p\times q}$ be any three matrices.

Then $(AB)C=A(BC)$ where $(AB)C$ and $A(BC)$ both exist and defined because each of these is of order $m\times q$.

(ii) Matrix Multiplication in General is not commutative:

Case 1: If A is $m\times n$ matrix and B is $n\times p$ matrix, then the product AB exists whereas BA does not exists, since we know that AB can be calculated only if the number of columns in A is equal to the number of rows in B.

Case 2: If A is $m\times n$ matrix and B is $n\times m$ matrix, then both AB and BA exist, but AB is $m\times m$ matrix while the matrix BA is $n\times n$ matrix. Since the matrices AB and BA are not of the same order, therefore $AB\neq BA$.

Case 3: If both A and B are square matrices of the same order, then AB as well as exist but are not necessarily equal.

For example: If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, and $B=\begin{bmatrix}3 & 1\\ 4 & 7\end{bmatrix}$ are two matrices, then

$$AB=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}3 & 1\\ 4 & 7\end{bmatrix}=\begin{bmatrix}11 & 15\\ 25 & 31\end{bmatrix} \text{ and } BA=\begin{bmatrix}3 & 1\\ 4 & 7\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}6 & 10\\ 25 & 36\end{bmatrix}$$

$\therefore AB\neq BA.$

But, if $A=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}$ are two matrices, then

$$AB=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -8\end{bmatrix} \text{ and } BA=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -8\end{bmatrix}$$

$\therefore AB=BA$. Hence, in general, $AB\neq BA$.

(iii) Distribution with Respect to Addition: Let A, B and C be three matrices of order $m\times n, n\times p$ and $n\times p$, respectively. Then

$$A(B+C)=AB+AC$$

This shows that order of $B+C$ is $n\times p$ and the order of the product A. $(B+C)$ is $m\times p$. In the same way the AB, AC and $AB+AC$ are each of order $m\times p$. Hence, multiplication of matrices is distributive with respect to addition of matrices.

(iv) Multiplicative Identity: If $A=[a_{ij}]$ is a square matrix of order $n\times n$ and I is the unit matrix of the same order, then $AI=IA=A$.

(v) Multiplication with Zero Matrix: If A is any matrix of order $m\times n$ and 0 is the zero matrix of order $n\times p$, then $A0=0=0A$.

(vi) $AB=0$ (null matrix) whereas neither A nor B is zero matrix.

For example, if $A=\begin{bmatrix}4 & 4\\ 3 & 3\end{bmatrix}$, and $B=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$, then

$$AB=\begin{bmatrix}4 & 4\\ 3 & 3\end{bmatrix}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$$

REMARKS: 1. If A is a square matrix, then $A^nA^m=A^mA^n=A^{n+m}$, where A^n means AAA...A, i.e., A multiplied n times.

2. If I is a unit matrix, the $I=I^2=I^3=\cdots=I^m$.

11.10 TRANSPOSE OF A MATRIX

Definition: The matrix obtained from any given matrix A, by interchanging its rows and columns is called the transpose of A and is usually denoted by A^t or, A^T. Thus, if $A=[a_{ij}]$ is a matrix of order $m\times n$, then $A^T=[a_{ij}]$ is the transpose matrix of order $n\times m$.

For example, if $A=\begin{bmatrix}2 & 1\\ 3 & 2\\ 4 & 5\end{bmatrix}_{3\times2}$, then $A^T=\begin{bmatrix}2 & 3 & 4\\ 1 & 3 & 5\end{bmatrix}_{2\times3}$

PROPERTIES OF TRANSPOSE OF A MATRIX

(i) The transpose of the transpose of a matrix is original matrix, i.e. $(A^T)^T=A$.

(ii) The transpose of the sum (or difference) of two matrices is the sum (or difference) of their transpose i.e., $(A+B)^T=A^T+B^T$.

(iii) If A is any $m\times n$ matrix, then $(kA)^T=k\,A^T$, where k is any number.

(iv) The transpose of the product of a product of two matrices is the product of their transposes taken in the reverse order, i.e., if A and B are two matrices of $m\times n$ and $n\times P$ respectively, then $(AB)^T=B^TA^T$.

Theorem-01: Show that positive odd integral power of a skew-symmetric matrix is skew-symmetric and positive even integral powers of a skew-symmetric matrix is symmetric.

Solution: Let A be skew-symmetric matrix. Then $A^T=-A$.

Since all positive integral powers of a symmetric matrix are symmetric, therefore

$$(A^n)^T=(A^T)^n \text{ for all } n\in N. \quad \text{or, } (A^n)^T=(-A)^n\ (\because A^T=-A)$$

$$\text{or, } (A^n)^T=(-1)^nA^n \quad \text{or, } (A^n)^T=\begin{cases}A^n \text{ if } n \text{ is even}\\ -A^n \text{ if } n \text{ is odd}\end{cases}$$

Hence A^n is symmetric if n is even and skew-symmetric if n is odd.

Theorem-02: Let A and B be symmetric matrices of the same order. Then, show that

(i) $A+B$ is a symmetric matrix

(ii) $AB-BA$ is a skew-symmetric matrix

Solution: Since A and B are symmetric matrices, therefore $A^T = A$ and $B^T = B$.

(i) $(A+B)^T = A^T + B^T = A+B \quad (\because A^T = A, B^T = B)$

Thus, $A+B$ is symmetric

(ii) $(AB-BA)^T = (AB)^T - (BA)^T$

$= B^T A^T - A^T B^T$ (By reversal law)

$= BA - AB = -(AB-BA) \quad [\because B^T = B, A^T = A]$

Thus $AB-BA$ is skew-symmetric.

11.11 ADJOINT OF A SQUARE MATRIX

Let $A = |a_{ij}|$ be a square matrix of order n and let A_{ij} be the co-factor of a_{ij} in determinant A. Then the transpose of the co-factor matrix $B = |A_{ij}|$ of elements of A is called the adjoint of A and is denoted by adj A. Thus, if $A = |a_{ij}|$ and $B = |A_{ij}|$, then

$$\text{adj } A = [A_{ij}]^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

PROPERTIES OF AD-JOINT

(i) If $A = |a_{ij}|$ is a square matrix of order n and I is an identity matrix of order n then

$A(adjA) = (adj\,A)A = |A|I.$

(ii) $|\text{adj}\,A| = |A|^{n-1}$.

(iii) $\text{adj}\,(A^T) = (\text{adj}\,A)^T$.

(iv) If A and B are two square matrices of same order, then $\text{adj}\,(AB) = \text{adj}\,B \times \text{adj}\,A$.

For example: Find the ad-joint of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 4 \end{bmatrix}$ and verify the result:

$A(\text{adj}\,A) = (\text{adj}\,A).A = |A|.I_3.$

Solution: $\text{adj}\,A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

where $A_{11} = (-1)^{1+1}\begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} = 15,\ A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} = 0,\ A_{13} = (-1)^{1+3} = \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} = -10.$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6,\ A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3,\ A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0.$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = -15,\ A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0,\ A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5.$

$\therefore \text{adj } A = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}^T = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

Also, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{vmatrix} = 1\begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} = 1(15) - 2(0) + 3(-10) = 15 - 30 = -15$

$\therefore A \times \text{adj } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix} = -15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3 \quad ...(i)$

Also $\text{adj}\,A \times A = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

$= \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix} = -15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3 \quad ...(ii)$

Hence, from (i) and (ii), adj $A \times A = A \times \text{adj } A = |A|I$. [Proved]

11.12 INVERSE OF SQUARE MATRIX

Singular and non-singular matrices: A square matrix A is said to be non-singular if $|A| \neq 0$; otherwise singular.

Definition: Let A be any square matrix of order n. If there exists a square matrix B of order n which satisfies the relation: $AB = BA = I_n$, where I_n is the identity matrix of order n, then B is called the inverse of A and is denoted by A^{-1}, i.e., $B = A^{-1}$ so that, $AA^{-1} = A^{-1}A = I_n$.

Remarks: 1. A matrix, which has inverse, is also called invertible.

2. If A is a square matrix such that $|A| \neq 0$, then A is invertible and $A^{-1} = \frac{\text{adj } A}{|A|}$.

The concept of inverse of a matrix is useful in solving the system of linear simultaneous equations, input-output analysis and regression analysis. We shall discuss here following two methods of finding the inverse of a given square matrix:

(i) Adjoint matrix method

(ii) Row transformation method

Remark: The necessary and sufficient condition for a matrix to be invertible is that it must be non-singular and only square matrices can have inverse.

PROPERTIES OF INVERSE

The useful properties of matrix inverse are summarized as follows:

(i) The inverse of a non-singular matrix is unique, that is, there is only one matrix for which the relation, $AB = BA = I$ holds.

(ii) If two n-order matrices A, B are such that $AB = I$, then A and B are non-singular. $A^{-1} = B, B^{-1} = A$ and $BA = I$.

(iii) If A, B are two n-order non-singular matrices, then $(AB)^{-1} = B^{-1}A^{-1}$ $(\text{not } A^{-1}B^{-1})$.

(iv) The inverse of the inverse is the original matrix A, i.e. $(A^{-1})^{-1} = A$.

(v) The inverse of the transpose of a matrix is the transpose of its inverse, i.e. $(A^T)^{-1} = (A^{-1})^T$.

(vi) If A is non-singular and $AB = O$, then $B = O$.

(vii) The identity matrix is its own inverse, i.e. $I^{-1} = I$.

METHODS OF COMPUTING THE INVERSE OF A MATRIX

(a) Adjoint Matrix Method

Given a non-singular matrix A of order n. We already know that

$$A.(\text{adj } A) = |A|.I = (\text{adj } A).A$$

$$\text{or, } \frac{A.(\text{adj } A)}{|A|} = \frac{(\text{adj } A).A}{|A|} = I \text{ Hence, } A^{-1} = \frac{\text{adj } A}{|A|}, \text{ provided } |A| \neq 0.$$

11.13 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

Definition The system of n equations in n unknowns $x_1, x_2, \ldots, x_n$ of the form

$$\begin{aligned} a_{1f}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn} &= b_n \end{aligned}$$

is known as a system of simultaneous (non-homogenous) linear equations, where $b_1, b_2, \ldots, b_n$ are constants.

Definition: By the solution of this system of equations we mean a set of values of $x_1, x_2, \ldots, x_n$ which satisfies simultaneously these n equations. When a system of equations has one or more solutions, it is said to be consistent, otherwise the system of equations is said to be inconsistent.

This system of equations can also be written in the matrix form as: $AX = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

so that, A, X, B are matrices of order $n \times n, n \times 1, n \times 1$, respectively, and the matrix A is known as coefficient matrix.

Remarks:

1. Any set of values of variables $x_1, x_2, \ldots, x_n$ which simultaneously satisfy the system of equations $AX = B$ is called a solution of the system.
2. If $b_1 = b_2 = \cdots = b_n = 0$, the set of equations is said to be homogeneous. A system of linear homogeneous equations has a non-trivial solution if the number of equations is less than the number of variables.
3. If $|A| \neq 0$, then A is non-singular and hence its inverse, A^{-1} exists and unique. Take the system of equations $AX = B$. Pre-multiplying both sides by A^{-1}, we have

$$A^{-1}(AX) = A^{-1}B$$

$$\text{or, } IX = A^{-1}B \qquad (\because A^{-1}A = I)$$

$$\text{or, } X = A^{-1}B = \frac{\text{adj } A}{|A|} \times B$$

which gives the solution of the system of simultaneous (non-homogenous) equations when the number of variables are equal to the number of equations.

CRITERION OF CONSISTENCY

If there is a system of n linear equations in n variables given by $AX = B$ and A is non-singular, then the system has unique solution given by $X = A^{-1}B$.

The following are criteria of consistency or inconsistency of a system of linear equations given $AX = B$, where A is square matrix.

(i) If $|A| \neq 0$, then system is consistent and has a unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$, the system of equations has either no solution or an infinite number of solutions.

(a) If $(\text{adj } A)B \neq 0$, the system has no solution and is therefore inconsistent.

(b) If $(\text{adj } A)B = 0$, the system is consistent and has infinitely many solutions.

METHODS OF SOLVING NON-HOMOGENEOUS LINEAR EQUATIONS

The solution of linear non-homogeneous simultaneous equations can be obtained by applying following methods:

1. Matrix Inverse Method
2. Gauss Elimination Method
3. Cramer's Rule

1. Inverse matrix method:

If $|A| \neq 0$, then A is non-singular and hence its inverse, A^{-1} exists and unique. Take the system of equations $AX = B$. Pre-multiplying both sides by A^{-1}, we have

$$A^{-1}(AX) = A^{-1}B$$

$$\text{or, } IX = A^{-1}B \qquad (\because A^{-1}A = I)$$

$$\text{or, } X = A^{-1}B = \frac{\text{adj } A}{|A|} \times B$$

which gives the solution of the system of simultaneous (non-homogenous) equations when the number of variables are equal to the number of equations.

