CHAPTER-12 DIFFERENTIATION AND USE IN BUSINESS PROBLEMS

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Basic concept of derivative
- How to calculate derivatives of different types of functions
- Derivative of a function by certain direct formula
- Determenation of maximum and minimum values of a given function
- Application of derivatives in business problem

12.1 INTRODUCTION

In earlier days, the term "Calculus" as a branch of mathematics was familiar to engineers and scientists only. Now with the increasing use of quantitative techniques in the solution of business and many economics problems, there is increasing use of calculus based quantitative models and techniques. Such models are used in production and operations management, marketing, financial management, economics etc.

The techniques of calculus are used in those situations where there is a need to estimate the rate at which things change. For example, calculus has role to play when one is interested in knowing "how sales volume or sales is affected when the prices changes or how the total cost, price etc. are effected when the volume or sales of output changes."

There are two branches of calculus –differential calculus and integral calculus. These two are reverse of each other. Differential calculus is concerned with determining the rate at which the value of the given function changes due to a unit change in one of the independent variable, while integral calculus is concerned with inverse problem of finding a function when its rate of change is given.

In the previous chapter we discussed the functional relationship of profit, revenue, cost etc. with the independent variables: price, units of production and advertising budget, etc. From these functional forms (or from) the curve of the function, the rate of change in dependent variables for various values of independent variables can be studied. Such studies are very useful for the application of differential calculus in solving problems related to business and economics and also serve as an essential tool in the study of optimization as well as maxima and minima of functions.

12.2 CONCEPT OF SLOPE AND RATE OF CHANGE

The term slope in differential calculus is used to measure the rate of change (or degree of steepness) of a function due to a unit change (increase or decrease) in the value of independent variable. The slope is denoted by m or $\tan\theta$, where θ is the angle of inclination of the given line with x-axis.

12.2.1 SLOPE OF A STRAIGHT LINE

Consider the case of total cost of producing an item, which is expressed as the functions of fixed (or set-up) cost plus a constant additional cost for each item produced. If fixed cost is Tk. 5 and variable cost per unit of item is Tk. 2, then the total cost y is given by: y or f(x) = 2x + 5, where x is the number of items produced. Obviously, x is the independent variable and y is the dependent variable. This equation has been graphed in Fig. 1. It represents a straight line.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two neighboring points on the graph of $f(x)$ or, $y = 2x + 5$. Suppose a very small change (also called increment) in the value of x and y is denoted Δ(delta). This change can be positive or negative. Δx represents change in the value of x and Δy represents the change in the value of y due to one unit change (i.e. Δx) in x, then the ratio

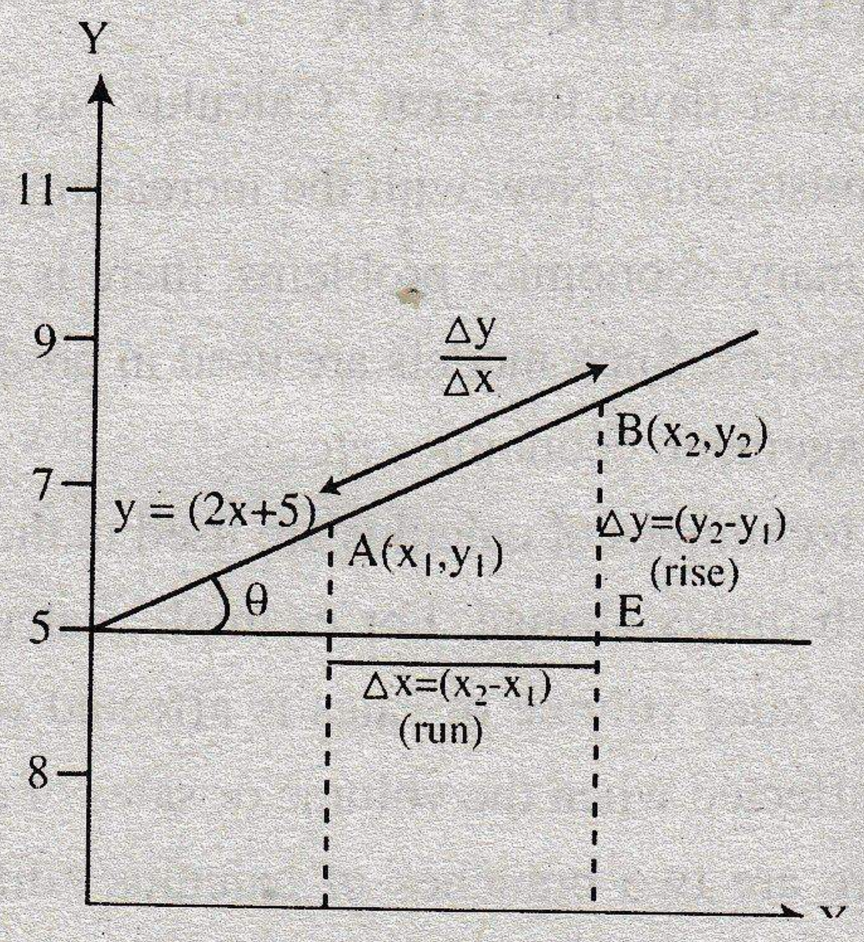


Fig. **Slope of Straight line**

$$m = \tan\theta = \frac{\text{rise}}{\text{run}}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11-7}{3-1} = 2 \text{ (coefficient of x)}$$

is called the rate of change in y due to one unit change in x, and also called the slope. It may be noticed that in case of a straight-line relationship, the slope is given by the coefficient of the independent variable. Thus, positive slope indicates that y increases when x increases and vice-versa. If the coefficient of independent variable is zero, then line is parallel to the x-axis. It must be noted that the slope of any given straight line remains same at any point on the line, i.e. the rate of change of y as changes in x is constant throughout the length of the line. However, for a non-linear function (i.e. curve) such change vary from point to point and hence it should be determined for each particular point of interest.

12.2.2 POSITIVE OR NEGATIVE SLOPES

A slope is said to be positive slope when dependent variable y increases as independent variable x increases. That is,

$$\text{Positive slope} = \frac{\text{Positive rise}}{\text{Positive run}}$$

But, if for a positive increase (run) in x, there is a negative increase (or rise) in y, and vice-versa is negative. That is,

$$\text{Negative slope} = \frac{\text{negative rise}}{\text{Positive run}}$$

For example, consider the equation of the line

$$y = 60 - 3x$$

This shows that slope is -3 (negative), which means value of y decrease as x increase and vice – versa.

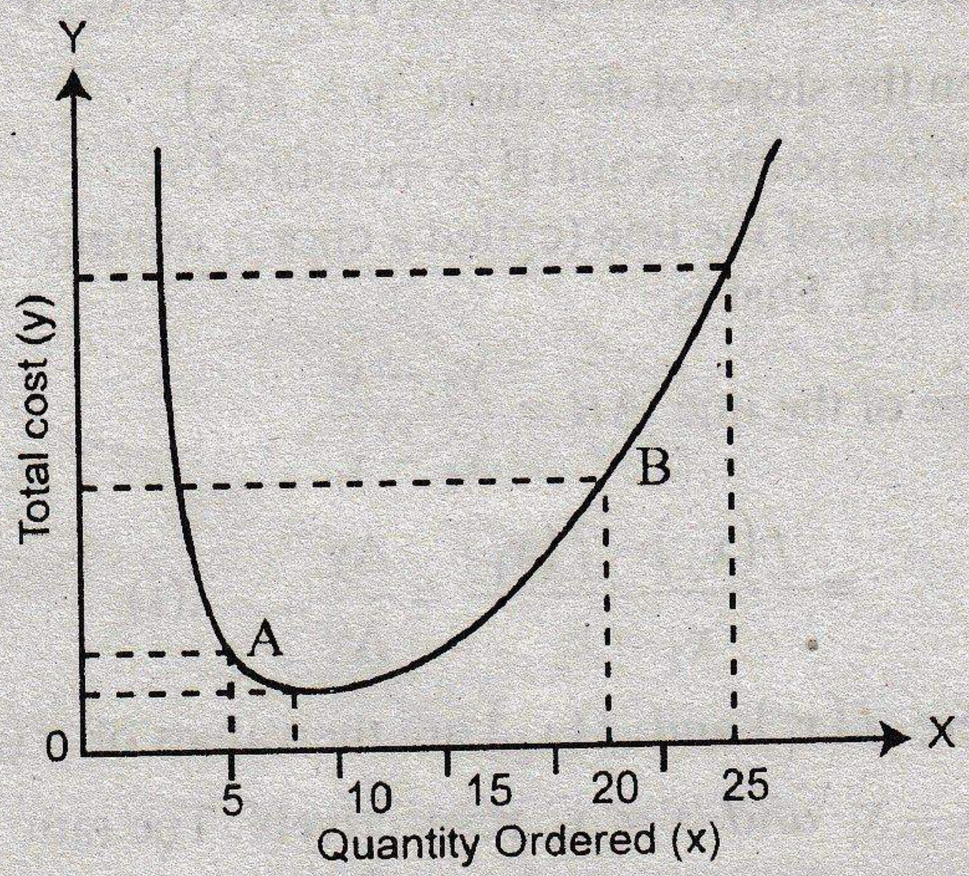


Fig-2. **Slope of a Curve (at a Point)**

12.2.3 SLOPE OF A CURVE (AT A POINT)

The slope of non-linear function (i.e. curve) changes at every point on the curve. Thus slope must be determined for each particular point of interest. Example: Consider the total cost function: $y = 0.5x + 20$, where y denote the total cost and x denotes the quantity ordered. The graph of this function is shown in Fig. 2.

For computing the slope at specified points, say A and B, we have to compute the ratio $\Delta y / \Delta x$.

That is, for x = 5 and x = 7

$$y = 0.5\times5 + 20/5 = 6.50 \text{ and } y = 0.5\times7 + 20/7 = 6.35$$

Thus, between $x = 5$ and $x = 7$, we have $\dfrac{\Delta y}{\Delta x} = \dfrac{6.35 - 6.50}{7 - 5} = -0.075$

For $x = 20$ *and* $x = 22$, we have

$$y = 0.5\times20 + 20/20 = 11 \textit{ and } y = 0.5\times22 + 20/22 = 11.90$$

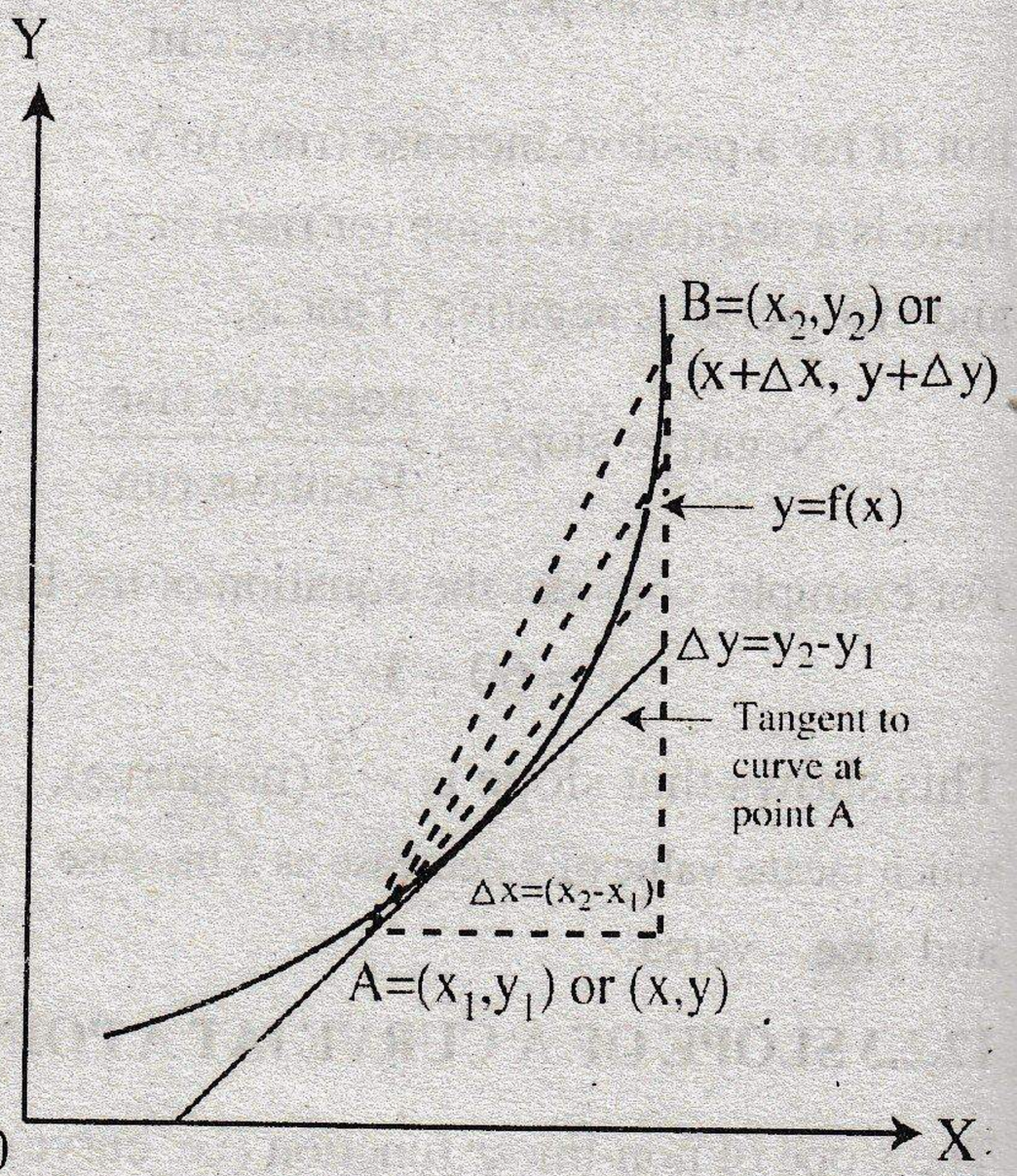
Thus, between $x = 20$ *and* $x = 22$, we have $\dfrac{\Delta y}{\Delta x} = \dfrac{11.90 - 11}{22 - 20} = 0.45$

These two values of the ratio $\Delta y / \Delta x$ at point A and B show that the slope of a curve is different at different point. However, absolute value in the lower range $x = 5$ and $x = 7$ is less than the value in the upper range $x = 20$ and $x = 22$. This shows that the value of y decreases as x increases from 5 to 7 but its value increase as x increase from 20 to 22.

12.3 CONCEPT OF DERIVATIVE

The term derivative is a generalized expression for measuring the rate of change or slope of function. Suppose that $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two points on the curve $y = f(x)$.

Then the slope of the curve $y = f(x)$ between points A and B is measured by the slope of the line (called a chord) joining A and B. That is,



$$\text{Slope of the line } AB = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \quad \ldots\ldots\ldots (i)$$

where $f(x_1)$ and $f(x_2)$ are the values of $f(x)$ at $x = x_1$ *and* $x = x_2$ respectively. The ratio $\Delta y / \Delta x$ is called the difference quotient of $y = f(x)$ *at* $x = x_1$.

Since $x_2 > x_1$, therefore $x_2 = x_1 + \Delta x_1$, where Δx_1 small change in x. Thus substituting for

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x_1) - f(x_1)}{(x_1 + \Delta x_1) - x_1} = \frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x_1} \quad \ldots\ldots\ldots (ii)$$

Equation (ii) represent the slope of the straight line AB rather that of the curve AB at a point A. Suppose now that the point A is held fixed while the point B is moved along the curve $y = f(x)$ towards the point A. Then the slope of the line AB gets closer to the tangent to the curve at A (tangent at a point is defined as the line that touches the curve only at that point and does not cross the curve at the point). This means Δx becomes very small when point B is extremely close to point A. In other words, slope of the secant line AB approach a constant limiting value which is also said to be the slope of the tangent to the curve at A or slope of the curve at A. This is written as:

$$\text{Slope of curve at A} = \lim_{\Delta x \to 0}\left\{\frac{\Delta y}{\Delta x}\right\} \text{ or, } \frac{dy}{dx} = \lim_{\Delta x \to 0}\left\{\frac{\Delta y}{\Delta x}\right\} = \lim_{\Delta x \to 0}\left\{\frac{f(x + \Delta x) - f(x)}{\Delta x}\right\}$$

Definition: A function defined over an interval $a \le x \le b$ is said to be derivable or differentiable at a point $x = a_0$ in the given interval, if

$$f'(a_0) = \lim_{h \to 0} \frac{f(a_0 + h) - f(a_0)}{h} \text{ exists.}$$

The above limit when it exists, is called the differential coefficient or derivative of $f(x)$ at $x = a_0$. The process or technique of finding the first derivative of a function is called differentiation.

If $y = f(x)$ is the given function, when some of the functions is addition to $\frac{dy}{dx}$ to denote the first derivative of $y = f(x)$ with respect to x are:

$D(y)$, $f'(x)$, $y'(x)$, y_1.

The derivative of a function $y = f(x)$ at a particular point, say x = a is written as:

$$\frac{dy}{dx}\Big|_{x=a},\quad f'(a)\ |D(y)|_{x=a}$$

REMARKS

1. Derivative at any point on a curve gives the value of the slope at that point and it measures the rate of change of the dependent variable y with respect to independent variable x.
2. If a function $y = f(x)$ is differentiable at a point $x = a_0$, then it is also continuous at that point, but the converse is not true.
3. The derivative of a function $y = f(x)$ at $x = a_0$ is also called instantaneous or marginal rate of change at $x = a_0$.

12.3.1 EXISTENCE OF DERIVATIVE

It is possible that a function does not have a derivative at a given point. This means that the limit

$$f'(a_0) = \lim_{h\to 0}\left\{\frac{f(a_0+h)-f(a_0)}{h}\right\}$$ fails to exist.

The necessary condition for a function $y = f(x)$ to have derivative at a given point is that the right hand derivative must be equal to the left hand derivative at that point. That is,

$$= \lim_{h\to 0^+}\left\{\frac{f(a_0+h)-f(a_0)}{h}\right\} = \lim_{h\to 0-}\left\{\frac{f(a_0-h)-f(a_0)}{-h}\right\}.$$

Here $h \to 0+$ *and* $h \to 0-$ means letting $h \to 0$ through positive and negative values of h respectively.

12.3.2 SOME STANDARD DERIVATIVES

Given below are some standard derivatives of various types of functions from the first principles.

1. Derivative of a constant. The derivative of a constant is zero. Let c be a constant such that $y = f(x) = c$ for all values of x, and $y + \Delta y = c$.

Then $\Delta y = 0$ *and* $\frac{\Delta y}{\Delta x} = 0(\Delta x \neq 0)$

Taking limit on both sides as $\Delta x \to 0$, we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left\{\frac{\Delta y}{\Delta x}\right\} = 0.$$

Hence, the derivative of a constant is always zero.

Examples: (i) If $y = 5$, then $\frac{dy}{dx} = 0$ (ii) If $y = 0$, then $\frac{dy}{dx} = 0$.

2. Derivative of linear function. $y = ax + b$.

Let $y = ax + b$, $y + \Delta y = a(x + \Delta x) + b$

Then $\Delta y = a\Delta x$ *and* $\frac{\Delta y}{\Delta x} = a$ (constant)

Taking limit on both sides as $\Delta x \to 0$, we get

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left\{\frac{\Delta y}{\Delta x}\right\} = 0. \text{ Hence } \frac{d}{dx}(ax + b) = a.$$

Examples: (i) If $f(x) = 10x + 2$, then $\frac{dy}{dx} = 10$ (ii) If $y = -5x$, then $\frac{dy}{dx} = -5$

3. Derivative of x^n, n being a positive number

Let $y = x^n$ and n being a positive integer, and $y + \Delta y = (x + \Delta x)^n$. Then

$$\Delta y = (x + \Delta x)^n - x^n = x^n\left(1 + \frac{\Delta x}{x}\right)^n - x^n = x^n\left[\left(1 + \frac{\Delta x}{x}\right)^n - 1\right]$$

$$= x^n\left[\left\{1 + n\left(\frac{\Delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\Delta x}{x}\right)^2 + \ldots \textit{higher powers of } x\right\} - 1\right]$$

$$= x^n\left[\left\{1 + n\left(\frac{\Delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\Delta x}{x}\right)^2 + \ldots \textit{higher powers of } x\right\} - 1\right]$$

and $$\frac{\Delta y}{\Delta x} = x^n\left[\frac{n}{x} + \frac{n(n-1)}{2!}\left(\frac{\Delta x}{x^2}\right) + \textit{terms involving } \Delta x\right]$$

Taking limit on both sides as $\Delta x \to 0$, we get

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} x^n\left[\frac{n}{x} + \frac{n(n-1)}{2!}\frac{\Delta x}{x^2} + \textit{terms involving } \Delta x\right] = x^n\frac{n}{x} = n.x^{n-1}$$

That is $\frac{d}{dx}(x^n) = nx^{n-1}$

Examples: (i) If $y = x^3$, then $\frac{dy}{dx} = 3x^{3-1} = 3x^2$ (ii) If $y = x$, then $\frac{dy}{dx} = 1.(x)^{1-1} = 1$

(iii) If $y = \frac{1}{x} = x^{-1}$, then $\frac{dy}{dx} = -1(x)^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

(iv) If $y = x^2 + 3x$, then $\frac{dy}{dx} = 2x + 3$.

4. Derivative of e^x

Let $y = e^x$ and $y + \Delta y = e^{x+\Delta x}$. Then

$$\Delta y = e^{x+\Delta x} - e^x = e^x . e^{\Delta x} - e^x = e^x \left[e^{\Delta x} - 1\right]$$

$$= e^x \left[\left\{1 + \Delta x + \frac{(\Delta x)^2}{2!} + \textit{terms involving higher powers of } \Delta x\right\} - 1\right]$$

$$= e^x \left[\Delta x + \frac{\Delta x^2}{2!} + \frac{(\Delta x)^3}{3!} + \textit{terms involving } \Delta x\right]$$

$$= e^x \Delta x \left[1 + \frac{\Delta x}{2!} + \frac{(\Delta x)^2}{3!} + \textit{terms involving } \Delta x\right]$$

Taking limit on both sides as $\Delta x \to 0$, we get

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!} \frac{\Delta x}{x^2} + \textit{terms involving } \Delta x\right] = x^n \frac{n}{x} = n.x^{n-1}$$

That is, $\frac{d}{dx}\left(e^x\right) = e^x$.

5. Derivative of $\log_e x$

We know that $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.7181$ (approx).

Let $y = \log_e x$, *and* $y + \Delta y = \log_e (x + \Delta x)$. Then

$$\Delta y = \log_e (x + \Delta x) - \log_e x = \log_e \left\{\frac{x + \Delta x}{x}\right\} \text{ (by definition)}$$

$$= \log_e \left\{1 + \frac{\Delta y}{x}\right\}$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_e \left(1 + \frac{\Delta x}{x}\right) = \frac{1}{x} \log_e \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}$$

Taking limit on both sides as $\Delta x \to 0$, we get

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)=\lim_{\Delta x\to 0}\left[\frac{1}{x}\log_e\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right]$$

$$=\frac{1}{x}\log_e\left[\lim_{\Delta x\to 0}\left(1+\frac{\Delta x}{x}\right)^{x/\Delta x}\right]=\frac{1}{x}\log_e(e)=\frac{1}{x}$$

That is, $\frac{d}{dx}(\log_e x)=\frac{1}{x}$.

6. Derivative of Trigonometric Functions

(a) Differential coefficient of $\sin x$

Let $y=\sin x,\ y+\Delta y=\sin(x+\Delta x)$.

$$\Delta y=\sin(x+\Delta x)-\sin x=2\cos\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)$$

$$\therefore\frac{\Delta y}{\Delta x}=\frac{2\cos\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}=\cos\left(x+\frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\right)$$

Taking limit on both sides as $\Delta x\to 0$, we get

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)=\lim_{\Delta x\to 0}\left[\cos\left(x+\frac{\Delta x}{2}\right).\frac{\sin(\Delta x/2)}{(\Delta x/2)}\right]$$

$$=\lim_{\Delta x\to 0}\left[\cos\left(x+\frac{\Delta x}{2}\right)\right]\lim_{\Delta x\to 0}\left[\frac{\sin(\Delta x/2)}{(\Delta x/2)}\right]=\cos x.1=\cos x$$

Hence, $\frac{d}{dx}(\sin x)=\cos x$

Similarly, $\frac{d}{dx}(\cos x)=-\sin x,\ \frac{d}{dx}(\tan x)=\sec^2 x,\ \frac{d}{dx}(\cos ecx)=-\cos ec\,x\cot x$

$$\frac{d}{dx}(\sec x)=\sec x\tan x,\ \frac{d}{dx}(\cot x)=-\cos ec^2 x$$

12.3.3 GENERAL RULE FOR DIFFERENTIATION

In this section, general rules for differentiation polynomial algebraic, logarithmic, exponential, inverse implicit and composite functions are stated and illustrated.

(a) Derivative Scalar Multiple of a Function

If $y = cu$, where c is a constant and u is differentiable function of x, then

$$\frac{dy}{dx} = \frac{d}{dx}(cu) = c\frac{du}{dx}$$

That is, the derivative of the product of a constant and a differentiable function is the product of the constant and the derivative of the function.

Examples: (i) If $y = 5x^4, \frac{dy}{dx} = 5(4)x^{4-1} = 20x^3$

(ii) If $y = \frac{2}{\sqrt{x}} = 2x^{-1/2}, \frac{dy}{dx} = 2\left(-\frac{1}{2}\right)x^{\frac{1}{2}-1} = -x^{-3/2}$

(b) Derivative of a sum (or Difference) of Functions

If $y = u \pm v \pm w \pm$ where u, v, w are differentiable function of x

then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \ldots\ldots$ That is, the derivative of the sum (or difference) of a finite number of differentiable functions is the sum of their derivatives.

Examples: (i) If $y = x^2 - 2x$, then

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - 2\frac{d}{dx}(x) = 2x - 2.$$

(ii) If $y = 4x^3 + 3x^2 + x + 2$, then

$$\frac{dy}{dx} = 4\frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(2) = 12x^2 + 6x + 1 + 0 = 12x^2 + 6x + 1$$

(iii) If $y = 7e^x + 4\log x + x^3$, then

$$\frac{dy}{dx} = 7\frac{d}{dx}(e^x) + 4\frac{d}{dx}(\log x) + \frac{d}{dx}(x^3) = 7e^x + (4/x) + 3x^2$$

(c) Derivative of the product of two functions

If $y = uv$, where u and v are differentiable functions of x, then

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

That is.the derivative of the product of two functions is equal to
(first function)×(derivative of second function)+(second function) ×(derivative of first function).

(d) Derivative of the quotient of two functions

If $y=\frac{u}{v}$, where u, and v are differentiable functions of x, then

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

That is, the derivative of the quotient of two functions equals: (denominator) × (derivative of numerator)-(numerator) × (derivative of denominator), then this difference divided by the square of the denominator.

12.4 DERIVATIVE OF A COMPOSITE FUNCTION (CHAIN RULE)

If $y=f(t)$ is a differential function of t and t is a function of x, i.e. $t=g(x)$ is a differential function of x, then $y=f[g(x)]$ is a differential function of x. The derivative of y with respect to x is given by

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

This rule is some times also called the chain rule of differentiation.

Proof: Let $y=y+\Delta y=f(t)+f(t+\Delta t)$

$$t=g(x), t+\Delta t=g(x+\Delta x)$$

Then $\Delta y=f(t+\Delta t)-f(t)=\frac{f(t+\Delta t)-f(t)}{\Delta t}.\Delta t=\frac{f(t+\Delta t)-f(t)}{\Delta t}[g(x+\Delta x)-g(x)]$

$$\therefore\frac{\Delta y}{\Delta x}=\frac{f(t+\Delta t)-f(t)}{\Delta t}.\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

As $\Delta x\to 0$ also . $\Delta t\to 0$. Thus we have

$$\frac{dy}{dx}=\lim_{\Delta t\to 0}\left(\frac{\Delta y}{\Delta x}\right)=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta t}.\frac{\Delta t}{\Delta x}\right)=\lim_{\Delta t\to 0}\left(\frac{\Delta y}{\Delta t}\right)\lim_{\Delta x\to 0}\left(\frac{\Delta t}{\Delta x}\right)$$

$$\therefore\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

Remark: In general, if y is a function of t, t is a function of u and u is the function of w and so on, then

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{du}\times\frac{du}{dw}\times\ldots$$

For Example-1: Differentiae with respect to x, the following functions

(a) $y=\log(\tan e^x)$, (b) $y=\log\sin x^2$

Solution:

(a) Let $t=\tan e^x$, so that $y=\log t$. Further let $u=e^x$, then $t=\tan u$. Now

$$\frac{dy}{dt}=\frac{dy}{dt}\frac{dt}{du}.\frac{du}{dx}=\frac{d}{dt}(\log t)\times\frac{d}{du}(\tan u)\times\frac{d}{dx}(e^x)=\frac{1}{t}\times\sec^2 u\times e^x$$

$$=\frac{1}{\tan(ex)}.\sec^2(e^x)e^x=\frac{e^x\sec^2(e^x)}{\tan e^x}$$

(b) $\frac{dy}{dx}=2x\cot x^2$. Left as an exercise for the reader to verify.

For example-2. Difference the following functions with respect to x,

(a) $\cos\left[2\sin^2(x^3)\right]$ (b) $\cos\left\{\sin\sqrt{ax+b}\right\}$

Solution: (a) Let $y=\cos 2t$ *and* $t=\sin^2 u$, *where* $u=x^3$

$$\therefore\frac{dy}{dt}=-2\sin 2t,\frac{dt}{du}=2\sin u\frac{d}{du}(\sin u)=2\sin u\cos u \text{ and } \frac{du}{dx}=3x^2$$

By using chain rule, we get

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{du}\times\frac{du}{dx}=-2\sin 2t\times 2\sin u\cos u\times 3x^2$$

$$=-12x^2\sin\left\{2\sin^2(x^3)\right\}\sin(x^3)\cos(x^3)$$

(b) Let $y=\cos\left\{\sin\sqrt{(ax+b)}\right\}$

$$\therefore\frac{dy}{dx}=-\sin\left\{\sin\sqrt{(ax+b)}\right\}\frac{d}{dx}\sin\sqrt{ax+b}$$

$$=-\sin\left\{\sin\sqrt{(ax+b)}\right\}\cos\sqrt{(ax+b)}\times\frac{a}{2\sqrt{(ax+b)}}$$

$$=-\frac{a}{2\sqrt{(ax+b)}}\sin\left\{\sin\sqrt{(ax+b)}\right\}\cos\sqrt{(ax+b)}$$

12.5 DERIVATIVE OF EXPONENTIAL FUNCTIONS

If $y=a^u$, *where* $u=f(x)$ is a differentiable function of x and $a>0$ is any real number, then the generalized exponential function $y=a^u$ defined by

$$y=a^u=e^{\log a^u}=e^{u\log a}$$

and $$\frac{dy}{dx}=\frac{d}{dx}(a^u)=\frac{d}{dx}(e^{u\log a})=e^{u\log a}\frac{d}{dx}(u\log_e a)=(\log a)e^{u\log a}\frac{du}{dx}$$

In particular, $\frac{d}{dx}(a^x)=a^x\log a;(a>0)$.

For examples: (i) Let $y=2^x$, then $\frac{dy}{dx}=2^x\log_e 2$

(ii) Let $y = (10)^{x^2 - x}$

and $\frac{dy}{dx} = 10^{x^2-x}(\log_e 10)\frac{d}{dx}(x^2 - x) = 10^{x^2-x}\log_e 10 \times (2x-1) = (2x-1)(10)^{x^2-x}\log_e 10$

(iii) *Let* $y = a^x x^a, \frac{dy}{dx} = a^x(\log_e a)x^a + a^x a x^{a-1} = a^x x^{a-1}(a + x\log e^a)$

Special case: If $a = e$, *then* $y = e^u$ and therefore $\frac{dy}{dx} = e^u . \frac{du}{dx}$

(iv) If $y = \frac{e^x}{x}$, *then* $\frac{dy}{dx} = \frac{x\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(x)}{x^2} = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$

12.6 DERIVATIVE OF LOGARITHMIC FUNCTIONS

If $y = \log a^u$, *where* $u = f(x)$ is a differentiable function of x and $a > 0$ is any real number then the generalized logarithmic function $y = \log_a u$ is defined as:

$$y = \log_a u = \frac{\log_e u}{\log_e a} \text{ and } \frac{dy}{dx} = \frac{d}{dx}(\log_a u) = \frac{1}{\log_e a} \times \frac{1}{u} \times \frac{du}{dx}$$

Illustration: If $y = (\log_e x^2)^3$, *then*

$$\frac{dy}{dx} = 3(\log x^2)^2 \frac{d}{dx}(\log_e x^2) = 3(\log_e x^2)^2\left(\frac{2x}{x^2}\right) = \frac{6}{x}(\log_e x^2)^2.$$

For Example: Differentiate with respect to x, the function $y = \log_e(e^{ax}\cos bx)$

Solution: We have $y = \log_e(e^{ax}\cos bx)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\{\log_e(e^{ax}\cos bx)\} = \frac{d}{dx}[\log_e(e^{ax}) + \log_e(\cos bx)] = \frac{d}{dx}\log_e(e^{ax}) + \frac{d}{dx}\log_e(\cos bx)$$

$$= \frac{d}{dx}(ax) + \frac{1}{\cos bx}\frac{d}{dx}(\cos bx) = a + \frac{1}{\cos bx}(-b\sin bx) = a - b\tan bx$$

Logarithmic differentiation

When a function $f(x)$ is given in the form

$$F(x) = \left[u(x)^{u(x)}\right] \text{ or, } u(x) \times y(x) \text{ or, } \frac{u(x)}{v(x)}$$

where u (x) and v (x) are both functions of x. For differentiating such function we take first logarithm of both sides and then differentiating the equation so obtained on both sides with respect to x. This method is known as logarithmic differentiation. In other words, the logarithmic differentiation is used in the following cases.

(i) When a function of x is raised to a power which is itself a function of x.

(ii) When a function is the product or quotient of a number of a function

12.7 DERIVATIVE OF INVERSE FUNCTIONS

The derivative of an inverse function is the reciprocal of the derivative of the original function. If $y = f(x)$ and $x = g(y)$ are both differentiable, then relationship between their derivatives is given by

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

Proof: Let δx and δy be the corresponding increments in x and y respectively. Then we have

$$\frac{\Delta y}{\Delta x}.\frac{\Delta x}{\Delta y} = 1$$

Taking limit on both sides as $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)\lim_{\Delta y \to 0}\left(\frac{\Delta x}{\Delta y}\right) = 1 \; or, \; \frac{dy}{dx}.\frac{dx}{dy} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{dx/dy}; \frac{dx}{dy} \neq 0.$$

For example: Find $\frac{dx}{dy}$ if $y = \frac{x-1}{x+1}$.

Solution: $\frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2} = \frac{(x+1).1-(x-1).1}{(x+1)^2} = \frac{2}{(x+1)^2}$

Thus $\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{2}(x+1)^2$

12.8 DERIVATIVE OF AN IMPLICIT FUNCTION

If the relationship between x and y is given by an equation $f(x, y) = 0$ involving both x and y and y can not be expressed in terms of x, then y is said to be an implicit function of x, say $y = f(x)$. Such function can be solved for y in terms of x by defining $f(x, y) = 0$ as:

$$\Phi\{x, f(x)\} = 0$$

To find, $\frac{dy}{dx}$ we differentiate each term on both sides of the equation with respect to x and then solve the resulting relation for $\frac{dy}{dx}$.

12.9 SUCCESSIVE (OR REPEATED) DIFFERENTIATION

The first derivative $\frac{dy}{dx}$ of the function $y=f(x)$ is, in general, a function of x and may be differentiate again with respect to x. The derivative of the first derivative $\frac{dy}{dx}$ is called the second derivative and is written as $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$. Similarly the derivative of second derivative is called the third derivative and so on.

In general, nth derivative denoted by $f^n(x)\, or, \frac{d^ny}{dx^n}$ of a function $y=f(x)$ is obtained by differentiating it n times. The following notations are generally used for successive differential coefficients of y :

$\frac{d^n}{dx^n}$ or, $f^n(x)$ or, y_n or, D^ny where n is a positive integer.

ILLUSTRATIONS

Illustration-01: Find first derivative of the following functions:

(a) $7x^4+3x^3-9x+5$, **(b)** $x+\frac{4}{x}-\frac{2}{x^7}$.