2. Gauss elimination method (or Triangular form reduction method)

This method is also known as **pivotal reduction method.** According to this method the co-efficient matrix $|a_{ij}|$ for a system of simultaneous equations (homogeneous or non-homogeneous) is reduced in such a way that $a_{ij} = 0$ for $j > i$.

This method can be used even if the co-efficient matrix is singular or non-singular, the number of equations equals to the number of unknowns or otherwise.

3. Cramer's Rule: Please see the determinant section, i.e. section 11.5

ILLUSTRATIONS (Matrices)

Illustration-11: If $A=\begin{bmatrix}2 & -1 & 3\\ 4 & 2 & 0\\ -2 & 7 & 3\end{bmatrix}$ and $B=\begin{bmatrix}6 & -2 & 7\\ 8 & 0 & 9\\ 3 & 1 & -5\end{bmatrix}$, find a matrix X such that $2A+3X=5B$.

Solution: Given that $2A+3X=5B$. Therefore

$$3X=5B-2A=5\begin{bmatrix}6 & -2 & 7\\ 8 & 0 & 9\\ 3 & 1 & -5\end{bmatrix}-2\begin{bmatrix}2 & -1 & 3\\ 4 & 2 & 0\\ -2 & 7 & 3\end{bmatrix}$$

$$=\begin{bmatrix}30 & -10 & 35\\ 40 & 0 & 45\\ 15 & 5 & -25\end{bmatrix}-\begin{bmatrix}4 & -2 & 6\\ 8 & 4 & 0\\ -4 & 14 & 6\end{bmatrix}=\begin{bmatrix}26 & -8 & 29\\ 32 & -4 & 45\\ 19 & -9 & -31\end{bmatrix}$$

$$\text{or, } X=\frac{1}{3}\begin{bmatrix}26 & -8 & 29\\ 32 & -4 & 45\\ 19 & -9 & -31\end{bmatrix}=\begin{bmatrix}26/3 & -8/3 & 29/3\\ 32/3 & -4/3 & 45/3\\ 19/3 & -9/3 & -31/3\end{bmatrix}$$

Illustration-12: Construct a 2×2 matrix $A=[a_{ij}]$ whose elements are given by

$$a_{ij}=(i+2j)^2/2.$$

Solution: Here $a_{ij}=(i+2j)^2/2, 1\le i\le 2$ and $1\le j\le 2$. Therefore,

$$a_{11}=\frac{(1+2\times1)^2}{2}=\frac{9}{2},\ a_{12}=\frac{(1+2\times2)^2}{2}=\frac{25}{2}$$

$$a_{21}=\frac{(2+2\times1)^2}{2}=8 \text{ and } a_{22}=\frac{(2+2\times2)^2}{2}=18$$

So the required matrix is $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}9/2 & 25/2\\ 8 & 18\end{bmatrix}$

Illustration-13: If $\begin{bmatrix}a+b & 2\\ 5 & ab\end{bmatrix}=\begin{bmatrix}6 & 2\\ 5 & 8\end{bmatrix}$, find the values of a and b.

Solution: Since the corresponding elements of two equal matrices are equal, therefore

$$\begin{bmatrix}a+b & 2\\ 5 & ab\end{bmatrix}=\begin{bmatrix}6 & 2\\ 5 & 8\end{bmatrix}$$

$$\text{or, } a+b=6 \text{ and } ab=8\Rightarrow a+(8/a)=6 \qquad [\because ab=8 \text{ or, } b=8/a]$$

$$\Rightarrow a^2+8=6a\Rightarrow a^2-6a+8=a \text{ or, } (a-4)(a-2)=0, \text{i.e.} a=2,4.$$

Putting $a=2$ and 4 in $ab=8$, we obtain $b=4$ for $a=2$ and $b=2$ for $a=4$

Hence, $a=2, b=4$ or $a=4, b=2$.

Illustration-14: Find x, y, z, t if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix}+3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}=3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

Solution: The given matrix equation can be written as

$$\begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix}+\begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix}=\begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3 & 2z-3 \\ 2y & 2t+6 \end{bmatrix}=\begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

Equating corresponding terms on both sides, we get

$$2x+3=9, 2z-3=15, 2y=12 \text{ and } 2t+6=18$$

Solving them, we get $x=3, z=9, y=6$ and $t=6$ [Ans]

Illustration-15: Find X and Y if $X+Y=\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X-Y=\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

Solution: We have $X+Y=\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X-Y=\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Therefore $(X+Y)+(X-Y)=\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}+\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$\Rightarrow 2X=\begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix}=\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} \Rightarrow X=\frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}=\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

and $(X+Y)-(X-Y)=\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}-\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$\Rightarrow 2Y=\begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix}=\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow Y=\frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Thus, $X=\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$ and $Y=\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Illustration-16: When $A=\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ and $B=\begin{bmatrix} i & -1 \\ -1 & -i \end{bmatrix}$ where $i=\sqrt{-1}$, determine AB.

Also compute BA.

Solution: $AB=\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} i & -1 \\ -1 & -i \end{bmatrix}=\begin{bmatrix} 1.i-i1 & -1.1-i.i \\ -i.i-i.1 & i.1-i.1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$BA=\begin{bmatrix} i & -1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}=\begin{bmatrix} 2i & -2 \\ -2 & -2i \end{bmatrix}$$

Illustration-17: Show that $A=\begin{bmatrix}1 & 0 & -2\\ 2 & 2 & 4\\ 0 & 0 & 2\end{bmatrix}$ satisfies the equation $A^2-3A+2I_3=0$

Solution: $A^2=A.A=\begin{bmatrix}1 & 0 & -2\\ 2 & 2 & 4\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & -2\\ 2 & 2 & 4\\ 0 & 0 & 2\end{bmatrix}$

$$=\begin{bmatrix}1+0+0 & 0+0+0 & -2+0-4\\ 2+4+0 & 0+4+0 & -4+8+8\\ 0+0+0 & 0+0+0 & 0+0+4\end{bmatrix}=\begin{bmatrix}1 & 0 & -6\\ 6 & 4 & 12\\ 0 & 0 & 4\end{bmatrix}$$

$$\therefore A^2-3A+2I_3=\begin{bmatrix}1 & 0 & -6\\ 6 & 4 & 12\\ 0 & 0 & 4\end{bmatrix}-3\begin{bmatrix}1 & 0 & -2\\ 2 & 2 & 4\\ 0 & 0 & 2\end{bmatrix}+2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1-3+2 & 0+0+0 & -6+6+0\\ 6-6+0 & 4-6+2 & 12-12+0\\ 0+0+0 & 0+0+0 & 4-6+2\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0$$

Hence, $A^2-3A+2I_3=0$.

Illustration-18: Find the value of x such that: $\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}=0$

$$\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}=0 \text{ or, } \begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}7+2x\\ 12+x\\ 21+2x\end{bmatrix}=0$$

or, $7+2x+12x+x^2+21+2x=0$ or, $x^2+16x+28=0$

or, $(x+14)(x+2)=0$ i.e. $x=-2$ or -14.

Illustration-19: If $A=\begin{bmatrix}1 & 0\\ -1 & 7\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, then find k so that $A^2=8A+kI$.

Solution: We have, $A=\begin{bmatrix}1 & 0\\ -1 & 7\end{bmatrix}$. Therefore

$$A^2 = AA = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

and $$8A + kI = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} + k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

Thus, $$A^2 = 8A + kI \Rightarrow \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

Therefore, $1 = 8 + k$ and $56 + k = 49$.

Solving the two equations, we get $k = -7$.

Illustration-20: Let $(x) = x^2 - 5x + 6$. if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Solution: First, we note that by $f(A)$ we mean the matrix polynomial $A^2 - 5A + 6I_3$. That is, to obtain $f(A)$, x is replaced by A and the constant term is multiplied by the identity matrix of order same as that of A. Now,

$$A^2 = AA = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 1 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$-5A = \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix}$$

$$6I_3 = 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Therefore,

$$f(A) = A^2 - 5A + 6I_3 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Illustration-21: If 'a' is a non-zero real or complex number and $A = \begin{bmatrix} a & 1 \\ 0 & 1 \end{bmatrix}$, then use principle of mathematical induction to prove that $A^n = \begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix}$ for every positive integer n.

Solution: We shall prove the result by mathematical induction. For $n=1$, matrix A^n becomes

$$A=\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}=\begin{bmatrix} a^1 & 1a^{1-1} \\ 0 & a^1 \end{bmatrix}$$

So, the result is true for $n=1$.

Let the result be true for any positive integer $n=k$.

Then, $$A^k=\begin{bmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{bmatrix} \quad \ldots (i)$$

Now $$A^{K+1}=A^k\times A=\begin{bmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{bmatrix}\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

$$=\begin{bmatrix} a^k.a+0.ka^{k-1} & a^k.1+ka^{k-1}.a \\ a.0+0.a^k & 0.1+a^k.a \end{bmatrix}$$

$$=\begin{bmatrix} a^{k+1} & a^k+ka^k \\ 0 & a^{k+1} \end{bmatrix}=\begin{bmatrix} a^{k+1} & (k+1)a^k \\ 0 & a^{k+1} \end{bmatrix}$$

Thus, the result is true for $n=k+1$.

Hence, by mathematical induction, the result is true for all positive integer values of n.

Illustration-22: If $A=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then verify $A^n=\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ where n is any positive integer.

Solution: We shall prove the result by mathematical induction. For $n=1$, matrix A^n becomes

$$A=\begin{bmatrix} 1+2(1) & -4(1) \\ 1 & 1-2(1) \end{bmatrix}=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

Thus, result is true for $n=1$

Let us assume that the result is true for any positive integer, $n=k$. Then

$$A^k=\begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

Now $$A^{k+1}=A^k.A=\begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}\times\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 3+2k & -4-4k \\ 1+k & -1-2k \end{bmatrix}=\begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$$

Thus, the result is true for $n=k+1$.

Hence, by mathematical induction, the result is true for all positive integer values of n.

Illustration-23: Prove that the matrix A given by $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies the relation

$$A^2-(a+d)A=(bc-ad)I,$$ where I is a unit matrix of order two.

Solution: $A^2=A.A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}=\begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix}$

$$A^2-(a+d)A=\begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix}-(a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$=\begin{bmatrix} a^2+bc & -a(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix}+\begin{bmatrix} -a(a+d) & -b(a+d) \\ -c(a+d) & -d(a+d) \end{bmatrix}$$

$$=\begin{bmatrix} a^2+bc-b(a+d) & b(a+d)-b(a+d) \\ c(a+d)-c(a+d) & bc+d^2-d(a+d) \end{bmatrix}$$

$$=\begin{bmatrix} bc-ad & 0 \\ 0 & bc-ad \end{bmatrix}=(bc-ad)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=(bc-ad)I.$$

Illustration-24: If $A=\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}, B=\begin{bmatrix} 7 & 8 \\ 4 & 3 \end{bmatrix}$, then show that $(A+B)^T=A^T+B^T$.

Solution: $A+B=\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}+\begin{bmatrix} 7 & 8 \\ 4 & 3 \end{bmatrix}=\begin{bmatrix} 3+7 & 4+8 \\ 6+4 & 2+3 \end{bmatrix}=\begin{bmatrix} 10 & 12 \\ 10 & 5 \end{bmatrix}$

$$\therefore (A+B)^T=\begin{bmatrix} 10 & 10 \\ 12 & 5 \end{bmatrix}$$

Again $A=\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}, A^T=\begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix}$

$$B=\begin{bmatrix} 7 & 8 \\ 4 & 3 \end{bmatrix}, B^T=\begin{bmatrix} 7 & 4 \\ 8 & 3 \end{bmatrix}$$

$$A^T+B^T=\begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix}+\begin{bmatrix} 7 & 4 \\ 8 & 3 \end{bmatrix}=\begin{bmatrix} 3+7 & 6+4 \\ 4+8 & 2+3 \end{bmatrix}=\begin{bmatrix} 10 & 10 \\ 12 & 5 \end{bmatrix}$$

Hence, $(A+B)^T=A^T+B^T$.

Illustration-25: If $A=\begin{bmatrix}1 & -2 & 3\\ -4 & 2 & 5\end{bmatrix}, B=\begin{bmatrix}1 & 3\\ -1 & 0\\ 2 & 4\end{bmatrix}$, then show that $(AB)^T=B^TA^T$.