Solution: **(a)** Let $y=7x^4+3x^3-9x+5$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}\left(7x^4+3x^3-9x+5\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(7x^4\right)+\frac{d}{dx}\left(3x^3\right)-\frac{d}{dx}(9x)+\frac{d}{dx}(5)$$

$$\Rightarrow \frac{dy}{dx}=4.7x^{4-1}+3.3x^{3-1}-9.1x^{1-1}+0 \Rightarrow \frac{dy}{dx}=28x^3+9x^2-9 \text{ [Ans.]}$$

(b) Let $y=x+\frac{4}{x}-\frac{2}{x^7} \Rightarrow y=x+4x^{-1}-2x^{-7}$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}\left(x+4x^{-1}-2x^{-7}\right) \Rightarrow \frac{dy}{dx}=\frac{d}{dx}(x)+\frac{dy}{dx}\left(4x^{-1}\right)-\frac{dy}{dx}\left(2x^{-7}\right)$$

$$\Rightarrow \frac{dy}{dx}=1+4(-1)x^{-1-1}-2(-7)x^{-7-1}$$

$$\Rightarrow \frac{dy}{dx}=1-4x^{-2}+14x^{-8} \Rightarrow \frac{dy}{dx}=1-\frac{4}{x^2}+\frac{14}{x^8} \text{ [Ans.]}$$

Illustration-02: Find first derivative of the following functions:

$$\text{(a)}\ \frac{3x^2+5x}{7x+4}, \qquad \text{(b)}\ \frac{(2x+1)(3x+1)}{4x+1}.$$

Solution: (a) Let $y=\dfrac{3x^2+5x}{7x+4} \Rightarrow \dfrac{d}{dx}(y)=\dfrac{d}{dx}\left(\dfrac{3x^2+5x}{7x+4}\right)$

$$\Rightarrow \frac{dy}{dx}=\frac{(7x+4)\dfrac{d}{dx}(3x^2+5x)-(3x^2+5x)\dfrac{d}{dx}(7x+4)}{(7x+4)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{(7x+4)(6x+5)-(3x^2+5x)(7)}{(7x+4)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{42x^2+35x+24x+20-21x^2-35x}{(7x+4)^2} \Rightarrow \frac{dy}{dx}=\frac{21x^2+24x+20}{(7x+4)^2}\ \text{[Ans.]}$$

(b) Let $y=\dfrac{(2x+1)(3x+1)}{4x+1} \Rightarrow \dfrac{d}{dx}(y)=\dfrac{d}{dx}\dfrac{6x^2+5x+1}{4x+1}$

$$\Rightarrow \frac{dy}{dx}=\frac{(4x+1)\dfrac{d}{dx}(6x^2+5x+1)-(6x^2+5x+1)\dfrac{d}{dx}(4x+1)}{(4x+1)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{(4x+1)(12x+5)-(6x^2+5x+1)(4)}{(4x+1)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{48x^2+20x+12x+5-24x^2-20x-4}{(4x+1)^2} \Rightarrow \frac{dy}{dx}=\frac{24x^2+12x+1}{(4x+1)^2}\ \text{[Ans.]}$$

Illustration-03: Find first derivative of the following functions:

$$\text{(a)}\ (5x^3+6x^2+11x+7)^{11}, \qquad \text{(b)}\ \frac{1}{\sqrt[3]{2x^4+3x^3-5x+6}}.$$

Solution: (a) Let $y=(5x^3+6x^2+11x+7)^{11}$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(5x^3+6x^2+11x+7)^{11}$$

$$\Rightarrow \frac{dy}{dx}=11(5x^3+6x^2+11x+7)^{11-1}\frac{d}{dx}(5x^3+6x^2+11x+7)$$

$$\Rightarrow \frac{dy}{dx}=11(5x^2+6x^2+11x+7)^{10}(15x^2+12x+11)\ \text{[Ans.]}$$

(b) Let $y = \dfrac{1}{\sqrt[3]{2x^4+3x^3-5x+6}}$.

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(2x^4+3x^3-5x+6\right)^{-\frac{1}{3}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3}\left(2x^4+3x^3-5x+6\right)^{\frac{1}{3}-1}\frac{d}{dx}\left(2x^4+3x^3-5x+6\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{3}\left(2x^4+3x^3-5x+6\right)^{\frac{4}{3}}\left(8x^3+9x^2-5\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{3}\cdot\frac{1}{\left(2x^4+3x^3-5x+6\right)^{\frac{4}{3}}}\cdot\left(8x^3+9x^2-5\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\left(8x^3+9x^2-5\right)}{3\sqrt[3]{\left(2x^4+3x^3-5x+6\right)^4}}$$ [Ans.]

Illustration-04: Find first derivative of the following functions:

(a) $3\sqrt{x}\sin x$, (b) $x^3\tan x$.

Solution: (a) Let $y = 3\sqrt{x}\sin x \Rightarrow \dfrac{d}{dx}(y) = 3\dfrac{d}{dx}\left[\sqrt{x}\sin x\right]$

$$\Rightarrow \frac{dy}{dx} = 3\left[\sqrt{x}\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}\left(\sqrt{x}\right)\right] \Rightarrow \frac{dy}{dx} = 3\left[\sqrt{x}\cos x+\sin x.\frac{1}{2\sqrt{x}}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3\left[\frac{2x\cos x}{2\sqrt{x}}+\frac{\sin x}{2\sqrt{x}}\right] \Rightarrow \frac{dy}{dx} = \frac{3}{2\sqrt{x}}\left(2x\cos x+\sin x\right)$$ [Ans.]

(b) Let $y = x^3\tan x \Rightarrow \dfrac{d}{dx}(y) = \dfrac{d}{dx}\left[x^3\tan x\right]$

$$\Rightarrow \frac{dy}{dx} = x^3\frac{d}{dx}(\tan x)+\tan x\frac{d}{dx}\left(x^3\right) \Rightarrow \frac{dy}{dx} = x^3.\sec^2 x+\tan x.3x^2$$

$$\Rightarrow \frac{dy}{dx} = 3x^2\tan x+x^3\sec^2 x \Rightarrow \frac{dy}{dx} = x^2\left(3\tan x+x\sec^2 x\right)$$ [Ans.]

Illustration-05: (i) If $y = e^x\cos x$, (ii) $y = x^3\log x$, then find $\dfrac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(e^x\cos x) = e^x\frac{d}{dx}(\cos x)+\cos x\frac{d}{dx}(e^x) = -e^x\sin x+\cos x e^x = e^x\left(\cos x-\sin x\right).$$

(ii)If $y = x^3 \log x$, *then find* $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 \log x\right) = x^3 \frac{d}{dx}(\log x) + \log x \frac{d}{dx}\left(x^3\right) = x^3 \times \frac{1}{x} + \log x \times 3x^2 = x^2(1 + 3\log x).$$

Illustration-06: (i) If $y = \frac{e^x}{\log x}$, *then*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{\log x}\right) = \frac{\log x \frac{d}{dx}\left(e^x\right) - e^x \frac{d}{dx}(\log x)}{(\log x)^2} = \frac{\log x.e^x - e^x(1/x)}{(\log x)^2}$$

(ii) If $y = \frac{4}{x^3}$, *then*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{4}{x^3}\right) = \frac{x^3 \frac{d}{dx}(4) - 4\frac{d}{dx}\left(x^3\right)}{\left(x^3\right)^2} = \frac{x^3(0) - 4\left(3x^2\right)}{x^6} = \frac{-12x^2}{x^6} = \frac{-12}{x^4}.$$

Illustration-07: (a) If $y = \sqrt{\frac{1-x}{1+x}}$ prove that $(1-x)^2 \frac{dy}{dx} + y = 0$

(b) If $y = \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}$, show that $\frac{dy}{dx} + \sec^2\left(\frac{\pi}{4} - x\right) = 0$

Solution: (a) $y = \sqrt{\frac{1-x}{1+x}} = \sqrt{\frac{1-x}{1+x} \times \frac{1-x}{1-x}} = \frac{1-x}{\sqrt{1-x^2}}$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{1-x^2}(0-1) - (1-x)\left\{\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)\right\}}{\left(\sqrt{1-x^2}\right)^2} = \frac{-\sqrt{1-x^2} + x(1-x)\left(1-x^2\right)^{-1/2}}{1-x^2}$$

$$= \frac{-\sqrt{1-x^2} + x\frac{(1-x)}{\sqrt{1-x^2}}}{1-x^2}$$

or, $\left(1-x^2\right)\frac{dy}{dx} = -\sqrt{1-x^2} + \frac{x(1-x)}{\sqrt{1-x^2}} = \frac{-\left(1-x^2\right) + x(1-x)}{\sqrt{1-x^2}} = \frac{-1+x^2+x-x^2}{\sqrt{1-x^2}} = \frac{-(1-x)}{\sqrt{1-x^2}} = -y$

Hence, $\left(1-x^2\right)\frac{dy}{dx} + y = 0.$

(b) $y = \sqrt{\frac{1-\sin 2x}{1+\sin 2x}} = \left[\frac{\cos^2 x + \sin^2 x - 2\sin x \cos x}{\cos^2 x + \sin^2 x + 2\sin x \cos x}\right]^{1/2} = \left[\frac{(\cos x - \sin x)^2}{(\cos x + \sin x)^2}\right]^{1/2} = \frac{\cos x - \sin x}{\cos x + \sin x} = \frac{1-\tan x}{1+\tan x}$,

on dividing by $\cos x = \dfrac{\tan(\pi/4) - \tan x}{\tan(\pi/4) + \tan x} = \tan\left(\dfrac{\pi}{4} - x\right)$

$$\therefore \frac{dy}{dx} = \sec^2\left(\frac{\pi}{4} - x\right)\frac{d}{dx}\left(\frac{\pi}{4} - x\right) = -\sec^2\left(\frac{\pi}{4} - x\right) or \frac{dy}{dx} + \sec^2\left(\frac{\pi}{4} - x\right) = 0.$$

Illustration-08: Find $\dfrac{dy}{dx}$ *where* $y = (x^2 + 3)^3$.

Solution: Given $y = (x^2 + 3)^3$.

Therefore, $\dfrac{dy}{dx} = 3(x^2 + 3)^2 \dfrac{d}{dx}(x^2 + 3) = 3(x^2 + 3)^2(2x) = 6x(x^2 + 3)^2$

Illustration-09: If $\sin y = x\sin(a + y)$, *show that* $\dfrac{dy}{dx} = \dfrac{\sin^2(a + y)}{\sin a}$

Solution: Given, $\sin y = x\sin(a + y) \Rightarrow x = \dfrac{\sin y}{\sin(a + y)}$

$$\therefore \frac{dx}{dy} = \frac{\sin(a + y)\cos y - \sin y\cos(a + y)}{\sin^2(a + y)} = \frac{\sin(a + y - y)}{\sin^2(a + y)}$$

$[\because \sin(A - B) = \sin A\cos B - \cos A\sin B]$

$$= \frac{\sin a}{\sin^2(a + y)}. \quad \therefore \frac{dy}{dx} = \frac{\sin^2(a + y))}{\sin a} \ (proved)$$

Illustration-10: If $y = x^3 \log\left(\dfrac{1}{x}\right)$ prove that $x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3x^2 = 0$

Solution: We have $y = x^3 \log\left(\dfrac{1}{x}\right)$

Differentiating with respect to x by parts, we get

$$\frac{dy}{dx} = x^3 \frac{d}{dx}\left[\log\left(\frac{1}{x}\right)\right] + \log\left(\frac{1}{x}\right)\frac{d}{dx}(x^3)$$

$$= x^3(x)\left(-\frac{1}{x^2}\right) + \log\left(\frac{1}{x}\right)(3x^2) = -x^2 + 3x^2 \log\left(\frac{1}{x}\right)$$

$$\therefore x\frac{dy}{dx} = -x^3 + 3x^3 \log\left(\frac{1}{x}\right) = -x^3 + 3y$$

Differentiating both sides again, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-3x^2+3\frac{dy}{dx}$$

$$\text{or, } x\frac{d^2y}{dx^2}-2\frac{dy}{dx}+3x^2=0$$

Illustration-11: If $y=\log\left\{x+\sqrt{\left(x^2+a^2\right)}\right\}$ prove that $\left(a^2+x^2\right)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$

Solution: Given $y=\log\left\{x+\sqrt{\left(x^2+a^2\right)}\right\}$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{\left(x^2+a^2\right)}}\left\{1+\frac{1}{2}\frac{1}{\sqrt{x^2+a^2}}.2x\right\}=\frac{1}{\sqrt{x^2+a^2}}$$

Squaring and cross multiplying, we get, $\left(x^2+a^2\right)\left(\frac{dy}{dx}\right)^2=1$

Differentiating again, we have

$$\left(a^2+x^2\right)2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)^2=0$$

Dividing both sides by $2\frac{dy}{dx}$, we get

$$\left(a^2+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$$

Illustration-12: Differentiate x^x with respect to x.

Solution: Let $y=x^x \Rightarrow \log y=\log x^x$ [Taking 'log' on both sides]

$$\Rightarrow \log y=x\log x \Rightarrow \frac{d}{dx}(\log y)=x\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(x)$$

$$\Rightarrow \frac{1}{y}.\frac{d}{dx}(y)=x.\frac{1}{x}+\log x.1 \Rightarrow \frac{dy}{dx}=y(1+\log x) \Rightarrow \frac{dy}{dx}=x^x(1+\log x)$$ [Ans.]

Illustration-13: Differentiate the following functions w.r.t.x:

(a) $(1+x)^{2x}$, (b) $x^{1/x}$, (c) $\left(x^x\right)^x$.

Solution: (a) Let $y=(1+x)^{2x}$

$\Rightarrow \log y=\log(1+x)^{2x}$ [Taking 'log' on both sides]

$$\Rightarrow \log y = 2x\log(1+x) \Rightarrow \frac{d}{dx}(\log y) = 2x\frac{d}{dx}[\log(1+x)] + \log(1+x)\frac{d}{dx}(2x)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = 2x.\frac{1}{1+x}(0+1) + \log(1+x).2$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2x}{1+x} + 2\log(1+x)\right] \Rightarrow \frac{dy}{dx} = 2y\left[\frac{x}{x+1} + \log(x+1)\right]$$

$$\Rightarrow \frac{dy}{dx} = 2(1+x)^{2x}\left[\frac{x}{x+1} + \log(x+1)\right] \text{ [Ans.]}$$

(b) Let $y = x^{1/x}$

$\Rightarrow \log y = \log x^{1/x}$ [Taking 'log' on both sides] $\Rightarrow \log y = \frac{1}{x}\log x$

$$\Rightarrow \frac{d}{dx}(\log y) = \frac{1}{x}\frac{d}{dx}(\log x) + \log x\frac{d}{dx}\left(\frac{1}{x}\right) \Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{x}.\frac{1}{x} + \log x.\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x^2} - \frac{\log x}{x^2}\right] \Rightarrow \frac{dy}{dx} = x^{1/x}\left[\frac{1-\log x}{x^2}\right] \text{ [Ans.]}$$

(c) Let $y = (x^x)^x \Rightarrow \log y = \log x^{x^2}$ [Taking 'log' on both sides] $\Rightarrow \log y = x^2\log x$

$$\Rightarrow \frac{d}{dx}(\log y) = x^2\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x^2) \Rightarrow \frac{1}{y}.\frac{dy}{dx} = x^2.\frac{1}{x} + \log x.(2x)$$

$$\Rightarrow \frac{dy}{dx} = y[x + 2x\log x] \Rightarrow \frac{dy}{dx} = yx[1 + 2\log x] \Rightarrow \frac{dy}{dx} = (x^x)^x.x[1 + 2\log x]$$

$$\Rightarrow \frac{dy}{dx} = x^{x^2}.x^1[1 + 2\log x] \Rightarrow \frac{dy}{dx} = x^{x^2+1}(1 + 2\log x) \text{ [Ans.]}$$

Illustration-14: If $y = x^{x^x}$, *find* $\frac{dy}{dx}$.

Solution: We have $y = x^{x^x}$. Taking logarithm on both sides, we get $\log y = x^x \log x$

Differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = x^x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x^x)$$

$$= x^x(1/x) + \log x[x^x(1+\log x)]$$

$$\therefore \frac{dy}{dx} = x^{x^x}\left[x^{x-1} + \log x.x^x(1+\log x)\right]$$

Illustration-15: If $y = 3^x e^{2x} \log(x^3+1)$, *find* $\frac{dy}{dx}$.

Solution: We have $y = 3^x e^{2x} \log(x^3+1)$. Taking logarithm on both sides, we get

$$\log y = \log(3^x) + \log(e^{2x}) + \log[\log(x^3+1)]$$

Differentiating both sides with respect to x we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3^x}\frac{d}{dx}(3^x) + \frac{1}{e^{2x}}\frac{d}{dx}(e^{2x}) + \frac{1}{\log(x^3+1)}\frac{d}{dx}[\log(x^3+1)]$$

$$= \frac{1}{3^x}3^x \log 3 + \frac{1}{e^{2x}} \times 2(e^{2x}) + \frac{1}{\log(x^3+1)} \times \frac{1}{x^3+1} \times (3x^2)$$

$$= \log 3 + 2 + \frac{1}{\log(x^3+1)} \times \frac{3x^2}{x^3+1}$$

$$\therefore \frac{dy}{dx} = y\left[\log 3 + 2 + \frac{1}{\log(x^3+1)} \times \frac{3x^2}{x^3+1}\right]$$

$$\therefore \frac{dy}{dx} = 3^x e^{2x} \log(x^3+1)\left[\log 3 + 2 + \frac{1}{\log(x^3+1)} \times \frac{3x^2}{x^3+1}\right]$$

$$\therefore \frac{dy}{dx} = 3^x e^{2x} \log(x^3+1)\left[\log 3 + 2 + \frac{3x^2}{(x^3+1)\log(x^3+1)}\right]$$

Illustration-16: Find $\frac{dy}{dx}$. from the equation: $\log(y/x) = \sec x$.

Solution: Given $\log\left(\frac{y}{x}\right) = \log y - \log x = \sec x$

Differentiating both sides w. r. to x, we get

$$\frac{1}{y}.\frac{dy}{dx} - \frac{1}{x} = \sec x \tan x$$

$$or, \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \sec x \tan x$$

$$\text{or, } \frac{dy}{dx} = y\left(\frac{1}{x} + \sec x \tan x\right) = \frac{y}{x}(1 + x \sec x \tan x)$$

$$= e^{\sec x}(1 + x \sec x \tan x) \qquad \left[\because \log(y/x) = \sec x \Rightarrow y/x = e^{\sec x}\right]$$

Illustration-17: If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Solution: Given $x^y = e^{x-y}$, Taking logarithm on both sides, we get

$$y\log x = x - y \Rightarrow y(1+\log x) = x \Rightarrow y = \frac{x}{1+\log x}$$

$$\therefore \frac{dy}{dx} = \frac{(1+\log x).1 - x(1/x)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

Illustration-18: If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Solution: Given $x\sqrt{1+y} + y\sqrt{1+x} = 0$,

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

Squaring both sides, we get

$$\Rightarrow x^2(1+y) = y^2(1+x)$$

$$\text{or, } (x^2 - y^2) + (x^2y - y^2x) = 0$$

$$\text{or, } (x^2 - y^2) + xy(x-y) = 0$$

$$\text{or, } (x-y)(x+y) + xy(x-y) = 0$$

$$\text{or, } x + y + xy = 0 (\because x \neq y)$$

$$\text{or, } y(1+x) + x = 0$$

$$\text{or, } y = -\frac{x}{1+x}$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = -\left[\frac{(1+x).1 - x.1}{(1+x)^2}\right] = -\frac{1}{(1+x)^2}$$

Illustration-19: If $y = x^{y^x}$, prove that $\frac{dy}{dx} = \frac{y\log y(x\log x\log y + 1)}{(1 - x\log y)x\log x}$

Solution: Since $y = x^{y^x}$, therefore $\log y = y^x \log x$

Taking logarithm on both sides again, we get

$$\log(\log y) = \log(y^x \log x) = x\log y + \log(\log x)$$

Differentiating both sides with respect to x, we get

$$\frac{1}{\log y}.\frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{y}.\frac{dy}{dx} + 1.\log y + \frac{1}{\log x}.\frac{1}{x}$$

$$\text{or, } \frac{dy}{dx}\left(\frac{1}{y\log y}-\frac{x}{y}\right)=\log y+\frac{1}{x\log x}$$

$$\text{or, } \frac{dy}{dx}\left(\frac{1-x\log y}{y\log y}\right)=\frac{x\log x\log y+1}{x\log x}$$

$$\therefore \frac{dy}{dx}=\frac{y\log y(1+x\log x\log y)}{x\log x(1-x\log y)}$$

Illustration-20: Find first derivative of the following functions:

(a) $\frac{\sin 2x}{1+\cos 2x}$, (b) $\frac{\cos x}{\cos x+\sin x}$.

Solution:

(a) Let $\frac{\sin 2x}{1+\cos 2x}$,

$$\Rightarrow y=\frac{\sin x}{\cos x}\Rightarrow y=\tan x\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(\tan x)\Rightarrow \frac{dy}{dx}=\sec^2 x \text{ [Ans.]}$$

(b) Let $\frac{\cos x}{\cos x+\sin x}\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left[\frac{\cos x}{\cos x+\sin x}\right]$

$$\Rightarrow \frac{dy}{dx}=\frac{(\cos x+\sin x)\frac{dy}{dx}(\cos x)-\cos x\frac{dy}{dx}(\cos x+\sin x)}{(\cos x+\sin x)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{(\cos x+\sin x)(-\sin x)-\cos x(-\sin x+\cos x)}{(\cos x+\sin x)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\sin x\cos x-\sin^2 x+\sin x\cos x-\cos^2 x}{(\cos x+\sin x)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\sin^2 x-\cos^2 x}{(\cos x+\sin x)^2}\Rightarrow \frac{dy}{dx}=\frac{-(\sin^2 x+\cos^2 x)}{(\cos x+\sin x)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-1}{(\cos x+\sin x)^2} \text{ [Ans.]} \qquad [\because \sin^2 x+\cos^2 x=1]$$

Illustration-21: If $y=\sin(2\sin^{-1}x)$, show that $\frac{dy}{dx}=2\sqrt{\frac{1-y^2}{1-x^2}}$.

Solution: Given $y=\sin(2\sin^{-1}x)\Rightarrow \sin^{-1}y=2\sin^{-1}x$

$$\Rightarrow \frac{d}{dx}(\sin^{-1}y)=\frac{d}{dx}(2\sin^{-1}x)\Rightarrow \frac{1}{\left(\sqrt{1-y^2}\right)}\frac{dy}{dx}=2.\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = 2.\frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-y^2}}{1} \Rightarrow \frac{dy}{dx} = 2.\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = 2\sqrt{\frac{1-y^2}{1-x^2}}$$ [Shown]

Illustration-22: Find first derivative of the following functions:

(a) $\log \sin x$, (b) $e^{\log \sin x}$

Solution: (a) Let $y = \log \sin x \Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(\log \sin x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}.\cos x$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = \cot x$$ [Ans.]

(b) Let $y = e^{\log \sin x}$

$\Rightarrow \log y = \log e^{\log \sin x}$ [Taking 'log' on both sides]

$\Rightarrow \log y = \log \sin x.\log e \Rightarrow \log y = \log \sin x.1 [\because \log e = 1]$

$\Rightarrow \log y = \log \sin x \Rightarrow y = \sin x$ [Omitting 'log' on both sides]

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(\sin x) \Rightarrow \frac{dy}{dx} = \cos x$$ [Ans.]

Illustration-23: Differentiate $\sin(\log x)$ *w.r.t.* $\tan(e^x)$.

Solution: Let $y = \sin(\log x) \Rightarrow \frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)]$

$$\Rightarrow \frac{dy}{dx} = \cos(\log x).\frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{\cos(\log x)}{x} \Rightarrow \frac{dy}{dx} = \cos(\log x)/x$$

Also, let $z = \tan(e^x) \Rightarrow \frac{dz}{dx} = \frac{d}{dx}[\tan(e^x)] \Rightarrow \frac{dz}{dx} = (\sec^2 e^x)e^x$

$$\Rightarrow \frac{dz}{dx} = e^x(\sec^2 e^x) \therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} \Rightarrow \frac{dy}{dz} = \frac{\cos(\log x)/x}{e^x(\sec^2 e^x)}$$ [Ans.]

Illustration-24: Find $\frac{dy}{dx}$ in the case of $x = a(t - \sin t), y = a(1 - \cos t)$.

Solution: Given $x = a(t - \sin t)$

$$\Rightarrow \frac{dx}{dt} = \frac{d}{dt}[a(t - \sin t)] \Rightarrow \frac{dx}{dt} = a\left[\frac{d}{dt}(t) - \frac{d}{dt}(\sin t)\right]$$

$$\Rightarrow \frac{dx}{dt} = a(1 - \cos t) \Rightarrow \frac{dx}{dt} = a.2\sin^2\frac{t}{2} \left[\because 1 - \cos t = 2\sin^2\frac{t}{2}\right]$$

Also, given $y = a(1-\cos t) \Rightarrow \frac{dx}{dt} = a2\sin^2\frac{t}{2}\left[\because 1-\cos t = 2\sin^2\frac{t}{2}\right]$

$$\Rightarrow \frac{dy}{dt} = \frac{dx}{dt}\left[2a\sin^2\frac{t}{2}\right] \Rightarrow \frac{dy}{dt} = 2a.\left(2\sin\frac{t}{2}\right)\left(\cos\frac{t}{2}\right)\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{dy}{dt} = 2a\sin\frac{t}{2}\cos\frac{t}{2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \frac{dy}{dx} = \frac{2a\sin\frac{t}{2}\cos\frac{t}{2}}{a.2\sin^2\frac{t}{2}} \Rightarrow \frac{dy}{dx} = \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \Rightarrow \frac{dy}{dx} = \cot\frac{t}{2}$$ [Ans.]

Illustration-25: Differentiate with respect to x of the following:

(a) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, (b) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

Solution: (a) Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \Rightarrow y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$ [Putting $x = \tan\theta$]

$$\Rightarrow y = \cos^{-1}(\cos 2\theta)\left[\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

$$\Rightarrow y = 2\theta \Rightarrow y = 2\tan^{-1}x\left[\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(2\tan^{-1}x\right) \Rightarrow \frac{dy}{dx} = 2.\frac{1}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$ [Ans.]

(b) Let $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \Rightarrow y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$ [*Putting* $x = \tan\theta$]

$$\Rightarrow y = \tan^{-1}(\tan 2\theta)\left[\because \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right]$$

$$\Rightarrow y = 2\theta \Rightarrow y = 2\tan^{-1}x\left[\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(2\tan^{-1}x\right) \Rightarrow \frac{dy}{dx} = 2.\frac{1}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$ [Ans.]

Illustration-26: Differentiate with respect to x of the following:

(a) $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$, (b) $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$,

Solution: (a) Let $y = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

$$\Rightarrow y = \tan^{-1}\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)\left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}, 1+\cos x = 2\cos^2\frac{x}{2}\right]$$

$$\Rightarrow y = \tan^{-1}\left(\tan\frac{x}{2}\right) \Rightarrow y = \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{2}$$ [Ans.]

(b) Let $y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right) \Rightarrow y = \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$ $[Putting\ x = \cot\theta]$

$$\Rightarrow y = \tan^{-1}\left(\sqrt{\tan^2\frac{x}{2}}\right) \Rightarrow y = \tan^{-1}\left(\tan\frac{x}{2}\right) \Rightarrow y = \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{2}$$ [Ans.]

Illustration-27: If $y = \sin x\cos x(\log x).e^x.\tan^{-1}x.x^n$, *find* $\frac{dy}{dx}$.

Solution: Given $y = \sin x\cos x(\log x).e^x.\tan^{-1}x.x^n$

$$\Rightarrow \log y = \log\sin x + \log\cos x + \log(\log x) + x\log e + \log\tan^{-1}x + n\log x$$

$$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx}(\log\sin x) + \frac{d}{dx}(\log\cos x) + \frac{d}{dx}\{\log(\log x)\} + \frac{d}{dx}(x.1) + \frac{d}{dx}(\log\tan^{-1}x) + \frac{d}{dx}(n\log x)\left[\because \log e = 1\right]$$

$$\Rightarrow \frac{1}{y}.\frac{d}{dx}(y) = \frac{1}{\sin x}.\frac{d}{dx}(\sin x) + \frac{1}{\cos x}.\frac{d}{dx}(\cos x) + \frac{1}{\log x}.\frac{d}{dx}(\log x) + \frac{d}{dx}(x) + \frac{d}{\tan^{-1}x}.\frac{d}{dx}(\tan^{-1}x) + n.\frac{1}{x}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{\sin x}(\cos x) + \frac{1}{\cos x}(-\sin x) + \frac{1}{\log x}.\frac{1}{x} + 1 + \frac{1}{\tan^{-1}x}.\frac{1}{1+x^2} + \frac{n}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} + \frac{1}{x\log x} + 1 + \frac{d}{\tan^{-1}x}.\frac{1}{1+x^2} + \frac{n}{x}\right]$$ [Ans.]

Illustration-28: If $y = (2-x)\left(\frac{3-x}{1+x}\right)^{1/2}$, *find* $\frac{dy}{dx}$.

Solution: Given $y = (2-x)\left(\frac{3-x}{1+x}\right)^{1/2} \Rightarrow \log y = \log(2-x) + \frac{1}{2}\log(3-x) - \frac{1}{2}\log(1+x)$

$$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx}[\log(2-x)] + \frac{1}{2}\frac{d}{dx}[\log(3-x)] - \frac{1}{2}\frac{d}{dx}[\log(1+x)]$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{1}{(2-x)}(0-1)+\frac{1}{2}.\frac{1}{(3-x)}(0-1)-\frac{1}{2}.\frac{1}{1+x}(0+1)$$

$$\Rightarrow \frac{dy}{dx}=y\left[-\frac{1}{(2-x)}-\frac{1}{2(3-x)}-\frac{1}{2(1+x)}\right]$$

$$\Rightarrow \frac{dy}{dx}=(2-x)\left(\frac{3-x}{1+x}\right)^{1/2}\left[\frac{1}{(x-2)}+\frac{1}{2(x-3)}-\frac{1}{2(1+x)}\right] \text{ [Ans.]}$$

Illustration-29: Differentiate the following functions w.r.t.x:

(a) $\log\left[e^{x}\left(\frac{x-2}{x+3}\right)^{3/4}\right]$, (b) $e^{5/x}.\left(\sqrt{2x^{2}-1}\right)$, (c) $\frac{x^{2}e^{5x}}{(3x+1)^{1/2}(2x-1)^{1/3}}$.

Solution: (a) Let $y=\log\left[e^{x}\left(\frac{x-2}{x+3}\right)^{3/4}\right]$

$$\Rightarrow y=\log e^{x}+\frac{3}{4}\log(x-2)-\frac{3}{4}\log(x+3)$$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(x.\log e)+\frac{3}{4}\frac{d}{dx}[\log(x-2)]-\frac{3}{4}\frac{d}{dx}[\log(x+3)]$$

$$\Rightarrow \frac{dy}{dx}=1+\frac{3}{4}.\frac{1}{(x-2)}(1-0)-\frac{3}{4}.\frac{1}{(x+3)}(1+0)\Rightarrow \frac{dy}{dx}=1+\frac{3}{4(x-2)}-\frac{3}{4(x+3)} \text{ [Ans.]}$$

(b) *Let* $y=e^{5/x}.\left(\sqrt{2x^{2}-1}\right)$

$$\Rightarrow \log y=\frac{5}{x}\log e+\frac{1}{2}\log(2x^{2}-1)\Rightarrow \frac{d}{dx}(\log y)=\frac{d}{dx}\left(\frac{5}{x}\right)+\frac{1}{2}\frac{d}{dx}[\log(2x^{2}-1)]$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=-\frac{5}{x^{2}}+\frac{1}{2}.\frac{1}{(2x^{2}-1)}(4x)$$

$$\Rightarrow \frac{dy}{dx}=y\left[-\frac{5}{x^{2}}+\frac{2x}{(2x^{2}-1)}\right]\Rightarrow \frac{dy}{dx}=e^{5/x}.\left(\sqrt{2x^{2}-1}\right)\left[\frac{-5(2x^{2}-1)+2x(x^{2})}{x^{2}(2x^{2}-1)}\right]$$

$$\Rightarrow \frac{dy}{dx}=e^{5/x}.\frac{-10x^{2}+5+2x^{3}}{x^{2}\sqrt{2x^{2}-1}}\Rightarrow \frac{dy}{dx}=e^{5/x}.\frac{2x^{3}-10x^{2}+5}{x^{2}\sqrt{2x^{2}-1}} \text{ [Ans.]}$$

(c) *Let* $y=\frac{x^{2}e^{5x}}{(3x+1)^{1/2}(2x-1)^{1/3}}$.

$$\Rightarrow \log y=2\log x+5x\log e-\frac{1}{2}\log(3x+1)-\frac{1}{3}\log(2x-1)$$

$$\Rightarrow \frac{d}{dx}(\log y)=\frac{d}{dx}\left[2\log x+5x\log e-\frac{1}{2}\log(3x+1)-\frac{1}{3}\log(2x-1)\right]$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{2}{x}+5-\frac{1}{2}.\frac{1}{3x+1}(3+0)-\frac{1}{3}.\frac{1}{2x-1}(2-0)$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2e^{5x}}{(3x+1)^{1/2}(2x-1)^{1/3}}\left[5+\frac{2}{x}-\frac{3}{2(3x+1)}-\frac{2}{3(2x-1)}\right] \text{ [Ans.]}$$

Illustration-30: Differentiate the following functions w.r.t.x:

(a) $x^{\log x}$, (b) $x^{\log(\log x)}$.

Solution: (a) Let $y=x^{\log x} \Rightarrow \log y=\log x^{\log x} \Rightarrow \log y=\log x.\log x$

$$\Rightarrow \frac{d}{dx}(\log y)=\log x\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\log x.\frac{1}{x}+\log x.\frac{1}{x} \Rightarrow \frac{dy}{dx}=y\left[\frac{1}{x}\log x+\frac{1}{x}\log x\right]$$

$$\Rightarrow \frac{dy}{dx}=y.\frac{1}{x}[\log x+\log x] \Rightarrow \frac{dy}{dx}=x^{\log x}x^{-1}(2\log x) \Rightarrow \frac{dy}{dx}=2x^{\log x-1}\log x\,[Ans]$$

(b) *Let* $y=x^{\log(\log x)} \Rightarrow \log y=\log x^{\log(\log x)} \Rightarrow \log y=\log(\log x).\log x$

$$\Rightarrow \frac{d}{dx}(\log y)=\log(\log x)\frac{d}{dx}(\log x)+\log x\frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{\log(\log x)}{x}+\log x.\frac{1}{\log x}.\frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}=y\left[\frac{\log(\log x)}{x}+\frac{1}{x}\right] \Rightarrow \frac{dy}{dx}=x^{\log(\log x)}\left(\frac{1+\log(\log x)}{x}\right) \text{ [Ans.]}$$

Illustration-31: If $y=(\sin x)^{\log x}+(x)^{\sin x}$, *find* $\frac{dy}{dx}$.

Solution: Given $y=u+v$ [Putting $u=(\sin x)^{\log x}$ and $v=(x)^{\sin x}$]

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

Here $u=(\sin x)^{\log x} \Rightarrow \log u=\log x.\log(\sin x)$

$$\Rightarrow \frac{d}{dx}(\log u)=\log x\frac{d}{dx}[\log(\sin x)]+\log(\sin x).\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}.\frac{d}{dx}(u)=\log x.\frac{1}{\sin x}.\cos x+\log(\sin x).\frac{1}{x}$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{\cos x}{\sin x}\log x + \frac{\log(\sin x)}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^{\log x}\left[\cot x \log x + \frac{\log \sin x}{x}\right]$$

Also $v = (x)^{\sin x} \Rightarrow \log v = \sin x.\log x$

$$\Rightarrow \frac{d}{dx}(\log v) = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{1}{v}.\frac{d}{dx}(v) = \sin\frac{1}{x} + \log x.\cos x$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{\sin x}{x} + \cos x \log x\right] \Rightarrow \frac{dv}{dx} = (x)^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\left[\cot x \log x + \frac{\log \sin x}{x}\right] + (x)^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$ [Ans]

Illustration-32: Find $\frac{dy}{dx}$ of the following: (a) $x^2 + y^2 - 2x = 0$ (b) $x^2 + 3xy + y^2 = 4$.

Solution: (a) Given $x^2 + y^2 - 2x = 0 \Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(2x)$

$$\Rightarrow \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(2x) \Rightarrow 2x + 2y\frac{d}{dx}(y) = 2$$

$$\Rightarrow 2y\frac{dy}{dx} = 2 - 2x \Rightarrow \frac{dy}{dx} = \frac{2(1-x)}{2y} \Rightarrow \frac{dy}{dx} = \frac{1-x}{y}$$ [Ans.]

(b) *Given* $x^2 + 3xy + y^2 = 4 \Rightarrow \frac{d}{dx}(x^2 + 3xy + y^2) = \frac{d}{dx}(4)$

$$\Rightarrow \frac{d}{dx}(x^2) + 3x\frac{d}{dx}(y) + y\frac{d}{dx}(3x) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

$$\Rightarrow 2x + 3x\frac{dy}{dx} + y.3 + 2y\frac{d}{dx}(y) = 0 \Rightarrow 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = -(2x + 3y)$$

$$\Rightarrow (3x + 2y)\frac{dy}{dx} = -(2x + 3y) \Rightarrow \frac{dy}{dx} = -\frac{2x + 3y}{3x + 2y}$$ [Ans.]