Solution: $AB=\begin{bmatrix}1 & -2 & 3\\ -4 & 2 & 5\end{bmatrix}\begin{bmatrix}1 & 3\\ -1 & 0\\ 2 & 4\end{bmatrix}=\begin{bmatrix}9 & 15\\ 4 & 8\end{bmatrix}\therefore(AB)^T=\begin{bmatrix}9 & 4\\ 15 & 8\end{bmatrix}$

Again, $B^T=\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & 4\end{bmatrix}$ and $A^T=\begin{bmatrix}1 & -4\\ -2 & 2\\ 3 & 5\end{bmatrix}$

$\therefore B^TA^T=\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & 4\end{bmatrix}\begin{bmatrix}1 & -4\\ -2 & 2\\ 3 & 5\end{bmatrix}=\begin{bmatrix}9 & 4\\ 15 & 8\end{bmatrix}=(AB)^T$

Hence, $(AB)^T=B^TA^T$.

Illustration-26: Verify that $B^TA^T=(AB)^T$, where $A=\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2\\ 1 & 1\\ 2 & 0\end{bmatrix}$.

Solution: (Try yourself)

Illustration-27: Find the values of a, b, c if matrix A given by

$$A=\frac{1}{2}\begin{bmatrix}a & 2 & 2\\ 2 & 1 & b\\ 2 & c & 1\end{bmatrix}$$

Obeys the law $AA^T=I$ *where*, A^T is the transpose of A, and I is the unit matrix of order 3.

Solution: (Try yourself)

Illustration-28: Give

$A=\begin{bmatrix}1 & 0 & 3\\ 2 & -2 & 1\end{bmatrix}, B=\begin{bmatrix}3 & 4 & 1\\ 0 & -1 & 5\\ 1 & 2 & -2\end{bmatrix}$ and $C=\begin{bmatrix}2\\ -1\\ 4\end{bmatrix}$ Calculate AB, B^TA^T, AC, C^TA^T

Solution: (Try yourself)

Illustration-29: Given, $A=\begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix}; B=\begin{bmatrix}1 & 0 & 9\\ 6 & 1 & 1\end{bmatrix}$ and $C=\begin{bmatrix}3 & 8\\ 0 & 1\end{bmatrix}$

Verify that (i) $(AB)^T = B^T A^T$ (ii) $(A+C)^T = C^T + A^T$

Solution: (Try yourself)

Illustration-30: Find the inverse of the matrix, $A=\begin{bmatrix}2 & 2 & 2\\ 2 & 1 & 1\\ -7 & 2 & -3\end{bmatrix}$

Solution: $|A| = 2\begin{vmatrix}1 & 1\\ 2 & -3\end{vmatrix} - 2\begin{vmatrix}2 & 1\\ -7 & -3\end{vmatrix} + 0\begin{vmatrix}2 & 1\\ -7 & 2\end{vmatrix} = 2(-3-2)-2(-6+7) = -12 \neq 0.$

Since $|A| \neq 0$, matrix A is non-singular. The co-factors A_{ij} of A are

$$A_{11} = (-1)^{1+1}\begin{vmatrix}1 & 1\\ 2 & -3\end{vmatrix} = -5,\ A_{12} = (-1)^{1+2}\begin{vmatrix}2 & 1\\ -7 & -3\end{vmatrix} = -1, A_{13} = (-1)^{1+3}\begin{vmatrix}2 & 1\\ -7 & 2\end{vmatrix} = 11.$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix}2 & 0\\ 2 & -3\end{vmatrix} = 6,\ A_{22} = (-1)^{2+2}\begin{vmatrix}2 & 0\\ -7 & -3\end{vmatrix} = -6,\ A_{23} = (-1)^{2+3}\begin{vmatrix}2 & 2\\ -7 & 2\end{vmatrix} = -18$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}2 & 0\\ 1 & 1\end{vmatrix} = 2,\ A_{32} = (-1)^{3+2}\begin{vmatrix}2 & 0\\ 2 & 1\end{vmatrix} = -2,\ A_{33} = (-1)^{3+3}\begin{vmatrix}2 & 2\\ 2 & 1\end{vmatrix} = -2.$$

The adjoint of A is:

$$\text{adj}A = \begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix} = \begin{bmatrix}-5 & 6 & 2\\ -1 & -6 & -2\\ 11 & -18 & -2\end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{12}\begin{bmatrix}-5 & 6 & 2\\ -1 & -6 & -2\\ 11 & -18 & -2\end{bmatrix} = \begin{bmatrix}5/12 & -1/2 & -1/6\\ 1/12 & 1/2 & 1/6\\ -11/12 & 3/2 & 1/6\end{bmatrix}$$

(b) Row Transformation Method

We consider the partitioned matrix $[A|I]$, where A is non-singular matrix. Applying a sequence of row transformations (operations) in such a manner that matrix A is changed to I and I is changed to A^{-1}.

Illustration-31: Find the inverse of matrix A by using row transformation method:

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 2\\ 3 & 3 & 4\end{bmatrix}$$

Solution: The inverse of matrix A can be obtained directly by applying row transformations (operations) on the following matrix:

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 2\\ 3 & 3 & 4\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \because R_2' \to R_2' = R_2 - 2R_1, \quad R_3' \to R_3 - 3R_1$$

Iteration I:
$$\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & -4\\ 0 & -3 & -5\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix} \quad \because R_2 \to (-1)R_2$$

Applying row operations: $R_1 \to R_1 - 2R_2$ and $R_3' \to R_3 + 3R_2$, continuing in the same way till LHS matrix become an identity matrix as shown below:

Iteration 2:
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-6/7 & -1/7 & -5/7\\ 2/7 & 5/7 & -4/7\\ 3/7 & 1/7 & 1/7\end{bmatrix}$$

The inverse of A is given by the right-hand side matrix.

Hence, $A^{-1} = \begin{bmatrix}-6/7 & -1/7 & -5/7\\ 2/7 & 5/7 & -4/7\\ 3/7 & 1/7 & 1/7\end{bmatrix}$

Illustration-32: Show that $A=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$ satisfies the equation $x^2 - 6x + 17 = 0$.

Hence find A^{-1}.

Solution: We have $A=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$

$$\therefore A^2 = A \times A = \begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix} \times \begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix} = \begin{bmatrix}-5 & -18\\ 18 & 7\end{bmatrix}$$

$$-6A = (-6)\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix} = \begin{bmatrix}-12 & 18\\ -18 & -24\end{bmatrix}$$

$$17I = 17\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}17 & 0\\ 0 & 17\end{bmatrix}$$

Now, $A^2-6A+17I_2=\begin{bmatrix}-5 & -18\\ 18 & 7\end{bmatrix}+\begin{bmatrix}-12 & 18\\ -18 & -24\end{bmatrix}+\begin{bmatrix}17 & 0\\ 0 & 17\end{bmatrix}$

$=\begin{bmatrix}-5-12+17 & -18+18+0\\ 18-18+0 & 7-24+17\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$

Hence, the matrix A satisfies the equation $x^2-6x+17=0$.

Now, $A^2-6A+17I_2=0\Rightarrow A^2-6A=-17I_2$.

or, $A^{-1}(A^2-6A)=-17A^{-1}I_2$

or, $+A^{-1}A^2-6A^{-1}A=-17(A^{-1}I_2)$

or, $A-6I_2=-17A^{-1}$

or, $A^{-1}=-\frac{1}{17}(A-6I_2)=-\frac{1}{17}(6I_2-A)$

$=\frac{1}{17}\left\{\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}-\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}\right\}=\frac{1}{17}\begin{bmatrix}4 & 3\\ -3. & 2\end{bmatrix}$

Illustration-33: Use matrix method to examine the following system of equations for consistency or inconsistency : 2x-3y=3, and 4x-y=11.

Solution: The given equations in the matrix form can be written as:

$$\begin{bmatrix}2 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}3\\ 11\end{bmatrix} \quad or, \quad Ax=B$$

where $A=\begin{bmatrix}2 & -3\\ 4 & -1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, $B=\begin{bmatrix}3\\ 11\end{bmatrix}$

Since $|A|=\begin{vmatrix}2 & -3\\ 4 & -1\end{vmatrix}=10(\neq 0)$, i.e. A is non-singular. A^{-1} exists and the system is consistent and has a unique solution given by

$AX=B$ or, $X=A^{-1}B$. .....(i)

To determine A^{-1} first we determine matrix of co-factors of each element of matrix A as follows:

$A_{11}=(-)^{1+1}(-1)=-1$ $\qquad$ $A_{12}=(-1)^{1+2}\ 4=-4$

$A_{21}=(1)^{2+1}(-3)=3$ $\qquad$ $A_{22}=(-1)^{2+2}\ 2=2$

$\therefore$ adj $A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}^T=\begin{bmatrix}A_{11} & A_{21}\\ A_{12} & A_{22}\end{bmatrix}=\begin{bmatrix}-1 & 3\\ -4 & 2\end{bmatrix}$

$A^{-1}=\frac{adj\ A}{|A|}=\frac{1}{10}\begin{bmatrix}-1 & 3\\ -4 & 2\end{bmatrix}=\begin{bmatrix}-1/10 & 3/10\\ -4/10 & 2/10\end{bmatrix}$ ....(*ii*)

From (i) and (ii), we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1/10 & 3/10 \\ -4/10 & 2/10 \end{bmatrix} \begin{bmatrix} 3 \\ 11 \end{bmatrix} = \begin{bmatrix} -3/10+33/10 \\ -12/10+22/10 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Hence, the unique solution is given by; $x = 3$ and $y = 1$.

Illustration-34: By matrix method solve the following equations:

$$5x - 6y + 4z = 15$$
$$7x + 4y - 3z = 19$$
$$2x + y + 6z = 46.$$

Solution: The given system of equations may be represented in the matrix natations as:

$$\begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

or, $AX = B$ or $X = A^{-1}B$

Since $|A| = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix} = 5\begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix} + 6\begin{vmatrix} 7 & -3 \\ 2 & 6 \end{vmatrix} + 4\begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix} = 135 + 288 - 4 = 419 \neq 0$

Hence, A is non-singular and A^{-1} exists. The system has the unique solution.

The matrix of co-factors of elements of A is given by

$$A_{11} = 27, \quad A_{12} = -48 \quad A_{13} = -1$$
$$A_{21} = 40, \quad A_{22} = 22, \quad A_{23} = -17$$
$$A_{31} = 2, \quad A_{32} = 43, \quad A_{33} = 62.$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{31} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

And $$A^{-1} = \frac{\text{Adj } A}{A} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix} \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Hence, the unique solution of the system is: $x = 3,\ y = 4,\ z = 6$.

Illustration-35: Show that the following system of equations is consistent.

$2x-y+3z=5,\ 3x+2y-z=7$ and $4x+5y-5z=9.$

Also, find the solution.

Solution: The given system of equation can be written in matrix form

$$\begin{bmatrix}2 & -1 & 3\\ 3 & 2 & -1\\ 4 & 5 & -5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}5\\ 7\\ 9\end{bmatrix} \text{ or, } AX=B,$$

where $A=\begin{bmatrix}2 & -1 & 3\\ 3 & 2 & -1\\ 4 & 5 & -5\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}5\\ 7\\ 9\end{bmatrix}$

Now, $|A|=\begin{bmatrix}2 & -1 & 3\\ 3 & 2 & -1\\ 4 & 5 & -5\end{bmatrix}=2(-10+5)+1(-15+4)+3(15-8)=0$

So, A is singular. Thus, The given system of equations is either inconsistent or it is consistent with infinitely many solutions according as (adj A) $B\neq 0$ or (adj A) $B=0$ respectively.

Let A_{ij} be the co-factors of elements a_{ij} $A=\lfloor a_{ij}\rfloor$ Then

$$A_{11}=(-1)^{1+1}\begin{vmatrix}2 & -1\\ 5 & -5\end{vmatrix}=-5, A_{12}=(-1)^{1+2}\begin{vmatrix}3 & -1\\ 4 & -5\end{vmatrix}=11, A_{13}=(-1)^{1+3}\begin{vmatrix}3 & 2\\ 4 & 5\end{vmatrix}=7.$$

$$A_{21}=(-1)^{1+2}\begin{vmatrix}-1 & 3\\ 5 & -5\end{vmatrix}=10, A_{22}=(-1)^{2+2}\begin{vmatrix}2 & 3\\ 4 & -5\end{vmatrix}=-22, A_{23}=(-1)^{2+3}\begin{bmatrix}2 & -1\\ 4 & 5\end{bmatrix}=-14,$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}-1 & 3\\ 2 & -1\end{vmatrix}=-5, A_{32}=(-1)^{3+2}\begin{vmatrix}2 & 3\\ 3 & -1\end{vmatrix}=11, A_{33}=(-1)^{3+3}\begin{vmatrix}2 & -1\\ 3 & 2\end{vmatrix}=7.$$

$$\therefore adj\ A=\begin{bmatrix}-5 & 11 & 7\\ 10 & -22 & 14\\ -5 & 11 & 7\end{bmatrix}^{T}=\begin{bmatrix}-5 & 10 & -5\\ 11 & -22 & 11\\ 7 & -14 & 7\end{bmatrix}$$

$$\therefore adj\ A\times B=\begin{bmatrix}-5 & 10 & -5\\ 11 & -22 & 11\\ 7 & -14 & 7\end{bmatrix}\times\begin{bmatrix}5\\ 7\\ 9\end{bmatrix}=\begin{bmatrix}-25+70-45\\ 55-154+99\\ 35-98+63\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Thus, $AX=B$ has infinitely many solutions. To find these solutions, we put $z=k$ in the first two equations and write them as follows:

$2x-y=5-3k$ and $3x+2y=7+k$

$$or, \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5-3k \\ 7+k \end{bmatrix} \; or, \; AX = B$$

where $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5-3k \\ 7+k \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4+3 = 7 \neq 0$ *and* *adj* $A = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

$\therefore \quad A^{-1} = \dfrac{1}{|A|}\text{adj } A = \dfrac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

Now, $X = A^{-1}B = \dfrac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 5-3k \\ 7+k \end{bmatrix} = \dfrac{1}{7}\begin{bmatrix} 10-6k+7+k \\ -15+9k+2k+14 \end{bmatrix}$

Or, $\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{\dfrac{17-5k}{7}}{\dfrac{11k-1}{7}}$ i.e. $x = \dfrac{17-5k}{7}, \; y = \dfrac{11k-1}{7}$

Illustration-36: Solve the following system of equation using Gauss elimination method

$$2x - y + 3z = 9; x + y + z = 6 \text{ and } x - y + z = 2.$$

Solution: The matrix equation of given equations can be written as

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} \; or, AX = B,$$

Applying row operations: $R_2' \to 2R_2 - R_1$ *and* $R_3' \to 2R_3 - R_1$, we get

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -5 \end{bmatrix}; \; R_3' \to R_3 + 3R_2$$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -12 \end{bmatrix}$$

The given system of equation can be rewritten as:

$$2x - y + 3z = 9 \qquad \ldots(i)$$

$$3y - z = 3 \qquad \ldots(ii)$$

$$-4z = -12 \qquad \ldots(iii)$$

From (iii), $z = 3$.

Putting $z=3$ in (ii) $3y-3=3$ or, $y=2$.

Also on putting values of $y, z,$ in (i) we get, $2x-2+9=9$ or, $x=1$.

Hence, the unique solution is: $x=1, y=2, z=3$.

Illustration-37: Solve the system of equations by using Gauss elimination method:

$$x+y+z=9$$
$$2x+5y+7z=52$$
$$2x+y-z=0.$$

Solution: The system of equations can also be represented as:

$$\begin{bmatrix}1 & 1 & 1\\ 2 & 5 & 7\\ 2 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 52\\ 0\end{bmatrix}$$

Applying elementary row operations: $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 2R_1$, we get

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 3 & 5\\ 0 & -1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 34\\ -18\end{bmatrix}; R_2' \to R_2 + 3R_3$$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & -4\\ 0 & -1 & -3\end{bmatrix}=\begin{bmatrix}9\\ -20\\ -18\end{bmatrix}$$

Rewriting equations as follows:

$$x+y+z=9$$
$$-4z=-20$$
$$-y-3z=-18$$

Solving these equations, we get $x=1, y=3,$ and $x=5$.

Illustration-38: The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number, we get 12. Using matrices find the numbers.

Solution: Let the three numbers be x, y and z respectively. Then, given that

$$x+y+z=6; x+2z=7 \text{ and } 3x+y+z=12$$

The above system of equations can be written in matrix form as:

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 2\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 7\\ 12\end{bmatrix} \text{ or, } AX=B$$

where $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 2\\ 3 & 1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}6\\ 7\\ 12\end{bmatrix}$

Now, $|A|=\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 2\\ 3 & 1 & 1\end{vmatrix}=1(0-2)-(1-6)+1(1-0)$

$=-2+5+1=4\neq 0$

So the above system of equations has a unique solution given by, $X=A^{-1}B$. Let A_{ij} be the co-factor of a_{ij} in $A=|a_{ij}|$. Then.

$A_{11}=-2, A_{12}=5, A_{13}=1, A_{21}=0, A_{22}=-2, A_{23}=2,$

$A_{31}=2, A_{32}=-1$ and $A_{33}=-1$

$\therefore$ adj $A=\begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}$. Thus $A^{-1}=\frac{1}{|A|}\text{adj}\,A=\frac{1}{4}\begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}$

and $X=A^{-1}B$ or $X=\frac{1}{4}\begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}6\\ 7\\ 12\end{bmatrix}$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-12+0+24\\ 30-14-12\\ 6+14-12\end{bmatrix}=\begin{bmatrix}3\\ 1\\ 2\end{bmatrix}$$

Equating terms on both sides, we get $x=3, y=1$ and $z=2$.

Hence, the three numbers are 3, 1 and 2 respectively.

Illustration-39: A mixture is to be made of three foods A, B, C. The three A, B, C contain nutrients P, Q, R as shown below:

	Ounces per pound of nutrient		
Food	Nutrient P	Nutrient Q	Nutrient R
A	1	2	5
B	3	1	1
C	4	2	1

How to form a mixture which will have 8 ounces of P, 5 ounces of Q and 7 ounces of R?

Solution: Let x, y and $z =$ pounds of food A, B and C, respectively be needed to form the mixture.

The required relationships can be written as:

$$x + 3y + 4z = 8$$
$$2x + y + 2z = 5$$
$$5x + y + z = 7$$

The above system of simultaneous linear equations can be written in matrix form as:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix} \text{ or, } AX = B$$

where $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$

Now $|A| = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 2 & 1 \end{bmatrix} = 1(1-2) - 3(2-10) + 4(2-5) = -1 + 24 - 12 = 11 \neq 0$

Let A be the cofactor of $[a_{ij}]$. Then,

$$A_{11} = -1, A_{12} = 8, A_{13} = -3$$
$$A_{21} = 1, A_{22} = -19, A_{23} = 14,$$
$$A_{31} = 22, A_{32} = 6 \text{ and } A_{33} = -5$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

Thus, $A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{11} \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$

Thus, the solution of the system of equations in given by

$$X = A^{-1}B = \frac{1}{11} \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+5+14 \\ 64-95+42 \\ -24+70-35 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Equating terms on both sides, we got $X=1, y=1$ and $z=1$.

Hence, the mixture is formed by mixing one proud of each of the foods A, B and C.

11.14 BUSINESS APPLICATION TO MATRICES

Problem-01: Food I has 3 units of vitamins A, 9 units of vitamin B and 12 units of vitamin C. Food II has 6,9 and 15 units of vitamin A, B and C respectively. Food III has 9,0 and 9 units of vitamin A, B and C respectively. The number of units of vitamin A, B and C required are 33, 27 and 60 respectively. Find the amount of three foods that will provide exactly these amounts. Use matrix method.

Solution: Let x, y and z = units of food I, II and III to be purchased respectively.

The summary of data is :

		Vitamin		
		A	*B*	*C*
Food	*I*	3	9	12
	II	6	9	15
	III	9	0	9
Requirement of vitamins		33	27	60

The system of equations can now be written as :

$$\begin{aligned} 3x+6y+9z &= 33 \\ 9x+9y &= 27 \\ 12x+15y+9z &= 60 \end{aligned}$$

Now by applying Cramer's rule, we can find the required values of x, y and z.

Problem-02: Memory Bakery Ltd. produces basic pastry mix types A, B and C. In the past, the mix of ingredients has been as shown below:

Pastry mix	Ingredients		
	Flour	Fat	Sugar
A	0	1	0
B	5.5	2.5	0.5
C	4.5	2	3

Due to changes in the consumer's tastes, it has been decided to change the mixes using the following amendment matrix:

Pastry mix	Ingredients		
	Flour	Fat	Sugar
A	5	1	0
B	-0.5	0.5	0.5
C	0.5	0	0

Obtain the matrix for the new mix.

Solution: The matrix of the new mix is obtained by the addition of the original mix matrix and the amendment matrix.

$$\begin{bmatrix} 5 & 1 & 1 \\ 5.5 & 2.5 & 0.5 \\ 4.5 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 0.5 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix} = \begin{matrix} \\ A \\ B \\ C \end{matrix} \begin{matrix} \text{Flour Fat . Sugar} \\ \begin{bmatrix} 5 & 2 & 1 \\ 5 & 3 & 1 \\ 5 & 2 & 3 \end{bmatrix} \end{matrix}$$

Problem-03: There are two families A and B. There are 4 men, 6 women and 2 children in family A and 2 men, 2 women and 4 children in family B. The recommended daily allowance for calories is: Man: 2400, women: 1900, child: 1800 and for proteins is: Man: 55 gm, women: 45 gm and child: 33 gm.

Represent the above information by matrices. Using matrix multiplication, calculate the total requirement of calories and proteins for each of the two families.

Solution: The members of the two families can be represented by the 2×3 matrix.

$$D = \begin{matrix} \\ A \\ B \end{matrix} \begin{matrix} \text{M W C} \\ \begin{bmatrix} 4 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix} \end{matrix}$$

and the recommended daily allowance of calories and proteins for each member can be represented by 3×2 matrix.

$$E = \begin{matrix} \\ M \\ W \\ C \end{matrix} \begin{matrix} \text{Calories proteins} \\ \begin{bmatrix} 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{bmatrix} \end{matrix}$$

The total requirement of calories and proteins for each of the two families is given by the matrix multiplication:

$$DE = \begin{bmatrix} 4 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{bmatrix} = \begin{matrix} \\ A \\ B \end{matrix} \begin{matrix} \text{Calories proteins} \\ \begin{bmatrix} 24600 & 556 \\ 15800 & 332 \end{bmatrix} \end{matrix}$$

Hence, family A requires 24,600 calories and 556 gm proteins and family B requires 15,800 calories and 332 gm proteins.

Problem-04: A finance company has offices located in every division, every district and every thana in a certain state in a country. Assume that are five divisions, 30 districts and 200 thanas in the state. Each office has one Head Clerk, one Cashier, one Clerk and one Peon. A divisional office has, in addition, One office superintendent, two clerks, one typist and one peon. A district office has, in addition one clerk and one peon. The basic monthly salaries are as follows: Office Superintendent Tk. 5000; Head Clerk, Tk. 2000; Cashier, Tk. 1750; Clerk and Typist, Tk. 1500 and Peon Tk. 1000. Using matrix notations find,

(i) the total number of posts of each kind in all the offices taken together,

(ii) the total basic monthly salary bill of each kind of office, and

(iii) the total basic monthly salary bill of all the offices taken together.

Solution: Let the matrix $A = [5 \quad 30 \quad 200]$ denotes the number of offices, and the manpower in the organization can be denoted by the 3×6 matrix B as under.

$$B = \begin{matrix} O & H & C & T & L & P \end{matrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} \textit{Divisional Office} \\ \textit{District Office} \\ \textit{Zonal Office} \end{matrix}$$

where, O = Office superintendent, H = Head clerk, C = Cashier, T = Typist, L = Clerk, P = Peon.

The basic monthly salaries for each category of person can be represented by 6×1 matrix E as:

$$E = \begin{bmatrix} 5000 \\ 2000 \\ 1750 \\ 1500 \\ 1500 \\ 1000 \end{bmatrix} \begin{matrix} O \\ H \\ C \\ T \\ L \\ P \end{matrix}$$

(i) Total number of posts of each type in all the offices taken together are given by matrix multiplication

$$AB = [5 \quad 30 \quad 200]_{1\times3} \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}_{3\times6} = \begin{matrix} \{o & H & C & T & L & P\} \\ [5 & 235 & 235 & 5 & 275 & 170] \end{matrix}$$

(ii) Total basic monthly salary bill of each type of office is given by

$$BE = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}_{3\times 6} \begin{bmatrix} 5000 \\ 2000 \\ 1750 \\ 1500 \\ 1500 \\ 1000 \end{bmatrix}_{6\times 1} = \begin{bmatrix} 16,750 \\ 8,750 \\ 6,250 \end{bmatrix} \begin{matrix} \textit{Divisional Office} \\ \textit{District Office} \\ \textit{Thana Office} \end{matrix}$$

(iii) Total salary bill of all the offices is given by

$$A(BE) = [5 \quad 30 \quad 200] \begin{bmatrix} 16,750 \\ 8,750 \\ 6,250 \end{bmatrix} = \text{Tk.}15,96,250.$$

Problem-05: Use matrix multiplication to divide Tk. 30,000 in two parts such that the total annual interest at 9% on the first part and 11% on the second part amounts Tk. 3060.