Illustration-33: Differentiate the following functions w.r.t.x: $\sin y = x\sin(a + y)$

Solution: Given $\sin y = x\sin(a+y) \Rightarrow x = \dfrac{\sin y}{\sin(a+y)} \Rightarrow \dfrac{d}{dy}(x) = \dfrac{d}{dy}\left[\dfrac{\sin y}{\sin(a+y)}\right]$

$$\Rightarrow \frac{dx}{dy} = \frac{sn(a+y)\frac{d}{dy}(\sin y) - \sin y \frac{d}{dy}[\sin(a+y)]}{\sin^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)(0+1)}{\sin^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin(a+y-y)}{\sin^2(a+y)} [\because \sin(A-B) = \sin A\cos B - \cos A\sin B]$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)} \Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$ [Ans.]

Illustration-34: If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ...\infty}}}$, *prove that* $\dfrac{dy}{dx} = \dfrac{\cos x}{2y-1}$.

Solution: Given $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ...\infty}}}$

$$\Rightarrow y = \sqrt{\sin x + y}\left[\textit{Putting } y = \sqrt{\sin x + \sqrt{\sin x + ...\infty}}\right]$$

$$\Rightarrow y^2 = \sin x + y$$ [Squaring on both sides]

$$\Rightarrow \frac{d}{dx}(y^2) = \frac{d}{dx}(\sin x + y) \Rightarrow 2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$ [Ans.]

Illustration-35: Find $\dfrac{d^2y}{dx^2}$ in the following: (a) $y = x^m e^{nx}$, (b) $y = x^2 \sin^{-1} x$.

Solution: (a) Given $y = x^m e^{nx} \Rightarrow \dfrac{d}{dx}(y) = \dfrac{d}{dx}\left(x^m e^{nx}\right)$

$$\Rightarrow \frac{dy}{dx} = x^m \frac{d}{dx}\left(e^{nx}\right) + e^{nx}\frac{d}{dx}\left(x^m\right) \Rightarrow \frac{dy}{dx} = x^m\left(ne^{nx}\right) + e^{nx}\left(mx^{m-1}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(nx^m + mx^{m-1}\right)e^{nx} \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\left(nx^m + mx^{m-1}\right)e^{nx}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(nx^m + mx^{m-1}\right)\frac{d}{dx}\left(e^{nx}\right) + e^{nx}\frac{d}{dx}\left(nx^m + mx^{m-1}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(nx^m + mx^{m-1}\right)\left(ne^{nx}\right) + e^{nx}\left\{mnx^{m-1} + \left(m^2 - m\right)x^{m-2}\right\}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(n^2x^m + mnx^{m-1} + mnx^{m-1} + m^2x^{m-2} - mx^{m-2}\right)e^{nx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = m(m-1)x^{m-2}e^{nx} + 2mnx^{m-1}e^{nx} + n^2x^me^{nx} \text{ [Ans.]}$$

(b) Given $y = x^2\tan^{-1}x \Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}\left(x^2\tan^{-1}x\right)$

$$\Rightarrow \frac{dy}{dx} = x^2\frac{d}{dx}\left(\tan^{-1}x\right) + \left(\tan^{-1}x\right)\frac{d}{dx}\left(x^2\right) \Rightarrow \frac{dy}{dx} = x^2\frac{1}{1+x^2} + \tan^{-1}x(2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{1+x^2} + 2x\tan^{-1}x \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{x^2}{1+x^2}\right] + \frac{d}{dx}\left[2x\tan^{-1}x\right]$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\left(1+x^2\right)2x - x^2(2x)}{\left(1+x^2\right)^2} + 2x.\frac{1}{1+x^2} + 2\tan^{-1}x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{2x+2x^3-2x^3}{\left(1+x^2\right)^2} + \frac{2x}{1+x^2} + 2\tan^{-1}x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left[\frac{2x}{\left(1+x^2\right)^2} + \frac{2x}{1+x^2}\right] + 2\tan^{-1}x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left[\frac{2x+2x\left(1+x^2\right)}{\left(1+x^2\right)^2}\right] + 2\tan^{-1}x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{4x+2x^3}{\left(1+x^2\right)^2} + 2\tan^{-1}x \Rightarrow \left(\frac{dy^2}{dx^2}\right) = \frac{2x\left(2+x^2\right)}{\left(1+x^2\right)^2} + 2\tan^{-1}x \text{ [Ans.]}$$

Illustration-36: (a) If $y = ax^3 + bx^2 + cx + d$, find y_3 and y_4.

(b) If $y = \frac{\log x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{2\log x - 3}{x^3}$.

Solution: (a) Given $y = ax^3 + bx^2 + cx + d$

$$\Rightarrow y_1 = \frac{d}{dx}\left(ax^3\right) + \frac{d}{dx}\left(bx^2\right) + \frac{d}{dx}(cx) + \frac{d}{dx}(d)$$

$$\Rightarrow y_1 = 3ax^2 + 2bx + c \Rightarrow y_2 = \frac{d}{dx}(3ax^2) + \frac{d}{dx}(2bx) + \frac{d}{dx}(c)$$

$$\Rightarrow y_2 = 6ax + 2b \Rightarrow y_3 = \frac{d}{dx}(6ax) + \frac{d}{dx}(2b) \Rightarrow y_3 = 6a \text{ [Ans.]}$$

and $y_3 = 6a \Rightarrow y_4 = \frac{d}{dx}(6a) \Rightarrow y_4 = 0$ [Ans.]

(b) Given $y = \frac{\log x}{x} \Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}\left[\frac{\log x}{x}\right] \Rightarrow \frac{dy}{dx} = \frac{x\frac{d}{dx}(\log x) - \log x \frac{d}{dx}(x)}{(x)^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{x.\frac{1}{x} - \log x(1)}{x^2} \Rightarrow \frac{dy}{dx} = \frac{1 - \log x}{x^2} \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{1 - \log x}{x^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x^2\frac{d}{dx}(1 - \log x) - (1 - \log x)\frac{d}{dx}(x^2)}{(x^2)^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x - 2x(1 - \log x)}{x^4} \Rightarrow \frac{d^2y}{dx^2} = \frac{-x - 2x + 2x\log x}{x^4} \Rightarrow \frac{d^2y}{dx^2} = \frac{2x\log x - 3x}{x^4}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x(2\log x - 3)}{x^4} \Rightarrow \frac{d^2y}{dx^2} = \frac{2\log x - 3}{x^3} \text{ [Shown]}$$

Illustration-37: If $y = ae^{mx} + be^{-mx}$, prove that $\frac{d^2y}{dx^2} = m^2y$.

Solution. Given $y = ae^{mx} + be^{-mx} \Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(ae^{mx}) + \frac{d}{dx}(be^{-mx})$

$$\Rightarrow \frac{dy}{dx} = mae^{mx} - mbe^{-mx} \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(mae^{mx}) - \frac{d}{dx}(mbe^{-mx})$$

$$\Rightarrow \frac{d^2y}{dx^2} = mmae^{mx} + mmbe^{-mx} \Rightarrow \frac{d^2y}{dx^2} = mm(ae^{mx} + be^{-mx}) \Rightarrow \frac{d^2y}{dx^2} = m^2y \text{ [Proved]}$$

Illustration-38: If $y = \sin ax$, prove that $\frac{d^2y}{dx^2} + a^2y = 0$.

Solution: Given $y = \sin ax \Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(\sin ax) \Rightarrow \frac{dy}{dx} = (\cos ax)a$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(a\cos ax) \Rightarrow \frac{d^2y}{dx^2} = (-a\sin ax)a \Rightarrow \frac{d^2y}{dx^2} = -a^2\sin ax \Rightarrow \frac{d^2y}{dx^2} = -a^2y$$

$$\Rightarrow \frac{d^2y}{dx^2}+a^2y=0 \qquad \text{[Proved]}$$

Illustration-39: If $y = A\sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$,

Solution: Given $y = A\sin(\log x) \Rightarrow y_1 = \frac{d}{dx}[A\sin(\log x) \Rightarrow y_1 = A\cos(\log x).\frac{1}{x}$

$$\Rightarrow y_1 = \frac{A\cos(\log x)}{x} \Rightarrow y_2 = \frac{d}{dx}\left[\frac{A\cos(\log x)}{x}\right] \Rightarrow y_2 = A.\frac{x.(-\sin\log x).\frac{1}{x} - \cos(\log x)\,(1)}{x^2}$$

$$\Rightarrow y_2 = A.\frac{-\sin\log x - \cos\log x}{x^2}$$

$$\therefore LHS = x^2y_2 + xy_1 + y$$

$$= x^2\left[A.\frac{-\sin\log x - \cos\log x}{x^2}\right] + x\left[\frac{A\cos\log x}{x}\right] + A\sin\log x$$

$$= x^2\left[A.\frac{-\sin\log x - \cos\log x}{x^2}\right] + x\left[\frac{A\cos\log x}{x}\right] + A\sin\log x$$

$$= -A\sin\log x - A\cos\log x + A\cos\log x + A\sin\log x = 0 = RHS \quad \text{[Proved]}$$

Illustration-40: If $y = e^x\cos 2x$, then show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$.

Solution: Given $y = e^x\cos 2x \Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}[e^x\cos 2x]$

$$\Rightarrow \frac{dy}{dx} = e^x\frac{d}{dx}(\cos 2x) + \cos 2x\frac{d}{dx}(e^x) \Rightarrow \frac{dy}{dx} = e^x(-\sin 2x)2 + \cos 2x.e^x$$

$$\Rightarrow \frac{dy}{dx} = -2e^x\sin 2x + e^x\cos 2x \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[-2e^x\sin 2x] + \frac{d}{dx}[e^x\cos 2x]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -2\left[e^x\frac{d}{dx}(\sin 2x) + \sin 2x\frac{d}{dx}(e^x)\right] + e^x\frac{d}{dx}(\cos 2x) + \cos 2x\frac{d}{dx}(e^x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -2[e^x(\cos 2x)2 + \sin 2x(e^x)] + e^x(-\sin 2x)2 + \cos 2x(e^x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -4e^x\cos 2x - 2e^x\sin 2x - 2e^x\sin 2x + e^x\cos 2x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -3e^x\cos 2x - 4e^x\sin 2x$$

$$\therefore LHS = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y$$

$$= -3e^x \cos 2x - 4e^x \sin 2x - 2\left(-2e^x \sin 2x + e^x \cos 2x\right) + 5e^x \cos 2x$$

$$= -3e^x \cos 2x - 4e^x \sin 2x + 4e^x \sin 2x - 2e^x \cos 2x + 5e^x \cos 2x$$

$$= -5e^x \cos 2x + 5e^x \cos 2x = 0 = RHS \text{ [Proved]}$$

Illustration-41: If $x = \frac{1-t}{1+t}$ and $y = \frac{2t}{1+t}$, find $\frac{d^2y}{dx^2}$.

Solution: *Given* $x = \frac{1-t}{1+t} \Rightarrow \frac{d}{dt}(x) = \frac{d}{dt}\left[\frac{1-t}{1+t}\right]$

$$\Rightarrow \frac{dx}{dt} = \frac{(1+t)\frac{d}{dt}(1-t) - (1+t)\frac{d}{dt}(1+t)}{(1+t)^2} \Rightarrow \frac{dx}{dt} = \frac{(1+t)(-1) - (1+t)(1)}{(1+t)^2}$$

$$\Rightarrow \frac{dx}{dt} = \frac{-1-t-1-t}{(1+t)^2} \Rightarrow \frac{dx}{dt} = \frac{-2-2t}{(1+t)^2} \Rightarrow \frac{dx}{dt} = \frac{-2(1+t)}{(1+t)^2} \Rightarrow \frac{dx}{dt} = \frac{-2}{(1+t)}$$

Also $y = \frac{2t}{1+t} \Rightarrow \frac{d}{dt}(y) = \frac{d}{dt}\left[\frac{2t}{1+t}\right]$

$$\Rightarrow \frac{dy}{dt} = \frac{(1+t)\frac{d}{dt}(2t) - 2t\frac{d}{dt}(1+t)}{(1+t)^2} \Rightarrow \frac{dy}{dt} = \frac{(1+t)(2) - 2t(1)}{(1+t)^2}$$

$$\Rightarrow \frac{dy}{dt} = \frac{2+2t-2t}{(1+t)^2} \Rightarrow \frac{dy}{dt} = \frac{2}{(1+t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \frac{dy}{dx} = \frac{\frac{2}{(1+t)^2}}{\frac{-2}{(1+t)}} \Rightarrow \frac{dy}{dx} = \frac{2}{(1+t)^2} \times \frac{(1+t)}{-2} \Rightarrow \frac{dy}{dx} = -\frac{1}{(1+t)}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy}{dx}\left[-\frac{1}{(1+t)}\right] \Rightarrow \frac{d^2y}{dx^2} = 0 \text{ [Ans.]}$$

EXERCISE -I

1. Find the first derivative with respect to x for each of the following functions:

(a) $y = 4e^x + 3$

(b) $y = 5e^x + 3\log x + x^3$

(c) $y = 7x^3 - \sin x$

(d) $y = x^5 + x^{3/4} + 5x^{-1/2}$

(e) $y = e^x \cos x$

(f) $y = x \log_e x$

(g) $y = e^x \log_e (2x+1)$

(h) $y = x^4 \sin x$

(i) $y = \dfrac{x}{(x+1)^{1/2}}$

(j) $y = \dfrac{e^x}{\log_e x}$

(k) $y = \dfrac{\cos x}{\log_e x}$

(m) $y = \dfrac{3}{(x^3-4)^{2/3}}$

(n) $y = \dfrac{\tan x}{x^2}$

(o) $y = \dfrac{2x^2+3}{\sqrt{x}}$

Answer

1. (a) $4e^x$

(b) $5e^x + \dfrac{3}{x} + 3x^2$

(c) $21x^2 - \cos x$

(d) $5x^4 + \dfrac{3}{4}x^{-1/4} - \dfrac{5}{2}x^{-3/2}$

(e) $-e^x \sin x + e^x \cos x$

(f) $1 + \log_e x$

(g) $\dfrac{2e^x}{2x+1} + e^x \log_e (2x+1)$

(h) $4x^3 \sin x + x^4 \cos x$

(i) $1 - \dfrac{1}{.2(x+1)}$

(j) $\dfrac{e^x \log_e x - \{e^x / x\}}{(\log_e x)^2}$

(k) $\dfrac{-\sin x \log_e x - \{\cos x / x\}}{(\log_e x)^2}$

(l) $-\dfrac{4}{(x-1)^2}$

(m) $\dfrac{2x^2}{(x^2-4)^{5/3}}$

(n) $\dfrac{x^2 \sec^2 x - 2x \tan x}{x^4}$

(o) $(6x^2 - 3)2x^{5/2}$

2. Determine $\dfrac{dy}{dx}$ for each of the following functions:

(a) $y = \log \sin x$

(b) $y = \sqrt{(ax^2) + bx + c}$

(c) $y = \sqrt{(1+x)/(1-x)}$

(d) $x = \log_e \left(\dfrac{1+\sin y}{1-\sin y}\right)^{1/2}$

(e) $(e^x + e^{-x})(e^x - e^{-x})$

(f) $y = \log(\log x)$

(g) $y = 3\sqrt{(4x^3 + 5x^2 - 7x + 6)}$

3. If $y = \dfrac{x}{\sqrt{1+x^2}}$. prove that $x^3 \dfrac{dy}{dx} = y^3$.

4. Prove that the function $y = \log[1/(1+x)]$ satisfies the relationship $x\dfrac{dy}{dx} + 1 = e^y$.

ANSWERS

2. (a) $\cot x$ (b) $(ax+b/2)\sqrt{ax^2+bx+c}$

(c) $\left[(1-x)\sqrt{1-x^2}\right]^{-1}$ (d) $\cos y$ (e) $-4\left(e^x - e^{-x}\right)^{-2}$

(f) $\dfrac{1}{x \log x}$ (g) $\dfrac{1}{3}\left(4x^3+5x^2-7x+6\right)^{-2/3}\left(12x^2+10x-7\right)$

5. Determine $\dfrac{dy}{dx}$ for each of the following function:

a. $y = \dfrac{e^x}{x}$ b. $y = x^{x^2}$ c. (i) $y = x^{e^x}$ (ii) $y = x^{e^{ax}}$

d. $y = x^{e^{x^2+1}}$ e. (i) $y = 2x^2 e^{x^2+3}$ (ii) $y = a^x \times x^a$

f. (i) $y = 16^{x^2-2x}$ (ii) $y = 3^{x+8}$

g. (i) $y = \dfrac{(2x+1)^2}{(x^2+2)^3}$ (ii) $y = \dfrac{x^2\sqrt{(4x+3)}}{3x+1}$

h. (i) $y = x^n e^x$ (ii) $y = a^x e^x$ (iii) $y = \dfrac{e^x}{x^2}$ (iv) $\dfrac{e^x}{1+x}$

i. (i) $y = \log_e\{\log(x^3+2x^2+5x+1)\}^3$ (ii) $y = \log_e\left|\sqrt{(1+\log x)} - \sin x\right|$

(iii) $y = \log_a(a^2 - x^2)^3$ (iv) $y = \log_5(x^3+x^2)^6$

j. $y = (x+1)^2 e^{5x} \log(x^2+1)$

k. If $y = \left(\dfrac{1+x}{1-x}\right)^r$, prove that $(1-x^2)\dfrac{dy}{dx} = 2ry$, where r is again positive integer.

l. If $(x-y)e^{x/(x-y)} = a$, prove that $y = \dfrac{dy}{dx} + x = 2y$.