Solution: Let the two parts be Tk. x and Tk. $(30,000-x)$ respectively. Let A be the matrix representing these two parts as:

$$\begin{matrix} & \text{Part I} & \text{Part II} \end{matrix}$$
$$A = [x \quad 30000-x]$$

Let R be the matrix representing the annual interest rates on two parts, i.e.

$$R = \begin{matrix} \text{Part I} \\ \text{Part II} \end{matrix} \begin{bmatrix} 0.09 \\ 0.11 \end{bmatrix}$$

The total annual interest on the two parts is given by the matrix multiplication as:

$$AR = [x \quad 30,000-x] \begin{bmatrix} 0.09 \\ 0.11 \end{bmatrix} = \text{Tk.}3,060 \text{ (Given)}$$

$$0.09x + 0.11(30,000-x) = 3060$$

$$\frac{9}{100}x + \frac{11}{100}(30000-x) = 3060$$

$$9x + 33,0000 - 11x = 30,6000, \text{ i.e. } x = 12,000.$$

Hence, one part is Tk. 1,200 and second part is $Tk.(30,000-12,000)Tk.18,000$.

Problem-06: A trust fund has Tk. 50,000 that is to be invested into two types of bonds. The first bond pays 5% interest per year and the second bond pays 6% interest per year. Using matrix algebra, determine how to divide Tk. 50,000 among the two types of bonds so as to obtain an annual interest of Tk. 2,780.

Solution: Let Tk. x and Tk. $(50,000-x)$ be invested in the first and second type of bond, respectively. The value of these bonds can be written in the form of a row matrix:

$$A = [x \quad 50{,}000 - x]_{1\times 2}$$

The amounts received as interest annual from two bonds can be written in the form of a column matrix: $B = \begin{bmatrix} 0.05 \\ 0.06 \end{bmatrix}_{2\times 1}$

Thus, total interest to be obtained annually from two bonds is given by

$$AB = [x \quad 50{,}000 - x]\begin{bmatrix} 0.05 \\ 0.06 \end{bmatrix} = 0.05x + 0.06(50{,}000 - x) = 3{,}000 - 0.01x$$

But given that the annual total interest is Tk. 2,780. Therefore

$$3{,}000 - 0.01x = 2780 \text{ or, } x = (3000 - 2780)/0.01 = 220\times 100 = \text{Tk.}22{,}000$$

Hence, the required amount to be invested into two types of bonds as : Tk. 22,000 and Tk. $(50{,}000 - 22{,}000) = \text{Tk.}28{,}000$ respectively.

Problem-07: Ms Lamia is a sole trader, manufacturing tables and chairs. Each table requires 5 hours of labour and 6 units of material. A Chair requires 3 labour hours and 3 units of material. If Ms. Lamia plans to produce 10 tables and 15 chairs in the next week, how many hours will he need to work and how much material will he require?

Solution: The labour requirement is $(10\times 5) + (15\times 3) = 95$ hours

The material requirements is $(10\times 6) + (15\times 3) = 105$ units.

The matrix solution would be:

Tables	Chairs	Labour	Materials	Labour	Materials

$$(10 \quad 15)_{1\times 2} \times \begin{pmatrix} 5 & 6 \\ 3 & 3 \end{pmatrix}_{2\times 2} = (95 \quad 105)_{1\times 2}$$

Remark: It may be noted that $\begin{pmatrix} 5 & 6 \\ 3 & 3 \end{pmatrix}_{2\times 2}\begin{pmatrix} 10 \\ 15 \end{pmatrix}_{5\times 1} = \begin{pmatrix} 140 \\ 75 \end{pmatrix}_{2\times 1}$ is incorrect as lobour hours are being added to units of material.

Problem-08: A firm produces different pump units, each of which requires same components shown below in a tabular form:

Pump	Housing	Impeller	Bolts	Couplings	Inlets	Aromoured hose
Type A	1	1	5	4	2	8 m.
Type B	1	1	7	3	2	4 m.
Type C	1	1	3	5	2	3 m.

The firm receives an order for 8 Type-A pump units, 4 Type-B units and 2 Type-C units. Using the notion of Matrix Multiplication, obtain the matrix whose elements may represent the quantities of each item required to make up the order.

Solution: The specifications of the different pump units with their components can be represented by the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 5 & 7 & 3 \\ 4 & 3 & 5 \\ 2 & 2 & 2 \\ 8 & 4 & 3 \end{pmatrix}_{6\times3}, \quad \begin{pmatrix} 8 \\ 4 \\ 3 \end{pmatrix}_{3\times1}$$

Where each column represents the type of the pump and each row represents the different components required. The firm has received order for8 8 type A. 4 type B, 2 type C units, This can be represented by the matrix.

Therefore the matrix multiplication of these two matrices gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 5 & 7 & 3 \\ 4 & 3 & 5 \\ 2 & 2 & 2 \\ 8 & 4 & 3 \end{pmatrix}_{6\times3} \times \begin{pmatrix} 8 \\ 4 \\ 3 \end{pmatrix}_{3\times1} = \begin{pmatrix} 1\times8+1\times4+1\times3 \\ 1\times8+1\times4+1\times3 \\ 5\times8+7\times4+3\times3 \\ 4\times8+3\times4+5\times3 \\ 2\times8+2\times4+2\times3 \\ 8\times8+4\times4+3\times3 \end{pmatrix}_{6\times1} = \begin{pmatrix} 15 \\ 15 \\ 77 \\ 59 \\ 30 \\ 89 \end{pmatrix}$$

The first element of matrix (=15) gives the number of components for housing, the second (=15) gives that of impeller and so on.

Problem-09: The following matrix gives the number of units of three products (P, Q and R) that can be processed per hour on three machines (A, B and C):

$$\begin{matrix} \\ P \\ Q \\ R \end{matrix} \begin{pmatrix} A & B & C \\ 10 & 12 & 15 \\ 13 & 11 & 20 \\ 16 & 18 & 14 \end{pmatrix}$$

Determine by using matrix algebra, how many units of each product con be produced, if the hours available on machines A, B and C are 54, 46 and 48 respectively.

Solution:

$$\text{Units of products} = \begin{matrix} \\ P \\ Q \\ R \end{matrix} \begin{bmatrix} A & B & C \\ 10 & 12 & 15 \\ 12 & 11 & 20 \\ 16 & 18 & 14 \end{bmatrix} \begin{bmatrix} 54 \\ 46 \\ 48 \end{matrix} \begin{matrix} A \\ B \\ C \end{matrix} = \begin{bmatrix} 540+552+720 \\ 702+506+960 \\ 864+828+672 \end{bmatrix} = \begin{bmatrix} 1812 \\ 2168 \\ 2364 \end{bmatrix} \begin{matrix} P \\ Q \\ R \end{matrix}$$

$\therefore$ 1812,2168 and 2368 units of product P, Q and R are produced respectively.

Problem-10: The following matrix gives the proportionate mix of constituents used for three fertilizers :

		Constituent			
		A	B	C	D
Fertilizer	1	0.5	0	0.5	0
	2	0.2	0.3	0	0.5
	3	0.2	0.2	0.1	0.5

(i) If sales are 100 tins (of one kilogram) per week, 20% being fertilizer 130% being fertilizer 2, and 50% fertilizer 3; how much of each constituent is used ?

(ii) If the cost of each constituent is 50 paisa, 60 paisa, 75 paisa, and 100 paisa per 100 grams, respectively, how much does a one-kilogram tin of each fertilizer cost?

(iii) What is the total cost per week?

Express the calculations and answers in matrix form.

Solution: (i) The sales of fertilizers per week can be expressed as the following matrix: $1000(.02 \quad 0.3 \quad 0.5) = (200 \quad 300 \quad 500)$

Thus $(200 \quad 300 \quad 500)\begin{pmatrix} 0.5 & 0 & 0.5 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{pmatrix} = (260 \quad 190 \quad 150 \quad 400)$

Requirements of constituents are:

A : 260, B : 190, C : 150, D : 400

(ii) Costs of each constituent are 50p, 60p, 75p, and 100p per 100 grams, i.e., 500p, 600p, 750p and 1000p per 1,000 grams (one kilogram of each constituent, respectively.

Thus $\begin{pmatrix} 0.5 & 0 & 0.5 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{pmatrix} \times \begin{pmatrix} 500 \\ 600 \\ 750 \\ 1000 \end{pmatrix} = \begin{pmatrix} 625 \\ 780 \\ 795 \end{pmatrix}$

Costs per 1-kg tin of fertilizer are:

1 : Tk. 6.25, 2 : Tk. 7.80 3 : Tk. 7.95.

(iii) The total cost of fertilizer if 1000 one-kilogram this are needed per week may be calculated by either :

$$(200 \quad 300 \quad 500)\begin{pmatrix} 625 \\ 780 \\ 795 \end{pmatrix} = (7,56,500)$$

or by $(260 \quad 190 \quad 150 \quad 400)\begin{pmatrix} 500 \\ 600 \\ 750 \\ 1000 \end{pmatrix} = (7,56,500)$

Hence, total cost per week is Tk. 7,565.

Problem-11: The total cost of manufacturing three types of motor car is given by the following table:

	Labour (hrs)	Materials (units)	Sub-contracted Work (Units)
Car A	40	100	50
Car B	80	150	80
Car C	100	250	100

Labour costs Tk. 20 per hour, units of materisls cost Tk. 5 each and units of sub-contracted work cost Tk. 10 per unit. Find the total cost of manufacturing 3,000; 2,000 and 1,000 vehicles of type A, B and C respectively.

Express the cost as a triple product of a three element row matrix , 3x3 matrix and a three element column matrix and perform the multiplication according to the same rules you used for 2×2 matrices.

Solution: Let matrix P represent labour hours, material cost and cost of sub-contracted work for three types of cars A, B, C repectively.

$$P=\begin{bmatrix}40 & 100 & 50\\ 80 & 150 & 80\\ 100 & 250 & 100\end{bmatrix}$$

Further let matrix Q represent labour cost per unit, material cost and cost of sub-contracted work.

$$Q=\begin{bmatrix}20\\ 5\\ 10\end{bmatrix}$$

The cost of each car A, B, C is now given by the column matrix

$$PQ\begin{bmatrix}40 & 100 & 50\\ 80 & 150 & 80\\ 100 & 250 & 100\end{bmatrix}\begin{bmatrix}20\\ 5\\ 10\end{bmatrix}=\begin{bmatrix}1800\\ 3150\\ 4250\end{bmatrix}$$

Let the number of cars A, B, C to be manufactured in that order be represented by the row matrix.

$$R=[3000 \quad 2000 \quad 1000]$$

Hence the total cost of manufacturing three cars A, B and C is given by the matrix.

$$RPQ=(3000 \quad 2000 \quad 1000)\begin{pmatrix}1800\\ 3150\\ 4250\end{pmatrix}=[1,59,50,000]$$

Problem-12: A manufacturer produces three products: P, Q and R which he sells in two markets. Annual sales volumes are indicated as follows:

Markets	Products		
	P	Q	R
I	10,000	2,000	18,000
II	6,000	20,000	8,000

If unit sale prices of P, Q and R are Tk. 2.50, 1.25 and 1.50 respectively find the total revenue in each market will the help of Matrix Algebra.
If the unit costs of the above 3 commodities are Tk. 1.80, 1.20 and 0.80 respectively, find his gross profits.
Solution: Total revenue in each market is obtained from the matrix product:

$$[2.50 \quad 1.25 \quad 1.50]\times\begin{bmatrix}10000 & 6000\\ 2000 & 20000\\ 18000 & 8000\end{bmatrix}=[54500 \quad 52000]$$

$$\text{Total cost} = [1.80 \quad 1.20 \quad 0.80]\times\begin{bmatrix}10000 & 6000\\ 2000 & 20000\\ 18000 & 8000\end{bmatrix}=[34800 \quad 41200]$$

Profit from market $A = 54500-34800=19700$
Profit from market $B = 52000-41200=10800$

Problem-13: In a certain city there are 25 colleges and 100 schools. Each school and college has 5 peons, 2 clerks and 1 cashier. Each college in addition has 1 accountant and 1 head-clerk. The monthly salary of each of them is as follows:
Peon-Tk. 300; Clerk-Tk. 500; Cashier-Tk. 600; Accountant-Tk. 700; and Head-clerk-Tk. 800
Using matrix notation, find
(a) the total number of posts of each kind in schools and colleges taken together
(b) the total monthly salary bill of each school and college separately, and
(c) the total monthly salary bill of all the schools and college taken together.
Solution: (a) Consider the row matrix of order 1×2

$$A=[25 \quad 100]$$

This represents the number of colleges and schools in that order.

$$\text{Let} \quad B=\begin{bmatrix}5 & 2 & 1 & 1 & 1\\ 5 & 2 & 1 & 0 & 0\end{bmatrix}$$

Where columns represent number of peons, clerks, cashier, accountant, head-clerk while rows represents colleges and schools in that order.
Then

$$AB=[25 \quad 100]_{1\times2}\times\begin{bmatrix}5 & 2 & 1 & 1 & 1\\ 5 & 2 & 1 & 0 & 0\end{bmatrix}_{2\times5}$$

$$=[625 \quad 250 \quad 125 \quad 25 \quad 25]_{1\times5}$$

where first element represents total number of peons, second represents total number of clerks, third represents total number of cashiers, fourth represents total number of accountants and fifth represents total number of head-clerks.
(b) Let the column matrix

$$C=\begin{bmatrix}300\\500\\600\\700\\800\end{bmatrix}$$

represent monthly salary of peon, clerk, cashier, accountant and head-clerk in that order. Then

$$BC=\begin{bmatrix}5 & 2 & 1 & 1 & 1\\ 5 & 2 & 1 & 0 & 0\end{bmatrix}_{2\times5}\times\begin{bmatrix}300\\500\\600\\700\\800\end{bmatrix}_{5\times1}=\begin{bmatrix}4600\\3100\end{bmatrix}_{2\times1}$$

Thus total monthly salary bill of each college is Tk. 4600 and of each school is Tk. 3100.