ANSWERS

5.

a. $\dfrac{e^x(x-1)}{x^2}$

b. $x^{x^2+1}(1+2\log_e x)$

c. (i) $x^{e^x}x^x\left(\dfrac{1}{x}+\log_e x\right)$ (ii) $x^{e^{ax}}\left\{e^{ax}\left(\dfrac{1}{x}+a\log_e x\right)\right\}$

d. $x^{e^{x^2+1}}\left(e^{x^2+1}\right)\left(\dfrac{1}{x}+2x\log_e x\right)$

e. (i) $4x(x^2+1)e^{x^2+3}$ (ii) $3^{x+8}\log e^3\ a^x x^a \log a + a^{x+1}x^{a-1}$

f.(i) $2(x-1)16^{x^2-2x}\log_e 16$

g. $(2x+1)(8-6x-8x^2)/(x^2+2)^4$

h. (i) $x^{n-1}e^x(x+n)$, (ii) $(1+\log a)a^x e^x$

(iii) $\dfrac{e^x(x-2)}{x^3}$ (iv) $\dfrac{xe^x}{(1+x)^2}$

i. (ii) $\dfrac{1}{\left\{\sqrt{(1+\log x)}-\sin x\right\}}\left[\dfrac{1}{2x\sqrt{(1+\log x)}}-\cos x\right]$ (iii) $\dfrac{6x\log_a e}{x^2-a^2}$ (iv) $\dfrac{6(3x^2+2x)\log_5 e}{x^3+x^2}$

6. If $y=\sin\left(2\sin^{-1}x\right)$, show that $\dfrac{dy}{dx}=\sqrt{\dfrac{1-y^2}{1-x^2}}$.

7. Find $\dfrac{dy}{dx}$, when (i) $x=e^t; y=\log t$

(ii) $x=a(t-\sin t), y=a(1-\cos t)$

(iii) $x=\log t+\sin t, y=e^t+\cos t$

8. If $x=t\log t, y=\dfrac{\log t}{t}$, *find* $\dfrac{dx}{dt}$, *also find* $\dfrac{dy}{dx}$ *when* $t=1$.

9. If $y=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, then prove that $\dfrac{dy}{dx}=\dfrac{2x}{1+x^2}$

10. Find $\dfrac{dy}{dx}$ for the following implicit functions:

(i) $x^3+4x^2y+3yx^2+y=c$

(ii) $x^2+3xy+3x^2-2x+y-12=0$

(iii) $x^3 - 4x^2y + 5xy^2 - y^2 = 27$

11. If $y = \dfrac{x}{\sqrt{1+x^2}}$ prove that $x^3 \dfrac{dy}{dx} = 2 - \dfrac{x}{y}$.

12. If $\dfrac{x}{x-y} = \log \dfrac{a}{x-y}$, prove that $\dfrac{dy}{dx} = 2 - \dfrac{x}{y}$.

13. If $\sin y = x \sin(a+y)$, prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

14. If $y.e^y = x$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x(1+y)}$.

15. Find $\dfrac{dy}{dx}$, if (i) $xy = e^{x+y}$ (ii) $x^y.y^x = k$, where k is constant

16. If $x^y = e^{x+y}$ show that $\dfrac{dy}{dx} = -\dfrac{\log x - 2}{(\log x - 1)^2}$.

17. If $(x-y)e^{x/(x-y)} = a$, prove that $y\dfrac{dy}{dx} + x = 2y$.

18. (a) If $x^y = y^x$, prove that $\dfrac{dy}{dx} = \dfrac{y(y - x\log y)}{x(x - y\log x)}$.

(b) If $x^p.y^q = (x+y)^{p+q}$, show that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

19. Find $\dfrac{dy}{dx}$ at $x = 1, y = 1$, *if* $x^3 - 2x^2y^2 + 5x + y - 5 = 0$.

20. Find the second derivative of the following:

a. x^2e^x b. $(ax+b)^2 \log x$ c. a^x

21. If $y = ae^{mx} + be^{-mx}$, show that $\dfrac{d^2y}{dx^2} - m^2y = 0$.

22. If $y = \cos^{-1} x$, show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0$.

23. If $y = (\log x)^2$, prove that $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 2$.

12.10 INCREASING AND DECREASING FUNCTIONS

If $y = f(x)$, then y is said to be an increasing function of x at the point $x = a$ if $\frac{dy}{dx} > 0$ at $x = a$ i.e. $\left[\frac{dy}{dx}\right]_{x=a} > 0.$

Again, if $y = f(x)$, then y is said to be a decreasing function of x at the point $x = a$ if $\frac{dy}{dx} < 0$ at $x = a$ i.e. $\left[\frac{dy}{dx}\right]_{x=a} < 0.$

For example: Determine whether the following functions are increasing or decreasing at the point $x = 3$.

(a) $y = 125 - 9x,$ (b) $y = 3x^2 - 25,$

(c) $y = 2x^2 - 48x + 27$ (d) $y = (4x - 5)^2.$

Solution: (a) $y = 125 - 9x, \therefore \frac{dy}{dx} = -9$

Since $\frac{dy}{dx}\Big|_{x=3} = -9 < 0,$ therefore $f(x)$ is a decreasing function at $x = 3.$

(b) $y = 3x^2 - 25 \therefore \frac{dy}{dx} = 6x$

Since $\frac{dy}{dx}\Big|_{x=3} = 6 \times 3 = 18 > 0,$ therefore $f(x)$ is an increasing function at $x = 3.$

(c) $y = 2x^2 - 48x + 27 \therefore \frac{dy}{dx} = 4x - 48$

Since $\frac{dy}{dx}\Big|_{x=3} = 4 \times 3 - 48 = -36 < 0,$ therefore $\mathrm{f}(\mathrm{x})$ is a decreasing function at $x = 3.$

(d) $y = (4x - 5)^2 \therefore \frac{dy}{dx} = 2(4x - 5)4 = 8(4x - 5)$

Since $\frac{dy}{dx}\Big|_{x=3} = 8(4 \times 3 - 5) = 56 > 0,$ therefore $f(x)$ is an increasing function at $x = 3.$

12.11 DETERMINATION OF MAXIMA AND MINIMA OF DIFFERENT FUNCTIONS USING DIFFERENTIATION RULES

Working rule for finding the maximum and minimum values of a given function:

Step-I: For the given function $y = f(x)$, find $\frac{dy}{dx}$.

Step-II: Put $\frac{dy}{dx} = 0$ and find the value or values of x. Let these are: a, b, c,...

Step-III: Find $\frac{d^2y}{dx^2}$.

Step-IV: Put $x = a$ in $\frac{d^2y}{dx^2}$. (i) If $\frac{d^2y}{dx^2} > 0$, the function $f(x)$ has minimum value at $x = a$ and $y_{min} = f(a)$.

(ii) If $\frac{d^2y}{dx^2} < 0$, the function $f(x)$ has maximum value at $x = a$ and $y_{max} = f(a)$.

Similarly, test other values b, c,... of x found in step II.

Step-V: When $\frac{d^2y}{dx^2} = 0$ for a particular value $x = a$ (say), then we find $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$,... and put $x = a$ successively in these derivative and test for maxima and minima.

We tabulate the result as follows:

	Maximum	Minimum
Necessary Condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$
Sufficient Condition	$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$	$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

INFLECTION POINT

Inflection points can occur only where the second derivative equals zero or is undefined. The sign of the first derivative is immaterial for an inflection point. For an inflection point at a, the conditions are

(i) $\frac{dy}{dx} = 0$ (ii) $\frac{d^2y}{dx^2} = 0$ *at* $x = a$. (iii) $\frac{d^3y}{dx^3} \neq 0$ *at* $x = a$.

12.12 CONDITIONS FOR PROFIT MAXIMIZATION

We know that if $y = f(x)$, then for y to be maximum

$$\frac{dy}{dx} = f'(x) = 0 \text{ and } \frac{d^2y}{dx^2} = f''(x) < 0$$

Now assuming that we are given the total cost function along with the total revenue function, both in terms of output, i.e., given functions are:

Total cost function, $C = f(x)$

Total revenue function: $R = \Phi(x)$

Total profit: $P = R - C = \Phi(x) - f(x)$

For Profit (P) to be maximum, the conditions are:

First order condition:

$$\frac{dP}{dx} = \frac{dR}{dx} - \frac{dC}{dx} = 0 \quad \Rightarrow \frac{dR}{dx} - \frac{dC}{dx} = 0 \Rightarrow \frac{dR}{dx} = \frac{dC}{dx} \Rightarrow MR = MC$$

Thus, profit P is maximized at that quantity x for which marginal revenue equals marginal cost MR = MC means that slope of total revenue function = slope of total function.

Second order condition:

$$\frac{d^2P}{dx^2} = \frac{d^2R}{dx^2} - \frac{d^2C}{dx^2} < 0 \Rightarrow \frac{d^2R}{dx^2} < \frac{d^2C}{dx^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dR}{dx}\right) < \frac{d}{dx}\left(\frac{dC}{dx}\right) \Rightarrow \frac{d}{dx}(MR) < \frac{d}{dx}(MC)$$

Thus rate of change of MR (slope of MR) should be less than the rate of change of MC (slope of MC) at the profit maximizing output level.

Profit of a firm under various market conditions:

(a) **Perfect competition:** Under perfect competition, the price is constant. The profit maximization conditions is MR = MC, which is the condition for equilibrium of a firm and the rate of change of MR (slope of MR) should be less than the rate of change of MC (slope of MC) at the equilibrium output. Here rate of change of $MR = \frac{d}{dx}(MR)$. Rate of change of $MC = \frac{d}{dx}(MC)$.

(b) **Monopoly:** Under monopoly, the monopolist fixes the output-leaving price to be determined by demand conditions. The profit maximization conditions, obtained above, apply to this case also.

12.13 MARGINAL CONCEPT ANALYSIS

Total, Average and Marginal Cost

(a) Total cost C is represented as a function of output x,i.e., $C = f(x)$.

Total cost, TC = Total variable cost + Fixed cost.

(b) Average Cost: The average cost (AC) represents the cost per unit of production.

Mathematically, average cost $= \frac{C}{x} = \frac{f(x)}{x}$.

(c) Marginal Cost: Marginal cost (MC) represents the change in the total cost for each additional unit of production. MC is the first derivative of the total cost function, i.e.,

Marignal cost, $MC = \frac{dc}{dx}$

(d) Average Variable Cost, $\text{AVC} = \frac{\text{Total Average Cost}}{\text{x}}$

(e) Average fixed cost, $\text{AFC} = \frac{\text{Fixed Cost}}{\text{x}}$

RELATION BETWEEN AVERAGE COST AND MARGINAL COST

We know, $AC = \frac{C}{x}$, the slope of AC is given by

$$\frac{d}{dx}\left(\frac{C}{x}\right) = \frac{x\frac{dC}{dx} - C}{x^2} = \frac{1}{x}\left(\frac{dC}{dx} - \frac{C}{x}\right) = \frac{1}{2}(MC - AC)$$

Case I. When average cost curve slopes downwards, i.e., when AC is declining, its slope will be negative. In other words, $\frac{d}{dx}\left(\frac{C}{x}\right) < 0 \Rightarrow \frac{1}{2}(MC - AC) < 0 \Rightarrow MC < AC.$

Thus when AC curve slopes downward, MC curve will lie below AC curve.

Case II. When AC curve reaches a minimum point, its slope becomes zero, i.e.,

$$\frac{d}{dx}\left(\frac{C}{x}\right) = 0 \Rightarrow MC - AC = 0 \Rightarrow MC = AC$$

Thus MC curve and AC curve intersect at the point of minimum average cost.

Case III. When average cost curve rises upwards, its slope is positive i.e.

$$\frac{d}{dx}\left(\frac{C}{x}\right) > 0 \Rightarrow MC > AC\ .$$

Thus when AC curve slopes upwards, MC curve will be above AC curve.

Illustration-42: Find the point of inflection of the function $y = x^3 - 3x^2 + 3x$.

Solution: Given $y = x^3 - 3x^2 + 3x \therefore \frac{dy}{dx} = 3x^2 - 6x + 3$

Now $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 6x + 3 = 0 \Rightarrow 3(x-1)^2 = 0 \Rightarrow x = 1$

Therefore, $\frac{dy}{dx} = 0$ when $x = 1$

Thus $\frac{d^2y}{dx^2} = 6x - 6 \therefore \frac{d^2y}{dx^2}\Big|_{x=1} = 6 - 6 = 0$

Again $\frac{d^3y}{dx^3} = 6 \neq 0$

When $x = 1, y = 1^3 - 3 \times 1^2 + 3 \times 1 = 1$

Point of inflection $= (1,1)$ $[Ans]$

Illustration-43: Show that the function $y = xe^{-x}$ has a point of inflection at $x = 2$.

Solution: Given $y = xe^{-x} \therefore \frac{dy}{dx} = x(e^{-x})(-1) + e^{-x} = e^{-x}(1-x)$

Now $\frac{d^2y}{dx^2} = e^{-x}(-1) + (1-x)(-1)e^{-x} = e^{-x}(x-2)$

$$\frac{d^3y}{dx^3} = e^{-x}(1) + (x-2)(-1)e^{-x} = e^{-x}(3-x)$$

For $x = 2$, we have $\frac{d^2y}{dx^2}\Big|_{x=2} = e^{-2}(2-2) = 0$ and $\frac{d^3y}{dx^3}\Big|_{x=2} = e^{-2}(3-2) = \frac{1}{e^3} \neq 0$

Therefore $x = 2$ is the point of inflection. [Ans]

Illustration-44: Find the maximum and minimum values of the function: $x^3 - 2x^2 - 4x - 1$

Solution: Let $y = x^3 - 2x^2 - 4x - 1 \therefore \frac{dy}{dx} = 3x^2 - 4x - 4 \quad \therefore \frac{d^2y}{dx^2} = 6x - 4$

For maximum or minimum, $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 4x - 4 = 0 \Rightarrow 3x^2 - 6x + 2x - 4 = 0 \Rightarrow 3x(x-2) + 2(x-2) = 0$

$\Rightarrow (x-2)(3x+2) = 0 \Rightarrow x - 2 = 0$ or $3x + 2 = 0 \Rightarrow x = 2$ or $x = -\frac{2}{3}$

When $x=2$, then $\frac{d^2y}{dx^2}=6x-4=6(2)-4=8$, which is positive. So the function is minimum at $x=2$.

$\therefore$ Minimum value $(y)=x^3-2x^2-4x-1=(2)^3-2(2)^2-4(2)-1=8-8-8-1=-9$

When $x=-\frac{2}{3}$, then $\frac{d^2y}{dx^2}=6x-4=6\left(-\frac{2}{3}\right)-4=-8$, which is negative.

So the function is maximum at $x=-\frac{2}{3}$.

$\therefore$ Maximum value $(y)=x^3-2x^2-4x-1=\left(-\frac{2}{3}\right)^3-2\left(-\frac{2}{3}\right)^2-4\left(-\frac{2}{3}\right)-1$

$=-\frac{8}{27}-\frac{8}{9}+\frac{8}{3}-1=\frac{-8-24+72-27}{27}=\frac{13}{27}$

Answers: Maximum value $=\frac{13}{27}$ at $x=-\frac{2}{3}$, and Minimum value $=-9$ at $x=2$

Illustration-45: If $y=x^4-3x^3+3x^2+5x+1$, prove that $\frac{d^2y}{dx^2}$ is negative when x lies between $\frac{1}{2}$ and 1. What happens if $x=\frac{1}{2}$ or $x=1$.

Solution: First Part:

Given $y=x^4-3x^3+3x^2+5x+1$

$\therefore \frac{dy}{dx}=4x^3-9x^2-6x+5$ $\qquad$ $\therefore \frac{d^2y}{dx^2}=12x^2-18x+6$

Let $x=\frac{3}{4}$, which lies between $\frac{1}{2}$ and 1.

$\frac{d^2y}{dx^2}=12x^2-18x+6=12\left(\frac{3}{4}\right)^2-18\left(\frac{3}{4}\right)+6=-0.75$, which is negative. [Proved]

Second Part:

When $x=\frac{1}{2}$ then $\frac{d^2y}{dx^2}=12x^2-18x+6=12\left(\frac{1}{2}\right)^2-18\left(\frac{1}{2}\right)+6=3$, which is positive.

So the function is minimum at $x=\frac{1}{2}$. [Ans]

When $x=1$, then $\frac{d^2y}{dx^2}=12x^2-18x+6$

$=12(1)^2-18(1)+5$

$= 0,$ which is neither positive nor negative.

Here $\dfrac{d^3y}{dx^3} = 24x - 18 = 24(1) - 18 = 6,$ which is not zero.

So it has neither a maximum nor a minimum value at $x = 1$. It is a point of inflexion. [Ans.]

Illustration-46: The difference of two numbers is 100. The square of the larger one exceeds five times the square of the smaller one by an amount which is maximum. Find the numbers.

Solution: Let the larger number $= x$

The smaller number $= y$

According to the question, $x - y = 100$

Also, let the maximum value $= h$

From the condition,

$h = x^2 - 5y^2 \Rightarrow h = x^2 - 5(x-100)^2 \Rightarrow h = x^2 - 5(x^2 - 200x + 10000)$

$\Rightarrow \mathrm{h} = \mathrm{x}^2 - 5\mathrm{x}^2 + 1000\mathrm{x} - 50000 \Rightarrow h = -4x^2 + 1000x - 50000$

$\therefore \dfrac{dh}{dx} = -8x + 1000 \qquad \therefore \dfrac{d^2h}{dx^2} = -8$

For maximum or minimum, $\dfrac{dh}{dx} = 0 \Rightarrow -8x + 1000 = 0$

$\Rightarrow 8x^2 = 1000 \Rightarrow x = 125$

When $x = 125,$ then $\dfrac{d^2h}{dx^2} = -8,$ which is negative. So the function is maximum at $x = 125.$

Hence h is maximum at $x = 125$.

$\therefore$ Re*quired l*arg*er number* $(x) = 125$

$\therefore$ Required smaller number $(y) = x - 100 = 125 - 100 = 25$

Answers: Larger number = 125, Smaller number = 25.

Illustration-47: Find the maximum or minimum value of $2x^3 - 15x^2 + 36x + 12$.

Solution: Let $y = 2x^3 - 15x^2 + 36x + 12$

$\therefore \dfrac{dy}{dx} = 6x^2 - 30x + 36 \therefore \dfrac{d^2y}{dx^2} = 12x - 30$

For maximum or minimum, $\dfrac{dy}{dx} = 0$

$\Rightarrow 6x^2 - 30x + 36 = 0 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x^2 - 3x - 2x + 6 = 0$

$\Rightarrow x(x-3) - 2(x-3) = 0 \qquad \Rightarrow (x-3)(x-2) = 0$

$\Rightarrow x-3=0$ *or* $x-2=0$ $\Rightarrow x=3$ *or* $x=2$

When $x=2$, then

$\frac{d^2y}{dx^2}=12x-30=12(2)-30=-6$ which is negative. So the function is maximum at $x=2$.

$\therefore$ Required maximum value $(y)=2x^3-15x^2+36x+12$

$=2(2)^3-15(2)^2+36(2)+12 \quad =16-60+72+12 \quad =40$

When $x=3$, then

$\frac{d^2y}{dx^2}=12x-30=12(3)-30=6$ which is positive. So the function is minimum at $x=3$.

$\therefore$ Required maximum value $(y)=2x^3-15x^2+36x+12$

$=2(3)^3-15(3)^2+36(3)+12 \quad =54-135+108+12 \quad =39$

Answers: Max. value = 40 at $x=2$, and Min. value=39 at $x=3$

Illustration-48: Prove that the curve given by $3y=x^3-3x^2-9x+11$, has a maximum at $x=-1$, minimum at $x=3$, and point of inflexion when $x=1$.

Solution: (a) Let $3y=x^3-3x^2-9x+11$

$$\Rightarrow y=\frac{x^3-3x^2-9x+11}{3} \Rightarrow y=\frac{x^3}{3}-x^2-3x+\frac{11}{3}$$

$$\therefore \frac{dy}{dx}=x^2-2x-3 \therefore \frac{d^2y}{dx^2}=2x-2$$

For maximum or minimum,

$\frac{dy}{dx}=0 \Rightarrow x^2-2x-3=0 \quad \Rightarrow x^2-3x+x-3=0 \Rightarrow x(x-3)+(x-3)=0$

$\Rightarrow (x-3)(x+1)=0 \Rightarrow x-3=0$ *or* $x+1=0 \quad \Rightarrow x=3$ *or* $x=-1$

When $x=-1$, then

$\frac{d^2y}{dx^2}=2x-2=2(-1)-2=-4$, which is negative. So the function is maximum at $x=-1$. [Proved]

When $x=3$, then

$\frac{d^2y}{dx^2}=2x-2=2(3)-2=4$, which is positive. So the function is minimum at $x=3$. [Proved]

When $x=1$, then

$\frac{d^2y}{dx^2}=2x-2 \quad =2(1)-2=0$, which is neither positive nor negative.

Here $\frac{d^3y}{dx^3}=2$, which is not zero. So it has neither a maximum nor a minimum value at $x=1$.

It is a point of inflexion. [Proved]

12.14 BUSINESS PROBLEM SOLUTION

Problem-1: The cost c of manufacturing a certain article is given by the formula

$$C = 5 + \frac{48}{x} + 3x^2$$

Solution: Given $C = 5 + 48x^{-1} + 3x^2$

$$\therefore \frac{dC}{dx} = -48x^{-2} + 6x$$

$$\therefore \frac{d^2y}{dx^2} = 96x^{-3} + 6$$

For maximum or minimum,

$$\frac{dC}{dx} = 0 \qquad \Rightarrow -48x^{-2} + 6x = 0 \qquad \Rightarrow -\frac{48}{x^2} + 6x = 0$$

$$\Rightarrow \frac{-48 + 6x^3}{x^2} = 0 \qquad \Rightarrow -48 + 6x^3 = 0 \qquad \Rightarrow 6x^3 = 48$$

$$\Rightarrow x^3 = 8 \qquad \Rightarrow x^3 = 2^3 \qquad \Rightarrow x = 2$$

When $x = 2$, then

$$\frac{d^2y}{dx^2} = 96x^{-3} + 6 \qquad = \frac{96}{x^3} + 6 \qquad = \frac{96}{(2)^3} + 6$$

$= 18$, which is positive. So the function is minimum at $x = 2$.

So the required minimum cost, $C_{min} = 5 + \frac{48}{x} + 3x^2$

$$= 5 + \frac{48}{2} + 3(2)^2 = 5 + 24 + 12 = 41$$

Answers: Minimum value of C is , $C_{min} = 41$

Problem-2: A company finds that it can sell out a certain product that it produces, at the rate of Tk. 2 per unit. It estimates the cost function of the product to be

Tk. $\left[1000 + \frac{1}{2}.\left(\frac{q}{50}\right)^2\right]$ for q units produced.

(i) Find the expression for the total profit, if q units are produced and sold.
(ii) Find the number of units produced that will maximize profit.
(iii) What is the amount of this maximum profit?
(iv) What would be the profit if 6000 units are produed?

Solution:

(i) Given price per unit $(p) = 2$

cost function $(C)=\left[1000+\frac{1}{2}.\left(\frac{q}{50}\right)^2\right]=1000+\frac{1}{2}.\left(\frac{q^2}{2500}\right)\quad=1000+\frac{q^2}{5000}$

We know, Revenue (R) = price × quantity

$$=p\times q$$

$$=2q$$

$\therefore \text{Profit }(y)=\text{Revenue}(R)-\text{Cost}(C)$

$$\Rightarrow y=R-C \quad \Rightarrow y=2q-\left(1000+\frac{q^2}{5000}\right)$$

$\Rightarrow y=2q-1000-\frac{q^2}{5000}$, which is the required expression of total profit. [Ans.]

(ii) We have profit $(y)=2q-1000-\frac{q^2}{5000}$

$$\therefore \frac{dy}{dq}=2-\frac{2q}{5000}$$

$$\therefore \frac{d^2y}{dq^2}=\frac{-2}{5000}$$

For maximum or minimum,

$$\frac{dy}{dq}=0\Rightarrow 2-\frac{2q}{5000}=0\Rightarrow 2q=10000\Rightarrow q=5000$$

When $x=5000$, then

$\frac{d^2y}{dq^2}=-\frac{2}{5000}$, which is negative. So the profit function is maximum at $x=5000$.

Answers: Required units $(q)=5000$ units.

(iii) When $q=5000$, then

$$\therefore \text{Required maximum profit}\,(y)=2q-1000-\frac{q^2}{5000}=2(5000)-1000-\frac{(5000)^2}{5000}$$

$$=10000-1000-5000=Tk.4000$$

Answers: Maximum profit $=Tk.4000$.

(iv) When $q=6000$, then

$$\therefore \text{Required profit}\,(y)=2q-1000-\frac{q^2}{5000}=2(6000)-1000-\frac{(6000)^2}{5000}$$

$=12000-1000-7200=Tk.3800$ [Ans.]

Answers: Required profit $=Tk.3800$.

Problem-3: By an 'economic lot size', we mean a lot size (x) which minimizes the total cost (T) incurred in obtaining and storing material for a certain period to fulfill a given rate of demand for the material during the time period.
The material demanded is 10000 units per year, the cost of material is Tk. 2 per unit, the cost of replenishing the stock of material per order, regardless of the seze order (x), is Tk. 40 per order, and the cost of storing material is 10 percent per year on the taka value of the average inventory $\left(\frac{x}{2}\right)$ on hand.

(i) Show that $T = 20000 + \frac{400000}{x} + \frac{x}{10}$,

(ii) Find the economic lot size.

Solution: (i) We know,

Total Cost (T) = Fixed Cost (FC) + Variable Cost (VC)

$\Rightarrow T = FC + VC$ $\Rightarrow T = FC + \text{Odering Cost}(OC) + \text{Storing Cost}(SC)$

$[\because VC = Ordering\ Cost + Storing\ Cost]$

$\Rightarrow T = (2\times10000) + \frac{40\times10000}{x} + 2\times10\%\times\left(\frac{x}{2}\right)$ $\Rightarrow T = 20000 + \frac{400000}{x} + \frac{10}{100}\times x$

$\Rightarrow T = 20000 + \frac{400000}{x} + \frac{x}{10}$ [Shown]

(ii) We know, economic lot size = number of units at minimum cost

Here $T = 20000 + \frac{400000}{x} + \frac{x}{10}$ $= 20000 + 400000x^{-1} + \frac{x}{10}$

$\therefore \frac{dT}{dx} = -400000x^{-2} + \frac{1}{10}$ $\therefore \frac{d^2T}{dx} = 800000x^{-3}$

For maximum or minimum,

$\frac{dT}{dx} = 0$ $\Rightarrow -400000x^{-2} + \frac{1}{10} = 0$ $\Rightarrow -\frac{400000}{x^2} + \frac{1}{10} = 0$ $\Rightarrow \frac{1}{10} = \frac{400000}{x^2}$

$\Rightarrow x^2 = 4000000$ $\Rightarrow x^2 = (2000)^2$ $\Rightarrow x = 2000$

When $x = 2000$, then

$\frac{d^2T}{dx^2} = 800000x^{-3} = \frac{800000}{x^3} = \frac{800000}{(2000)^3}$

$= 0.0001$, which is positive. So the cost function is minimum at $x = 2000$.

$\therefore$ Required economic lot size =2000 units. [Ans.]

Problem-4: A firm has to produce 144000 units of an item per year. It costs Tk. 160 to make the factory ready for a product run of the item regardless of units x produced in a run. The cost of material is Tk. 5 per unit and the cost of storing the material is 50 paise per item per year on the

average inventory $\left(\frac{x}{2}\right)$ in hand. Show that the total cost C is given by

$$C = 720000 + \frac{23040000}{x} + \frac{x}{4}.$$

Find also the economic lot size, i.e., value of x for which C is minimum.

Solution: Part-I

We know,

Total Cost (C) = Fixed Cost (FC) + Variable Cost (VC)

$\Rightarrow C = FC + VC \qquad \Rightarrow C = FC + OC + SC$

$$\Rightarrow C = (5\times144000) + \frac{160\times144000}{x} + 0.50\times\left(\frac{x}{2}\right)$$

$$\Rightarrow C = 720000 + \frac{23040000}{x} + \frac{1}{2}\times\frac{x}{2}$$

$$\Rightarrow C = 720000 + \frac{23040000}{x} + \frac{x}{4} \text{ [Shown]}$$

Part II

Here $C = 720000 + \frac{23040000}{x} + \frac{x}{4}$

$= 720000 + 23040000x^{-1} + \frac{x}{4} \quad \therefore \frac{dC}{dx} = -23040000x^{-2} + \frac{1}{4} \quad \therefore \frac{d^2C}{dx^2} = 46080000x^{-3}$

For maximum or minimum, $\frac{dC}{dx} = 0 \quad \Rightarrow -23040000x^{-2} + \frac{1}{4} = 0 \quad \Rightarrow -\frac{23040000}{x^2} + \frac{1}{4} = 0$

$$\Rightarrow \frac{1}{4} = \frac{23040000}{x^2} \Rightarrow x^2 = 92160000 \Rightarrow x^2 = (9600)^2 \Rightarrow x = 9600$$

When $x = 9600$, then

$$\frac{d^2C}{dx^2} = 46080000x^{-3} = \frac{46080000}{x^3} = \frac{46080000}{(9600)^3}$$

$= 0.000052$, which is positive. So the cost function is minimum at $x = 9600$.

$\therefore$ Required economic lot size = 9600 units. [Ans.]

Problem-5: A company has to manufacture 36000 units of an item per year. It costs Tk. 250 to make the factory ready for a product run of the item regardless of units x produced in a run. The cost of material per unit made is Tk. 5 and it costs 50 paise per year for each unit for storing on

the average inventory $\left(\frac{x}{2}\right)$ in hand. Show that the total cost C is given by

$$C = \frac{250 \times 36000}{x} + 180000 + \frac{x}{4}.$$

Find also the economic lot size, i.e., value of x for which C is minimum.

Part I

Solution: We know,

$\text{Total Cost}(C) = \text{Fixed Cost}(FC) + \text{Variable Cost}(VC)$

$\Rightarrow C = FC + VC \qquad \Rightarrow C = FC + OC + SC$

$\Rightarrow C = (5 \times 36000) + \frac{250 \times 36000}{x} + 0.50 \times \left(\frac{x}{2}\right) \quad \Rightarrow C = 180000 + \frac{250 \times 36000}{x} + \frac{1}{2} \times \frac{x}{2}$

$\Rightarrow C = \frac{250 \times 36000}{x} + 180000 + \frac{x}{4}$ [Shown]

Part II

Here $C = \frac{250 \times 36000}{x} + 180000 + \frac{x}{4} \qquad = 9000000x^{-1} + 180000 + \frac{x}{4}$

$\therefore \frac{dC}{dx} = -9000000x^{-2} + \frac{1}{4} \qquad \therefore \frac{d^2C}{dx^2} = 18000000x^{-3}$

For maximum or minimum,

$\frac{dC}{dx} = 0 \Rightarrow -9000000x^{-2} + \frac{1}{4} = 0 \qquad \Rightarrow -\frac{9000000}{x^2} + \frac{1}{4} = 0$

$\Rightarrow \frac{1}{4} = \frac{9000000}{x^2} \Rightarrow x^2 = 36000000 \Rightarrow x^2 = (6000)^2$

$\Rightarrow x = 6000$

When $x = 6000$, then

$\frac{d^2C}{dx^2} = 18000000x^{-3} = \frac{18000000}{x^3} = \frac{18000000}{(6000)^3}$

$= 0.000083$, which is positive. So the cost function is minimum at $x = 6000$.

$\therefore$ Required economic lot size $= 6000$ units. [Ans.]

Problem-6: A company notices that higher sales of a particular item which it produces are achieved by lowering the price charged. As a result the total revenue from the sales at first rises as the number of units sold increases, reaches the highest point and then falls off. This pattern of total revenue is described by the relation

$y = 4000000 - (x - 2000)^2$

where y is the total revenue and x the number of units sold.

(i) Find, what number of units sold maximizes total revenue?
(ii) What is the amount of this maximum revenue?
(iii) What would be the total revenue if 2500 units were sold?

Solution: (i) Given $y = 4000000 - (x - 2000)^2$

$\Rightarrow y = 4000000 - (x^2 - 4000x + 4000000)$ $\Rightarrow y = 4000000 - x^2 + 4000x - 4000000$

$\Rightarrow y = -x^2 + 4000x$ $\therefore \frac{dy}{dx} = -2x + 4000$ $\therefore \frac{d^2y}{dx^2} = -2$

For maximum or minimum,

$\frac{dy}{dx} = 0$ $\Rightarrow -2x + 4000 = 0$ $\Rightarrow -2x + 4000 = 0$ $\Rightarrow 2x = 4000$ $\Rightarrow x = 2000$

When $x = 2000$, then

$\frac{d^2y}{dx^2} = -2$, which is negative. So the revenue function is maximum at $x = 2000$.

$\therefore$ Required revenue maximizing units $(x) = 2000$ units. [Ans.]

(ii) When $x = 2000$, then

$\therefore$ Required maximum revenue $(y) = 4000000 - (x - 2000)^2$

$= 4000000 - (2000 - 2000)^2$ $= Tk.4000000$ [Ans.]

(iii) When $x = 2500$ *then*

$\therefore$ Re*quired revenue* $(y) = 4000000 - (x - 2000)^2$ $= 4000000 - (2500 - 2000)^2$

$= 4000000 - 250000$ $= Tk.3750000\,[Ans.]$

Problem-7: A sitar manufacturer notices that he can sell x sitars per week at p taka each where $5x = 375 - 3p$. The cost of production is $\left(500 + 13x + \frac{1}{5}x^2\right)$ taka. Show that the maximum profit is obtained when the production is 30 sitars per week.

Solution: Given $C = 500 + 13x + \frac{1}{5}x^2$

Here $5x = 375 - 3p$ $\Rightarrow 3p = 375 - 5x$ $\Rightarrow p = \frac{375 - 5x}{3}$

$\Rightarrow p = \frac{375}{3} - \frac{5x}{3}$

We know, Revenue = price × quantity

$\Rightarrow R = p \times x$ $\Rightarrow R = \left(\frac{375}{3} - \frac{5x}{3}\right) \times x$ $\Rightarrow R = \frac{375x}{3} - \frac{5x^2}{3}$

$\therefore \text{Profit}(y) = \text{Revenue}(R) - \text{Cost}(C)$

$\Rightarrow y = R - C \quad \Rightarrow y = \left(\frac{375x}{3} - \frac{5x^2}{3}\right) - \left(500 + 13x + \frac{1}{5}x^2\right)$

$\Rightarrow y = \frac{375x}{3} - \frac{5x^2}{3} - 500 - 13x - \frac{1}{5}x^2 \quad \therefore \frac{dy}{dx} = \frac{375}{3} - \frac{10x}{3} - 13 - \frac{2}{5}x$

$\therefore \frac{d^2y}{dx^2} = -\frac{10}{3} - \frac{2}{5}$

For maximum or minimum,

$\frac{dy}{dx} = 0 \Rightarrow \frac{375}{3} - \frac{10x}{3} - 13 - \frac{2}{5}x = 0 \Rightarrow -\frac{10x}{3} - \frac{2}{5}x = 13 - \frac{375}{3}$

$\Rightarrow -\left(\frac{10x}{3} + \frac{2x}{5}\right) = \frac{39 - 375}{3} \Rightarrow -\left(\frac{50x + 6x}{15}\right) = -\frac{336}{3} \Rightarrow \frac{56x}{15} = \frac{336}{3}$

$\Rightarrow \frac{x}{5} = \frac{6}{1} \Rightarrow x = 30$

When $x = 30$, then

$\frac{d^2y}{dx^2} = -\frac{10}{3} - \frac{2}{5} = -\frac{56}{15}$, Which is negative. So the profit function is maximum at $x = 30$.

Problem-8: If the demand function of the monopolist is $3q = 98 - 4p$ and average cost is $3q + 2$, where q is output and p is the price, find maximum profit of the monopolist.

Solution: Given Average Cost $(AC) = 3q + 2$

$\therefore$ Total Cost $(TC) = AC \times q$

$\Rightarrow TC = (3q + 2) \times q \quad \Rightarrow TC = 3q^2 + 2q$

$\therefore Marginal\ Cost\ (MC) = \frac{d}{dq}(TC)$

$\Rightarrow MC = \frac{d}{dq}(3q^2 + 2q) \quad \Rightarrow MC = 6q + 2$

Here $3q = 98 - 4p \quad \Rightarrow 4p = 98 - 3q \Rightarrow p = \frac{98 - 3q}{4} \Rightarrow p = \frac{98}{4} - \frac{3q}{4}$

We know, Total Revenue $(TR) = \text{price} \times \text{quantity}$

$\Rightarrow TR = p \times q \quad \Rightarrow TR = \left(\frac{98}{4} - \frac{3q}{4}\right) \times q \quad \Rightarrow TR = \frac{98q}{4} - \frac{3q^2}{4}$

$\therefore Marginal\ Revenue\ (MR) = \frac{d}{dq}(TR)$

$$\therefore MR = \frac{d}{dq}\left(\frac{98q}{4} - \frac{3q^2}{4}\right) \Rightarrow MR = \frac{98}{4} - \frac{6q}{4}$$

Under monopoly business, profit will be maximum at $MC = MR$.

$\therefore MC = MR$

$$\Rightarrow 6q+2 = \frac{98}{4} - \frac{6q}{4} \quad \Rightarrow 6q+2 = \frac{98-6q}{4} \quad \Rightarrow 24q+8 = 98-6q \Rightarrow 30q = 90 \Rightarrow q = 3$$

So the maximum profit of the monopolist will be obtained at $q = 3$.

When $q = 3$, then

Required maximum profit $(y) = TR - TC$

$$= \left(\frac{98q}{4} - \frac{3q^2}{4}\right) - \left(3q^2 + 2q\right) \quad = \frac{98q}{4} - \frac{3q^2}{4} - 3q^2 - 2q = \frac{98(3)}{4} - \frac{3(3)^2}{4} - 3(3)^2 - 2(3)$$

$$= \frac{294}{4} - \frac{27}{4} - 27 - 6 \; = 73.5 - 6.75 - 27 - 6 \; = Tk.33.75 \text{ [Ans.]}$$

Problem-9: Find the profit maximizing output given the following revenue and cost functions: $R(q) = 1000q - 2q^2 \;\; C(q) = q^3 - 59q^2 + 1315q + 2000.$

Also find the maximum profit.