(c) The total monthly salary bill of all schools and colleges taken together is

$$A(BC)=[25 \quad 100]_{1\times2}\times\begin{bmatrix}4600\\3100\end{bmatrix}_{2\times1}=[1{,}15{,}000+3{,}10{,}000]_{1\times1}=[4{,}25{,}000]$$

Problem-14: A, B and C has Tk. 480, Tk. 760 and Tk. 710 respectively. They utilized the amounts to purchase three types of shares of prices x, y and z respectively. A purchases 2 shares of price x, 5 of price y and 3 of price z. B purchases 4 shares of price x, 3 of price y and 6 of price z, C purchases 1 share of price x, 4 of price y and 10 of price z. Find x, y and z.

Solution: We obtain the following set of simultaneous linear equations:

$$2x+5y+3z=480$$
$$4x+3y+6z=760$$
$$x+4y+10z=710$$

The above system of equations in the matrix notation is

$$\begin{bmatrix}2 & 5 & 3\\ 4 & 3 & 6\\ 1 & 4 & 10\end{bmatrix}\times\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}480\\760\\710\end{bmatrix}\quad\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2 & 5 & 3\\ 4 & 3 & 6\\ 1 & 4 & 10\end{bmatrix}^{-1}\times\begin{bmatrix}480\\760\\710\end{bmatrix}\qquad\ldots(1)$$

Now $A^{-1}=\dfrac{Adj\ A}{|A|}$; *where* $|A|=\begin{vmatrix}2 & 5 & 3\\ 4 & 3 & 6\\ 1 & 4 & 10\end{vmatrix}=-119$

And $\text{Adj}A = \begin{bmatrix} +6 & -38 & +21 \\ -34 & +17 & 0 \\ +13 & -3 & -14 \end{bmatrix}$ [By Calculation]

Form (1), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{119}\begin{bmatrix} +6 & -38 & +21 \\ -34 & +17 & 0 \\ +13 & -3 & -14 \end{bmatrix} \times \begin{bmatrix} 480 \\ 760 \\ 710 \end{bmatrix} = -\frac{1}{119}\begin{bmatrix} 6\times480-38\times760+21\times710 \\ -34\times480+17\times760+0\times710 \\ 13\times480-3\times760-14\times710 \end{bmatrix}$$

$$= -\frac{1}{119}\begin{bmatrix} -11090 \\ -3400 \\ -5980 \end{bmatrix} = \begin{bmatrix} 11090/119 \\ 3400/119 \\ 5980/119 \end{bmatrix}$$ Hence $x = \frac{11090}{119}, y = \frac{3400}{119}, z = \frac{5980}{119}$.

Problem-15: To control a certain crop disease it is necessary to use 8 units of chemical A, 14 units of chemical B and 13 units of chemical C. One barrel of spray P contains one unit of A, 2 units of B and 3 units of C. One barrel of spray Q contains 2 units of A, 3 units of B and 2 units of C. How many barrels of each type of spray should be used to control the disease?

Solution: To grasp the situation easily, let us tabulate the data as follows:

		Spray			Requirement in chemicals
		P	Q	R	
	A	1	2	1	8
Chemical	B	2	3	2	14
Quantity in each spray	C	3	2	2	13
		x	y	z	

Let x barrels of spray P, y barrels of spray Q and z barrels of spray R be used to control the disease. Then

$$x + 2y + z = 8$$
$$2x + 3y + 2z = 14$$
$$3x + 2y + 2z = 13$$

Writing the equations in the matrix form, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix}^{-1} \times \begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix}$$

Now $\begin{bmatrix}1 & 2 & 1\\ 2 & 3 & 2\\ 3 & 2 & 2\end{bmatrix}^{-1} = \begin{bmatrix}+2 & -2 & +1\\ +2 & -1 & 0\\ -5 & +4 & -1\end{bmatrix}$ [By Calculation]

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}+2 & -2 & +1\\ +2 & -1 & 0\\ -5 & +4 & -1\end{bmatrix} \times \begin{bmatrix}8\\ 14\\ 13\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 3\end{bmatrix} \quad \Rightarrow x = 1, y = 2 \text{ and } z = 3$$

Hence 1 barrel of the spray P, 2 barrels of Q and 3 barrels of spray R should be used to control of the disease.

Problem-16: The XYZ Bakery Ltd. produces three basic pastry mixes A. B and C. In the past the mix of ingredients has been as shown in the following matrix.

Type	Flour	Fat	Sugar
A	5	1	1
B	6.5	2.5	0.5
C	4.5	3	2

(All quantities in kilogram weight)

Due to changes in consumer tastes it has been decided to change the mixes using the following amendment matrix:

Type	Flour	Fat	Sugar
A	0	+1	0
B	-0.5	+0.5	-0.5
C	+0.5	0	0

Using matrix algebra you are required to calculate:

(i) the matrix for the new mix

(ii) the production requirements to meet an order for 50 units of type A, 30 units of type B and 20 units of type C of the new mix;

(iii) the amount of each type that must be made to totally use up 3700 kgs of flour, 1700 kgs of fat and 800 kgs of sugar that are at present in the stores.

Solution: (i) The new mix is given by the addition of the original mix matrix and the amendment matrix.

$$\begin{pmatrix}5 & 1 & 1\\ 6.5 & 2.5 & 0.5\\ 4.5 & 3 & 2\end{pmatrix} + \begin{pmatrix}0 & +1 & 0\\ -0.5 & +0.5 & +0.5\\ +0.5 & 0 & 0\end{pmatrix} = \begin{pmatrix}5 & 2 & 1\\ 6 & 3 & 1\\ 5 & 3 & 2\end{pmatrix}$$

Therefore, the answer to part (i) is

Type	Flour	Fat	Sugar
Type A	5	2	1
Type B	6	3	1
Type C	5	3	2

(ii) To determine the production requirements it is necessary to multiply the order vector by the new mix matrix,

$$(50 \quad 30 \quad 20)\begin{pmatrix}5 & 2 & 1\\ 6 & 3 & 1\\ 5 & 3 & 2\end{pmatrix}=(530 \quad 250 \quad 120)$$

$\therefore$ 530 kgs of flour, 250 kgs of fat, 120 kgs of sugar

(iii) $5X_1+6X_2+5X_3=3700$

$2X_1+3X_2+3X_3=1700$

$X_1+X_2+2X_3=800$

$$\begin{pmatrix}5 & 6 & 5\\ 2 & 3 & 3\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}X_1\\ X_2\\ X_3\end{pmatrix}=\begin{pmatrix}3700\\ 1700\\ 800\end{pmatrix}$$

$$\Rightarrow AX=B \qquad \Rightarrow X=A^{-1}B \Rightarrow \begin{pmatrix}X_1\\ X_2\\ X_3\end{pmatrix}=\begin{pmatrix}5 & 6 & 5\\ 2 & 3 & 3\\ 1 & 1 & 2\end{pmatrix}^{-1}\times\begin{pmatrix}3700\\ 1700\\ 800\end{pmatrix}$$

On simplification, we get $X_1=400, X_2=200$ and $X_3=100$.

BRIEF REVIEW

Definition

Matrix: A matrix is a rectangular array of numbers arranged in rows and columns enclosed by a pair of brackets $(\ )$ or $[\ ]$.

Row matrix: A matrix having a single row is called row matrix.
Column matrix: A matrix having a single column is called column matrix.
Square matrix: A matrix with the equal number of rows and columns is called a square matrix.
Equal matrices: Two matrices are said to be equal if and only if (i) they are of same order and (ii) each element of one is equal to the corresponding element of the other.
Diagonal matrix: A square matrix whose all elements except the diagonal elements are zero is called a diagonal matrix.
Scalar matrix: A square matrix whose all diagonal elements are identical and the rest of the elements are zero is called scalar matrix.
Identity/Unit matrix: A square matrix where diagonal elements are unity or one and remaining elements are zero is called identity matrix.
Null matrix: A matrix in which every elements are zero is called null matrix or zero matrix.
Sub matrix: A matrix which is obtained from a given matrix by deleting any number of rows and any number of columns is called a sub matrix of the given matrix.

Symmetric matrix: Any matrix A is said to be symmetric, if it is (i) a square matrix and (ii) $A = A^t$.

Skew Symmetric matrix: A square matrix A is said to be skew symmetric if $A = -A^t$ and every diagonal elements are zero.

Triangular matrix: A square matrix $A = (a_{ij})_{n\times n}$ is called upper triangular matrix if $a_{ij} = 0$ for $i > j$ and is called lower triangular matrix if $a_{ij} = 0$ for $i < j$.

Idempotent matrix: A square matrix A is said to be idempotent matrix if $A^2 = A$.

Nilpotent matrix: A square matrix A of order n is said to be nilpotent if $A^n = 0$ where n is positive integer and 0 is the null matrix.
Transpose of a matrix: A matrix obtained by interchanging rows into columns or columns into rows is called the transpose of the given matrix. The transpose of A is denoted by A^t or A^T.
Inverse of a matrix: If $AB = BA = I$ then A is called the inverse matrix of B or B is called the inverse matrix of A.

Singular matrix: A square matrix A is said to be singular matrix if $|A| = 0$.

Non Singular matrix: A square matrix A is said to be non singular matrix if $|A| \neq 0$.

Order of a matrix: The number of rows and number of columns of the matrix is called order of a matrix. The order of a matrix having m rows and n columns will be $m \times n$ (read as m by n).

Involuntary matrix: A square matrix A is said to be involuntary matrix if $A^2 = I$.
Condition for matrix addition or subtraction: Two matrices are possible for addition or subtraction if they are of same order.
Condition for matrix multiplication: Two matrices are conformable for multiplication if the number of columns in the first matrix is equal to the number of rows in the second matrix. That is if $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})n \times p$, then the multiplication of A and B denoted by AB and is defined by $C = (c_{ij})m \times p$.

Quiz Questions

Multiple Choice Questions

1. Which is not correct?
   (i) A matrix is not a number; (ii) [5] is a number; (iii) 5 is a number
   (iv) (5) is a matrix
2. An $m \times n$ matrix A is said to be a square matrix if.....
   (i) $m > n$; (ii) $m < n$; (iii) $m = n$; (iv) $m \neq n$
3. A square matrix $A = (a_{ij})$ is said to be symmetric if.....
   (i) $a_{ij} > a_{ji}$; (ii) $a_{ij} < a_{ji}$; (iii) $a_{ij} = a_{ji}$; (iv) $a_{ij} = -a_{ji}$
4. If $A = (a_{ij})$ *is a* 2×3 *matrix and* $B = (b_{ij})$ *is a* 3×2 matrix then what is true?
   (i) A+B is possible; (ii) A-B is possible; (iii) both (A+B and A-B) are possible;
   (iv) both (A+B and A-B) are impossible
5. If $A = (a_{ij})$ *is a* $m \times n$ *matrix and* $B = (b_{ij})$ *is a* $p \times q$ matrix then AB is possible when...
   (i) $m = p$; (ii) $m = q$; (iii) $n = p$; (iv) $n = q$
6. If $A = \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ then AB= ? (i) A; (ii) B; (iii) 0; (iv) I
7. The transpose of the transpose $(i., e. A^{T^T})$ *of a matrix A is* (i) A^T (ii) $-A^T$ (iii) A (iv) $-A$
8. If $A = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$ then what is the minor of the element a_{23} in A ?
   (i) $\begin{vmatrix} 7 & 9 \\ 1 & 3 \end{vmatrix}$; (ii) $\begin{vmatrix} 7 & 9 \\ 1 & 2 \end{vmatrix}$; (iii) $\begin{vmatrix} 8 & 7 \\ 2 & 1 \end{vmatrix}$; (iv) $\begin{vmatrix} 9 & 7 \\ 3 & 1 \end{vmatrix}$
9. A square matrix A is said to be non-singular if
   (i) $|A| = 0$; (ii) $|A| > 0$; (iii) $|A| \neq 0$; (iv) None of them
10. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then what is the inverse of A ?
    (i) $\frac{1}{|A|}\begin{pmatrix} d & b \\ c & d \end{pmatrix}$; (ii) $\frac{1}{|A|}\begin{pmatrix} d & b \\ -c & a \end{pmatrix}$; (iii) $\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; (iv) $\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
11. If $A = \begin{bmatrix} 1 & 3 \\ 5 & 0 \\ 3 & 9 \end{bmatrix}$ then the order of A is: (a) 2×3 (b) 2×2 (c) 3×2 (d) None of these.

12. If $X=\begin{bmatrix}2 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -3\end{bmatrix}$ then the X is a: (a) Identity (b) Diagonal (c) Null (d) Scalar matrix.

13. A square matrix $A=(a_{ij})_{n\times n}$ is said to be skew symmetric if

(a) $a_{ij}=a_{ij}$ (b) $a_{ij}=-a_{ij}$ (c) $a_{ji}=a_{ji}$ (d) $a_{ij}=-a_{ji}$.

14. For matrix addition which is correct ?

(a) Commutative (b) Associative (c) (i) and (ii) both (d) None of these.

15. if $A=\begin{bmatrix}2 & 4\\ 3 & 6\end{bmatrix}$ *and* $B=\begin{bmatrix}-2 & -4\\ -3 & -6\end{bmatrix}$ *than* $A+B$ *is*

(a) $\begin{bmatrix}4 & 8\\ 6 & 12\end{bmatrix}$ (*b*) $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ (*c*) $\begin{bmatrix}0 & 4\\ 3 & 0\end{bmatrix}$ (*d*) $\begin{bmatrix}2 & 8\\ 0 & 12\end{bmatrix}$

16. Which is correct for matrix multiplication?

(a) Associative (b) Commutative (c) (i) and (ii) both (d) None of these.

17. If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ *and* $B=\begin{bmatrix}x & y\\ y & z\\ z & x\end{bmatrix}$ *than* AB *is*

(a) $\begin{bmatrix}ax+by\\ cy+dz\end{bmatrix}$ (*b*) $\begin{bmatrix}ax+by+cz\\ bx+ay+cz\end{bmatrix}$ (c) Impossible (d) Possible.

18. Transpose of the matrix $P=\begin{bmatrix}3 & 1\\ 5 & 2\\ 6 & 4\end{bmatrix}$ is

(a) $\begin{bmatrix}3 & 5 & 6\\ 1 & 2 & 4\end{bmatrix}$ (*b*) $\begin{bmatrix}1 & 2 & 4\\ 3 & 5 & 6\end{bmatrix}$ (*c*) $\begin{bmatrix}1 & 3\\ 2 & 5\\ 4 & 6\end{bmatrix}$ (d) None of these.

19. If A and B are two matrices of $m\times n$ *and* $n\times p$ respectively then

(a) $(AB)^T=A^TB^T$ (*b*) $(AB)^T=A^TB$ (*c*) $(AB)^T=B^TA^T$ (*d*) $(AB)^T=AB^T$

20. If $X=\begin{bmatrix}2 & 3\\ 4 & 6\end{bmatrix}$ than X is an

(a) invertible matrix (b) vertible matrix (c) scalar matrix (d) None of these.

21. If A, B are two non-Singular matrices, then which one is correct?

(a) $(AB)^{-1}=A^{-1}B^{-1}$ (*b*) $(AB)^{-1}=B^{-1}A^{-1}$ (*c*) $(AB)^{-1}=A^{-1}B$ (*d*) None of these.

22. Which is the basic difference between matrix and determinant?

(a) Matrix has value but determinant has no value.
(b) Both of these has value.
(c) Determinate has value but matrix has no value.
(d) None of the above.

23. If two rows (or columns) of a determinant are identical, then the value of the determinant is ?
(a) Zero(b) a scalar quantity (c) Impossible (d) None of these.

24. The value of the determinate $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ -1 & 2 & 5 \end{vmatrix}$ is: (a) 1 (b) 7 (c) 3 (d) 14.

25. A is a square matrix of order $n \times n$ then A is said to be non singular if
(a) $|A|=0$ (b) $A=0$ (c) $|A| \neq 0$ (d) None of these.

- **Quiz Type Question (True/False)**

Which one of the following statement is true/false?

a. Matrix consists of a square presentation.
b. If $A^2 = 0$ then A is nilpotent matrix of order 3.
c. If A and B are two matrices of order $m \times n$ then $(AB)^{\Gamma} = A^{\Gamma} B^{\Gamma}$
d. If $|A| = 0$, then A is called non singular matrix.
e. If two rows or column of a determinant are identical then the value of determinant is zero.

Brief Questions

Write down the example of diagonal matrix.

1.
2. When two matrices are possible for multiplication?
3. Write down the order of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.
4. What is the value of $(A^t)^t = ?$
5. If $AB = BA = I$, what is the name of these two matrices.
6. If $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, what is the value of $|A|$.
7. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, Is it possible to calculate AB ?
8. If $A^2 = I$ What is the name of the matrix.
9. When two matrix are said to be equal?
10. Write down the example of upper and lower triangular Matrix.

EXERCISE -II

1. What do you mean by a matrix?
2. Write down the condition for matrix addition or subtraction.
3. When two matrices are conformable for addition or subtraction.
4. What do you mean by Multiplication of two matrices?
5. When two matrices are conformable for multiplication?
6. Define order dimension of a matrix. Discuss with example.
7. Define transpose of a matrix. Discuss the properties of transpose a matrix.
8. Define inverse of a matrix.
9. When a square matrix has inverse.
10. Define adjoin of a square matrix.
11. Define determinant. Discuss any five properties of determinant of a square matrix.
12. How you distinguish between matrix and determinant.
13. Define upper triangular and lower triangular matrix with example.

Conceptual, Analytical & Numerical Questions

1. If $A=\begin{bmatrix}0 & 2 & 3\\ 2 & 1 & 4\end{bmatrix}, B=\begin{bmatrix}7 & 6 & 3\\ 1 & 4 & 5\end{bmatrix}$, find the matrix $2A-3B$.

2. If $A=\begin{bmatrix}3 & -7 & 5\\ 4 & -2 & 10\\ -2 & 7 & 3\end{bmatrix}, B=\begin{bmatrix}6 & -2 & -7\\ -8 & 8 & 0\\ 5 & 1 & -3\end{bmatrix}$

   find the matrix X such that $3A+5X=7B$.

3. If $A=\begin{bmatrix}2 & -3 & 5\\ 4 & 6 & 9\end{bmatrix}$, find the matrix X such that $X+A=0$.

4. If $A=\begin{bmatrix}1 & 4 & 3\\ 5 & 3 & 6\end{bmatrix}, B=\begin{bmatrix}-2 & 0 & 6\\ 4 & 6 & 10\end{bmatrix}$, find the matrix X such that $A+B-X=0$.

5. Find X if $Y=\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$ and $2X+Y=\begin{bmatrix}1 & 0\\ -3 & 2\end{bmatrix}$

6. If $A=\begin{bmatrix}9 & 1\\ 4 & 3\end{bmatrix}, B=\begin{bmatrix}1 & 5\\ 7 & 12\end{bmatrix}$, find matrix X such that $3A+5B+2X=0$.

7. Find the value of x, y, z and a which satisfy the matrix equation

(a) $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$

(b) Find matrices X and Y, if $X+Y=\begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X-Y=\begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

(c) Find x, y, a and b if

(i) $\begin{bmatrix} 2x-3y & a-b & 3 \\ 1 & x+4y & 3a+4b \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 6 & 29 \end{bmatrix}$

(ii) $\begin{bmatrix} 3x+4y & 2 & x-2y \\ a+b & 2a-b & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 5 & -5 & -1 \end{bmatrix}$

8. Solve the matrix equation

(a) $2\begin{bmatrix} x & y \\ z & t \end{bmatrix} - 4\begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix} = 5\begin{bmatrix} 0 & 2 \\ 4 & 3 \end{bmatrix}$

(b) $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

(c) $3\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2t \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+t & 3 \end{bmatrix}$

9. If $A=\begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix}, B=\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, find the matrix C such the $A+B+C$ is zero matrix.

10. If $A=\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}$, find a and b such that $AB=BA$ .Compute $3A+5B$

11. (a) If $A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, find AB and BA and show that $AB \neq BA$

(b) Given $A=\begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}; B=\begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}$, calculate AB. Can you calculate BA?

12. If $A=\begin{bmatrix}1 & 4 & 5\\ 3 & 1 & 1\end{bmatrix}, B=\begin{bmatrix}3 & 2\\ 1 & 0\\ 4 & 5\end{bmatrix}, C=\begin{bmatrix}-2\\ 1\end{bmatrix}$ then find $(AB)^2C$. Is A^2B^2C possible?

13. Let $A=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}$ and find $f(x)=x^2-4x+7$. Show that $f(A)=0$. Use this result to find A^5.

14. If $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$ where $i^2=-1$, verify that $(A+B)^2=A^2+B^2$.

15. (a) If $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ show that $(A+B)(A-B)\neq A^2-B^2$.

(b) If $A=\begin{bmatrix}2 & 1\\ 3 & 10\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, then show that $(2B-A)(10B-A)=9B$.

16. If $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$, then evaluate the equation $A^2-3A+9I_3$.

17. Solve the matrix equation $\begin{bmatrix}5 & 4\\ 1 & 1\end{bmatrix}X=\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}$, where X is a 2×2 matrix.

18. Using matrices A, B, C where

$$A=\begin{bmatrix}1 & -1 & 1\\ 0 & 2 & 1\end{bmatrix}, B=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 1 & 1\end{bmatrix} \text{ and } C=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}$$ verify that $(AB)C=A(BC)$.

19. A manufacturer produces three products P,Q, and R, which he sells in two markets A and B. Annual sale volumes are indicated below:

Market	Product		
	P	Q	R
A	10,000	2,000	18,000
B	6,000	20,000	8,000

If unit sale prices of P, Q, R are Tk. 2.50, Tk. 1.25 and Tk. 1.50, respectively, find the total revenue in each market with the help of matrix algebra.

20. In a certain city there are 50 colleges and 400 schools. Each school and college has 18 peons. 5 clerks and 1 cashier. Each college in addition has one section officer and one librarian. The

monthly salary of each of them is as follows: Peon: Tk. 300, Clerk: Tk. 500, Cashier: 600, Section officer: Tk. 700, and Librarian: Tk. 900. Using matrix notation find:

(i) the total number of the posts of each kind in schools and colleges taken together.

(ii) the total monthly salary bill of all the schools and colleges taken together.

21. A man buys 8 dozen of mangoes, 10 dozen of apples and 4 dozen of bananas. Mangoes cost Tk. 18 per dozen, apples Tk. 9 per dozen and banana Tk. 6 per dozen. Represent the quantities bought by row matrix and the prices by a column matrix and hence obtain the total cost.

22. The total cost of manufacturing three of cars is given by the following table:

Car	Labour (hrs)	Material (units)	Sub-contracted work (units)
A	40	100	50
B	80	150	80
C	100	250	100

Labour costs Tk. 10 per hour, units of material cost Tk. 1.50 each and units of sub-contracted work cost Tk. 3 per unit. Find the total cost of manufacturing 3,000; 2,000 and 1,000 cars of type A, B and C respectively.

Express the cost as a triple product of a three-element row matrix, a 3×3 matrix and a three element column matrix and perform the multiplication.

23. A firm produces three products A, B and C which it sells in two markets. Annual sales in units are given below:

Market	Units sold		
	A	B	C
I	8,000	4,000	16,000
II	7,000	18,000	9,000

If the price per unit of A, B and C is Tk. 2.50, Tk. 1.25 and Tk. 1.50 respectively and the costs per unit are Tk. 1.70, Tk. 1.20 and Tk. 0.80 respectively, find the total profit in each market by using matrix algebra.

24. The following matrix gives the number of units of three products P, Q and R that can be processed per hour on three machines A, B and C:

$$\begin{array}{c} \\ P \\ Q \\ R \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 10 & 12 & 15 \\ 13 & 11 & 20 \\ 6 & 18 & 14 \end{bmatrix} \end{array}$$

Determine, by using matrix algebra, how many units of each product can be produced, if the hours available on machines A, B and C are 54, 46 and 48 respectively.

25. A manufacture produces three products, X, Y and Z which he sells in three markets I, II and III. Monthly sales and the sales prices are given as follows:

Market	Item sold			Sale price (in Tk.)		
	X	Y	Z	X	Y	Z
I	1000	1500	2000	2.00	3.00	4.00
II	2000	2800	400	2.50	2.80	3.70
III	3000	3500	4000	2.30	3.10	4.20

If the costs per item for the three products, X, Y and Z are Tk. 1.80, Tk. 2.50 and Tk. 3.40 respectively, find by using the matrix algebra his total profit.

26. Is the inverse of a non-singular symmetric matrix is symmetric matrix?

27. Find the inverse of the matrix $A=\begin{bmatrix} a+ib & c+id \\ -c+id & a-id \end{bmatrix}$, given that $a^2+b^2+c^2+d^2=1$.

28. Define a non-singular matrix. Show that the matrix $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 0 \\ 1 & 1 & 5 \end{bmatrix}$ is non-singular.

Also find its inverse.

29. Find the adjoint of the matrix, $A=\begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. Hence obtain A^{-1}.

30. Two matrices A and B are given by $A=\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 1/3 & -1/6 \\ 0 & 1/2 \end{bmatrix}$

(i) Verify that $AB=BA=I_2$. Hence write down A^{-1}

(ii) Write down $\left(A^{-T}\right)^T, A^T\left(A^T\right)^{-1}$. Hence show that $\left(A^T\right)^{-1}=\left(A^{-1}\right)^T$.

31. (a) If $A=\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1}=B^{-1}A^{-1}$.

(b) If $A=\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & -3 & -1 \end{bmatrix}$ are two matrices of order three, verify that $(AB)^{-1}=B^{-1}A^{-1}$.

32. Given that matrix, $A=\begin{bmatrix}7 & 3 & -3\\ 2 & 4 & 1\end{bmatrix}$. Calculate A^TA and $(A^TA)^{-1}$ and verify that $(A^TA)(A^TA)^{-1}=I$.

33. Given the matrix, $A=\begin{bmatrix}1 & 3 & 4\\ 3 & -1 & 6\\ -1 & 5 & 1\end{bmatrix}$, prove that $(A^T)^{-1}=(A^{-1})^T$.

34. A square matrix A is called orthogonal if the transpose of A is equal to its inverse, i.e. $A^T=A^{-1}$. Using this definition, verify that the following square matrix is orthogonal.

$$\begin{bmatrix}1/3 & 2/3 & 2/3\\ 2/3 & 1/3 & -2/3\\ -2/3 & 2/3 & -1/3\end{bmatrix}$$

35. Find the inverse of the matrix, $A=\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$. Also verify that $A^{-1}=A^3$.

36. Give $A=\begin{bmatrix}2 & -3\\ -4 & 8\end{bmatrix}$, compute A^{-1} and show that $2A^{-1}=9I-A$.

37. If $A=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$, find A^2 and show that $A^2=A^{-1}$

38. Let $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$. Prove that $A^2-4A-5I=0$. Hence obtain A^{-1}.

39. Find the inverse of the matrix, $A=\begin{bmatrix}a & b\\ c & (1+bc)/a\end{bmatrix}$ and show that $\alpha A^{-1}=(a^2+bc+1)I-\alpha A$.

40. (a) Find matrix A such that $A\begin{bmatrix}3 & 4\\ 6 & 2\end{bmatrix}=\begin{bmatrix}2 & 8\\ 9 & 4\end{bmatrix}$.

(b) Find the matrix X for which $\begin{bmatrix}1 & -4\\ 3 & -2\end{bmatrix}X=\begin{bmatrix}-16 & -6\\ 7 & 2\end{bmatrix}$

41. Solve by the matrix algebra the system of equations:

(a) $10x_1 - 2x_2 = 8; 6x_1 + 3x_2 = 7$

(b) $x + y = 10; 2x + 2y = 50.$

42. Use matrix method to solve the system of equations:

(a) $x + y + z = 6, x - y + z = 2, 2x + y - z = 1$

(b) $2x + 4y + 5z = 18, 2x - y + 8z = 13, 5x - 2y + 7z = 20$

(c) $2x - y + 3z = 9, x + 3y - z = 4, 3x + 2y + z = 10$

43. Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ and hence solve the system of equations:

$x - y = a, y - z = b, x + z = c$

44. Find the inverse of the matrix $\begin{bmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{bmatrix}$ and hence solve the system of equations:

$7x - y - z = 0, \quad 10x - 2y + z = 8. \quad 6x + 3y - 2z = 7.$

45. Solve the following equations by using inverse of matrix

(a) $2x - y + 3z = 9, \ -x + 2y + z = 6, \ 3x + y - 4z = -7$

(b) $2x + 3y - z = 9, \ x + y + z = 9, \ 3x - y - z = -1$

(c) $9x + 7y + 3z = 6, \ 5x + y + 4z = 1, \ 6x + 8y + 2z = 4.$

46. The prices of 3 commodities A, B and C in a shop are Tk. 5, Tk. 6 and Tk. 10 respectively. Customer X buys 8 units of A, 7 units of B and 6 units of C. Customer Y buys 6 units of A, 7 units of B and 8 units of C. Show in matrix notation, the price of the commodities, quantities bought and the amount spent.

47. Two types of food, 1 and 2 have a vitamin content in units per kg given by the following table:

	Vitamin A	Vitamin B
Food 1	3	7
Food 2	2	9

Express the vitamin content of 5 kg of food 1 and 6 kg of food 2 as a matrix product and evaluate it. If food 1 costs 30 paisa per kg and food 2 costs 35 paisa per kg, express the cost of 5 kg, 6 kg of foods, 1 2 respectively as a matrix product and evaluate it.

48. The prices (in Tk.) of three commodities X, Y, Z are x, y and z per unit respectively. A purchases 4 units of Z, and sells 3 units of X and 5 units of Y, B purchases 3 units

of Y and sells 2 units of X and 1 unit of Z, C purchases 1 unit of X and sells 4 units of Y and 6 units of Z. In the process A, B, C earn Tk. 6,000, Tk. 5,000 and Tk. 13,000 respectively. Using matrices, find the prices per unit of the three commodities.

49. An amount of Tk. 5,000 is put into three different investments at the rates of interest of 6%, 7% and 8% per annum respectively. The total annual income is Tk. 358. If the combined income from the first two investments is Tk. 70 more than the income from the third, find the amount of each investment by using matrix algebra.

50. A motor corporation has two types of factories each producing buses and trucks. The weekly production figures at each type of factory are as follows:

	Factory A	Factory B
Buses	20	30
Trucks	40	10

The corporation has 5 factories A and 7 factories B. Buses and Trucks sell at Tk. 50,000 and Tk. 40,000 respectively. Express in matrix form and hence evaluate:

(i) The total weekly production of buses and trucks.
(ii) The total market value of vehicles produced each week.

51. Mr. X has Tk. 35,000 that he has divided into three investments. Part of the money is invested in a savings account with an annual rate of 6%, part in 7% annual yield bonds, and the remainder in business. In 1998, when he lost 6% of the money that he invested in that business, his net income from all three investments was Tk. 660. If he invested Tk. 3000 more in the business than in the savings accounts, how much was invested in each. Use matrix algebra or determinants.

52. A builder develops a site by building 9 houses and 6 bungalows. On the average one house requires 16,000 units of materials and 2,000 hours of labour, one bungalows requires 50,000 units of materials and 4,800 hours of labour. Labour costs Tk. 5 per hour and each unit of material costs, on the average Tk. 10. Express in matrix form and hence evaluate:
(i) The total materials and labour used in completing the site.
(ii) The cost of building a house and a bungalow.
(iii) The total cost of developing the site.

53. One unit of commodity A is produced by combining 1 unit of land, 2 units of labour and 5 units of capital. One unit of B is produced by 2 units of land, 3 units of labour and 1 unit of capital. One unit of C is produced by 3 units of land, 1 unit of labor and 2 unit of capital. If the prices of A, B and C are Tk. 27, Tk. 16 and Tk. 19 respectively, find the rent R, wages W and rate of interest I.

54. Three products X, Y and Z are produced after being processed through three departments

D_1, D_2 and D_3. The following data are available:

Products	Hours required for a unit produced		
	D_1	D_2	D_3
X	2	5	1
Y	1	2	3
Z	2	2	3
Maximum time available in hours	1100	1800	1400

Find by matrix method the number of units produced for each product to have full utilization of capacity.

55. A firm produces two products P_1 and P_2, passing through two machines M_1 and M_2 before completion. M_1 can produce either 10 units of P_1 or 15 units of P_2 per hour. M_2 can produce 15 units of either product per hour. Find daily production of P_1 and P_2 if time available is 12 hours of machine M_1 and 10 hours of M_2 per day using matrix inversion.

56. A manufacturing unit produces three types of products A, B, C. The following matrix shows the sale of products in two different cities.

$$\begin{matrix} & A & B & C \\ \begin{bmatrix} \\ \\ \end{bmatrix} & \begin{matrix} 1200 \\ 900 \end{matrix} & \begin{matrix} 900 \\ 600 \end{matrix} & \begin{matrix} 600 \\ 300 \end{matrix} \end{matrix}$$

If cost price of each product A, B, C is Tk. 1000, Tk. 2000, Tk. 3000 respectively and selling price Tk. 1500, Tk. 3000, Tk. 4000 respectively, find the total profits using matrix algebra only.

57. A salesman has the following record of sales during three months for three items A, B and C which have different rates of commission.

Month	Sales of Units			Total Commission drawn (in Tk.)
	A	B	C	
January	90	100	20	800
February	130	50	40	900
March	60	100	30	850

Find out the rates of commission on items A, B and C.

58. One unit of food I contains 100 units of vitamins, 60 units of minerals and 80 calories. One unit of food II contains 150 units of vitamins, 60 units of minerals and

180 calories. One unit of food III contains 90 units of vitamins, 40 units of minerals and 100 calories. Diet requirement for a patient is 1100 units of vitamins, 500 units of minerals and 1200 calories. Find out either by matrix method or by determinant method how many units of each food be mixed to form the diet which would meet the requirements exactly.

ANSWERS

13. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ 16. $\begin{bmatrix} -6 & 1 & 4 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$

19. Let $X = \begin{bmatrix} 10{,}000 & 2{,}000 & 18{,}000 \\ 6{,}000 & 20{,}000 & 8{,}000 \end{bmatrix}$; $Y = [2.50 \quad 1.25 \quad 1.50]$

$XY = \begin{bmatrix} 54{,}500 \\ 52{,}000 \end{bmatrix}$; Total revenue, market A = Tk.54,500 and market B = Tk.52,000

20. X (Institution number) $= [50 \quad 400]$; Y (Salary post wise) $= [300 \quad 500 \quad 600 \quad 700 \quad 900]$

Z (Institution posts) $= \begin{bmatrix} 18 & 5 & 1 & 1 & 1 \\ 18 & 5 & 1 & 0 & 0 \end{bmatrix}$ College / School

XZ (Total number of posts) $= [8100 \quad 2250 \quad 450 \quad 50 \quad 50]$

(XZ) Y (Total monthly salary) = Tk.39,05,000

21. Tk. 258.

22. $X = \begin{bmatrix} 40 & 100 & 50 \\ 80 & 150 & 80 \\ 100 & 250 & 100 \end{bmatrix}$; $Y = \begin{bmatrix} 10 \\ 1.50 \\ 3 \end{bmatrix}$; $Z = [3000 \quad 2000 \quad 1000]$

XY (cost of each car) $= \begin{bmatrix} 700 \\ 1265 \\ 1675 \end{bmatrix}$; $Z(XY)$ Total cost of manufacturing) = Tk.63,05,000

23. Total profit $= \begin{bmatrix} 8{,}000 & 4{,}000 & 16{,}000 \\ 7{,}000 & 18{,}000 & 9{,}000 \end{bmatrix} \times \begin{bmatrix} 2.50 - 1.70 \\ 1.25 - 1.20 \\ 1.50 - 0.80 \end{bmatrix} = \begin{bmatrix} 17{,}800 \\ 12{,}800 \end{bmatrix}$ I / I Market

24. The number of hours available on machines A, B and C represented in matrix form as:

$$\begin{matrix} P \\ Q \\ R \end{matrix}\begin{bmatrix} 54 \\ 46 \\ 48 \end{bmatrix}$$. The number of units of each product that can be produced is given by

$$\begin{bmatrix} 10 & 12 & 15 \\ 13 & 11 & 20 \\ 16 & 18 & 14 \end{bmatrix}\begin{bmatrix} 54 \\ 46 \\ 48 \end{bmatrix} = \begin{bmatrix} 1812 \\ 2168 \\ 2364 \end{bmatrix}$$

Thus, Product $P = 1812$ units, Product $Q = 2168$ units and Product $R = 2364$ units.

25. Total profit $= (2150 + 3140 + 6800) =$ Tk.12,090

41. (a) $x_1 = 38/42, x_2 = 22/42$

(b) $|A| = 0, (\text{adj } A)B \neq 0$; system of equations is inconsistent

42. (a) $x = 1, y = 2, z = 3$;

(b) $x = 3, y = 1, z = 1$

(c) $x = 1, y = 2, z = 3$