Solution: We have, Profit, $P = R(q) - C(q)$

$$= \left(1000q - 2q^2\right) - \left(q^3 - 59q^2 + 1315q + 2000\right) = -q^3 + 57q^2 - 315q - 2000$$

First order condition: $\frac{dP}{dq} = 0$

$$\therefore \frac{dP}{dq} = \frac{d}{dq}\left(-q^3 + 57q^2 - 315q - 2000\right) = -3q^2 + 114q - 315$$

Now $\frac{dP}{dq} = 0 \Rightarrow -3q^2 + 114q - 315 = 0$

$$\Rightarrow q^2 - 38q + 105 = 0 \;\; \Rightarrow (q-3)(q-35) = 0 \;\; \therefore q = 3 \text{ or, } q = 35$$

Second order condition: $\frac{d^2P}{dq^2} = \frac{d}{dq}\left(-3q^2 + 114q - 315\right) = -6q + 114$

$$\therefore \left.\frac{d^2P}{dq^2}\right|_{q=3} = -6\times 3 + 114 = 96 > 0, \qquad \left.\frac{d^2P}{dq^2}\right|_{q=35} = -6\times 35 + 114 = -96 < 0$$

Hence the profit maximizing output is given by $q = 35$. [Ans.]

Maximum profit $= -(35)^3 + 57\times(35)^2 - 315\times 35 - 2000$

$= -42875 + 69825 - 11025 - 2000 = Tk.13925$ [Ans.]

Problem-10: A manufacturer produces x sets radio per week at a total cost of Tk. $\left(x^2+78x+2500\right)$. He is a monopolist and the demand function for his product is $x=\frac{600-p}{8}$ when the price is Tk. p per set. show that maximum net revenue i.e., profit is obtained when 29 sets are produced per week. What is the monopoly price?

Solution: Total cost $(TC)=x^2+78x+2500$

Marginal Cost $(MC)=\frac{dC}{dx}=\frac{d}{dx}\left(x^2+78x+2500\right)=2x+78$

Demand function is $x=\frac{600-p}{8}\Rightarrow 8x=600-p\Rightarrow p=600-8x$

Now total revenue for x sets, $R=p\times x=(600-8x)x=600x-8x^2$

Marginal revenue $(MR)=\frac{dR}{dx}=\frac{d}{dx}\left(600x-8x^2\right)=600-16x$

Net revenue will be maximum at the level of output, where $MR=MC$.

$$\therefore 2x+78=600-16x\Rightarrow 18x=600-78=522\Rightarrow x=\frac{522}{18}=29$$

Hence in order to maximize his profit, the manufacturer should manufacture 29 sets radio per week.

Also the monopoly price is given by $p=600-8x=600-8\times 29=Tk.368$. [Ans.]

Problem-11: The total revenue function of a firm is given as $R=21q-q^2$ and its total cost function as $C=\frac{1}{3}q^3-3q^2-7q+16$, where q is the output.

Find (i) the output at which the total revenue is maximum, and
(ii) the output at which the total cost is minimum.

Solution: (i) $R=21q-q^2$

Differentiating R w.r. to q, we have

$$\frac{dR}{dq}=\frac{d}{dq}\left(21q-q^2\right)=21-2q$$

Put $\frac{dR}{dq}=0\Rightarrow 21-2q=0\Rightarrow q=\frac{21}{2}=10.5$ is the necessary condition for revenue maximization.

To get the sufficient condition, we have $\frac{d^2R}{dq^2}=\frac{d}{dq}(21-2q)=-2$, a negative quantity, which means the revenue is maximum at $q=10.5$ [Ans.]

(ii) $C=\frac{1}{3}q^3-3q^2-7q+16$

Differentiating C. w.r. to q, we have

$$\frac{dC}{dq}=\frac{d}{dq}\left(\frac{1}{3}q^3-3q^2-7q+16\right)=\frac{1}{3}\times 3q^2-3\times 2q-7=q^2-6q-7$$

Now put $\frac{dC}{dq}=0\Rightarrow q^2-6q-7=0\Rightarrow (q-7)(q+1)=0$

$\therefore q=7$ or, $q=-1$ is the necessary condition for cost maximization or minimization. But $q=-1$ is not admissible as output cannot be negative.

To get the sufficient condition, we have $\frac{d^2C}{dq^2}=\frac{d}{dq}\left(q^2-6q-7\right)=2q-6$

$\left.\frac{d^2C}{dq^2}\right|_{q=7}=2\times 7-6=8$, a positive quantity which means that cost is minimum at $q=7$. [Ans.]

Problem-12: The demand function of a consumer good is $x=\frac{1}{3}(25-2p)$, where x is the number of units and p is the price. The average cost per unit is Tk. 40. Find:

(a) the revenue function R in terms of price p,
(b) the cost function C,
(c) the profit function P,
(d) the price per unit that maximizes the profit function, and
(e) the maximum profit.

Solution: (a) $R(p)=xp=\frac{1}{3}(25-2p)p=\frac{1}{3}\left(25p-2p^2\right)$

(b) $C(p)=40x=40\times\frac{1}{3}(25-2p)=\frac{40}{3}(25-2p)$

(c) $P(p)=R(p)-C(p)$

$$=\frac{1}{3}\left(25p-2p^2\right)-\frac{40}{3}(25-2p)=\frac{25p}{3}-\frac{2p^2}{3}-\frac{1000}{3}+\frac{80p}{3}$$

$$=\frac{1}{3}\left[-2p^2+105p-1000\right].....(i)$$

$$\frac{dP}{dp}=\frac{1}{3}\frac{d}{dp}\left(-2p^2+105p-1000\right)=\frac{1}{3}(-4p+105)$$

Solving the equation $\frac{dP}{dp}=0$, we have

$$\frac{1}{3}(-4p+105)=0\Rightarrow 4p=105\Rightarrow p=\frac{105}{4}=26.25$$

Using second derivative test, we have $\frac{d^2P}{dp^2}=\frac{1}{3}\frac{d}{dp}(-4p+105)=-\frac{4}{3}<0$

$\therefore$ Maximum profit is found when $p=26.25$ [Ans.]

(d) Maximum profit is [by the help of (1)]

$$P(p)=\frac{1}{3}\left[-2(26.25)^2+105(26.25)-1000\right]=126.04 \text{ [Ans.]}$$

Porblem-13: The demand function faced by a firm is $p=500-0.2x$ and its cost function is $C=25x+10{,}000$ where p = price, x = output and C = cost. Find the output at which the profit of the firm is maximum. Also find the price it will charge.

Solution: Revenue, $R(x)=p\times x=x(500-0.2x)=500x-0.2x^2$

Profit = Revenue-Cost

$$\therefore P(x)=R(x)-C(x)=500x-0.2x^2-(25x+10{,}000)$$
$$=475x-10{,}000-0.2x^2$$

For maximum or minimum:

$$\frac{dP}{dx}=\frac{d}{dx}\left(475x-10{,}000-0.2x^2\right)=475-0.2\times 2x=475-0.4x$$

$$\text{Put } \frac{dP}{dx}=0\Rightarrow 475-0.4x=0\therefore x=\frac{475}{0.4}=1187.50$$

$$\text{Also } \frac{d^2P}{dx^2}=\frac{d}{dx}(475-0.4x)=-0.4<0.$$

Hence the profit is maximum when the output, $x=1187.50$

At this level, the price is given by $p=500-0.2x=500-0.2(1187.50)=262.50$ [Ans.]

Problem-14: ABC Co. Ltd. is planning to market a new model of shaving razor. Rather than the set, the selling price of the razor based only on production cost estimates, management polls the retailers of the razors to see how many razors they would buy for various prices. From this survey, it is determined that the unit demand function (the relationship between the amount x each retailer would buy and the price p he is willing to pay) is

$$x=-1500p+30{,}000$$

The fixed costs to the company for production of the razors are found to be Tk. 28,000 and the cost for material and labor to produce each razor is estimated to be Tk. 8.00 per unit. What price should the company charge the retailers in orders to obtain a maximum profit?

Solution: Let x denotes the number of unit produced, C denotes the cost of production to the company, and p denotes the price per unit (in Taka).

Then the cost C is given by and the unit demand is $x=-1500p+30{,}000$

Substituting, we find that the cost function C (x) in terms of the price p per unit is

$$C(p) = 8\times(-1500p + 30{,}000) + 28{,}000$$
$$= 12{,}000p + 2{,}68{,}000.$$

The money derived from the sales of the shaving razors as a function of the price p per unit is the product of the number sold by the price per unit, i.e., the revenue function $R(x)$ is $R = (-1500p + 30{,}000)\times p$

$$= -15{,}00p^2 + 30{,}000p.$$

The profit P to the company is the difference between revenue (money derived from sales) and total cost, i.e., the profit function is:

$$P = R - C$$
$$= (-1500p^2 + 30{,}000p) - (12{,}000p + 2{,}68{,}000)$$
$$= -1500p^2 + 42{,}000p - 2{,}68{,}000.$$

The derivative of $P(x)$ is: $\frac{dP}{dp} = P'(p) = -3000p + 42000.$

Solving the equation, $P'(p) = 0$, we have $-3000p + 42000 = 0 \Rightarrow p = 14.$

Using second derivative test, we have $\frac{d^2P}{dp^2} = \frac{d}{dp}(-3000p + 42000) = -3000 < 0$

$\therefore$ Profit is Maximum when $p = 14$.

The profit for this price is $p = -1{,}500(14)^2 + 42{,}000\times 14 - 2{,}68{,}000 = Tk.26{,}000.$

The number of units sold at this price p is

$$x = (-1500)(14) + 30{,}000 = 9{,}000.$$ [Ans.]

Problem-15: A company making a single product has manufacturing and distribution divisions. Stock of finished goods is not held, all production being to order. The average net revenue per unit allowing for quantity discounts is Tk. $(100 - 0.01q)$ where q is the quantity sold. The average variable costs per unit for the two divisions are:

Manufacturing $= 10 + 0.015q$

Distribution $= 2 + 0.001q.$

The fixed costs per annual are: Manufacturing = Tk.40,000

Distribution = Tk.20,000

CALCULATE:

(a) the optimum annual production quantity to maximize the profit of the company,

(b) the profit of the company at the level of activity in (a) above,

(c) the annual production quantity to maximize the manufacturing division's profit if it has been instructed to transfer the product to the distribution division at Tk. 73 per unit.

(d) the profit of the company, showing the results of the two divisions at the level of activity in (c) above.

Solution: (a) Profit = Revenue – Cost = Revenue – (Variable costs + Fixed costs)

$$\therefore P = q(100-0.01q)-q(10+0.015q)-q(2+0.001q)-(40{,}000+20{,}000)$$

$$\Rightarrow P = 88q-0.026q^2-60{,}000.$$

For maximization, we have

$$\frac{dP}{dq}=\frac{d}{dq}\left(88q-0.026q^2-60{,}000\right)=88-0.052q$$

Now $\frac{dP}{dq}=0 \Rightarrow 88-0.052q=0 \Rightarrow q=1692$ units.

Also $\frac{d^2P}{dq^2}=\frac{d}{dq}(88-0.052q)=-0.052<0$

$\therefore$ Profit maximization output, $q=1692$ units.

(b) When $q=1692$, Profit $=88(1692)-0.026(1692)^2-60{,}000 = Tk.14{,}461.54$

(c) If the manufacturing division is to transfer out at Tk. 73 per unit, we can express their profit as Profit $=73\times$(Production quantity) – Manufacturing variable costs – Manufacturing fixed costs

$$\therefore P = 73q-q(10+0.015q)-40{,}000$$

$$\Rightarrow P = 63q-0.015q^2-40{,}000.$$

For maximization,

$$\frac{dP}{dq}=\frac{d}{dq}\left(63q-0.015q^2-40{,}000\right)=63-0.03q$$

Now $\frac{dP}{dq}=0 \Rightarrow 63-0.03q=0 \Rightarrow q=2100$ units

Also $\frac{dP^2}{dq^2}=\frac{d}{dq}(63-0.03q)=-0.03<0.$

$\therefore$ Annual production quantity, $q=2100$ units.

(d) Company profit (part 'a') $=88q-0.026q^2-60{,}000$.

If $q=2100$, Pr$ofit = 88(2100)-0.026(2100)^2-60{,}000 = Tk.10{,}140.$

Manufacturing Profit (part 'c') $=63q-0.015q^2-40{,}000$

If $q=2100$, Pr$ofit = 63(2100)-0.015(2100)^2-40{,}000 = Tk.26{,}150.$

Distribution profit = Revenue – Transfer cost and Department cost

$$= q(100-0.01q)-\{73q+q(2+0.001q)+20{,}000\}$$

$$= 100q-0.01q^2-73q-2q-0.001q^2-20{,}000$$

$$= -0.011q^2+25q-20{,}000$$

If $q = 2100$, $\text{Pr}ofit = -0.011(2100)^2 + 25(2{,}100) - 20{,}000 = -Tk.16{,}010.$
This shows that there is a loss of Tk. 16,010 from the distribution unit. [Ans.]

Problem-16: The production function of a commodity is given by

$$Q = 40F + 3F^2 - \frac{F^3}{3}$$

where Q is the total output and F is the units of input.
(i) Find the number of units of input required to give maximum output.
(ii) Find the maximum value of marginal product.
(iii) Verify that when the average product is maximum, it is equal to marginal product.

Solution: (i) $\frac{dQ}{dF} = \frac{d}{dF}\left(40F + 3F^2 - \frac{F^3}{3}\right) = 40 + 6F - \frac{3F^2}{3} = 40 + 6F - F^2$

For maximum or minimum (First order condition): $\frac{dQ}{dF} = 0$

$$40 + 6F - F^2 = 0 \quad \Rightarrow (F - 10)(F + 4) = 0 \quad \Rightarrow F = 10 \text{ or } F = -4$$

But, $F = -4$ is not admissible as input cannot be negative. Hence, $F = 10$.

Now $\frac{d^2Q}{dF^2} = \frac{d}{dF}(40 + 6F - F^2) = 6 - 2F$ (second order condition)

$$\frac{d^2Q}{dF^2}\Big|_{F=10} = 6 - 2\times 10 = -14 < 0.$$

Hence output is maximum when 4 units of input are used.

(ii) Marginal Product $MP = \frac{dQ}{dF} = 40 + 6F - F^2$

For maximum or minimum: $\frac{d(MP)}{dF} = 0$

Now $\frac{d(MP)}{dF} = \frac{d}{dF}(40 + 6F - F^2) = 6 - 2F \quad \therefore 6 - 2F = 0 \Rightarrow f = \frac{6}{2} = 3$

Also $\frac{d^2(MP)}{dF^2} = \frac{d}{dF}(6 - 2F) = -2 < 0.$

Hence value of marginal product is maximum when input is 3 units.
$\therefore$ Value of marginal product $= 40 + 6\times 3 - 3^2 = 49$ units.

(iii) Average product $(AP) = \frac{Q}{F} = \frac{40F + 3F^2 - \frac{1}{3}F^3}{F} = 40 + 3F - \frac{F^2}{3}$

For maximum or minimum: $\frac{d(AP)}{dF} = 0$

$$\frac{d(AP)}{dF}=\frac{d}{dF}\left(40+3F-\frac{F^2}{3}\right)=3-\frac{2F}{3}$$

Now $\Rightarrow \frac{d(AP)}{dF}=0\Rightarrow 3-\frac{2F}{3}=0\Rightarrow 2F=9\Rightarrow F=\frac{9}{2}=4.5$

Also $\frac{d^2(AP)}{dF^2}=\frac{d}{dF}\left(3-\frac{2F}{3}\right)=-\frac{2}{3}<0.$

$\therefore$ Average product is maximum when $F=4.5$

Average product $(when\ F=4.5)=40+3\times4.5-\frac{(4.5)^2}{3}=46.75$

Marginal product (when AP is maximum, i.e., $F=4.5)=40+27-\frac{81}{4}=\frac{187}{4}=46.75$ [Proved]

Problem-17: The total cost function of a firm is $C=\frac{1}{3}x^3-5x^2+28x+10$, where C is total cost and x is output. A tax at the rate of Tk. 2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by $p=2530-5x$, where Tk. p is the price per unit of output, find the profit maximizing output and price.

Solution: Total revenue function, $TR=(2530-5x)\times x=2530x-5x^2$

After imposition of a tax of Tk. 2 per unit, $TC=\frac{1}{3}x^3-5x^2+28x+10+2x$

$$=\frac{1}{3}x^3-5x^2+30x+10$$

Now Profit, $P=TR-TC=\left(2530x-5x^2\right)-\left(\frac{1}{3}x^3-5x^2+30x+10\right)$

For maximization, $\frac{dP}{dx}=\frac{d}{dx}\left[\left(2530x-5x^2\right)-\left(\frac{1}{3}x^3-5x^2+30x+10\right)\right]$

$$\Rightarrow \frac{dP}{dx}=(2530-10x)-\left(x^2-10x+30\right)=-x^2+2500$$

Now $\frac{dP}{dx}=0\Rightarrow -x^2+2500=0\Rightarrow x^2=2500\Rightarrow x=\pm50$

But $x=-50$ is not admissible as output can not be negative. $\therefore x=50.$

Now $\frac{d^2P}{dx^2}=\frac{d}{dx}\left(-x^2+2500\right)=-2x=-2\times50=-100<0.$

$\therefore$ Profit maximizing output is 50 units.

$\therefore$ When $x=50$, $price\ p=2530-5\times50=Tk.2280.$

Problem-18: Suppose the demand and total cost functions of a monopolist are $p = 20 - 4x \; and \; TC = 4x + 2$ respectively where p is price and x is quantity. If the government imposes tax at the rate of 20% of sales, determine the total tax revenue that the government will be able to collect.

Solution: Given $p = 20 - 4x \; and \; TC = 4x + 2$

$\therefore$ Total Revenue $= TR = px = (equal \;\; sign)\,(20 - 4x)x = 20x - 4x^2$

Tax $= 20\%$ of $TR = 20\% \times TR = \frac{1}{5}(20x - 4x^2)$

Total new cost $= TC + Tax = 4x + 2 + \frac{1}{5}(20x - 4x^2)$

$= 4x + 2 + 4x - \frac{4}{5}x^2 = -\frac{4}{5}x^2 + 8x + 2$

Now, profit P = Total Revenue – Total new cost

$= (20x - 4x^2) - \left(-\frac{4}{5}x^2 + 8x + 2\right) \quad = \frac{-16}{5}x^2 + 12x - 2$

$\therefore \frac{dP}{dx} = \frac{d}{dx}\left(-\frac{16}{5}x^2 + 12x - 2\right) = -\frac{32}{5}x + 12$

Now $\frac{dP}{dx} = 0$ gives $-\frac{32}{5}x + 12 = 0 \Rightarrow x = \frac{15}{8}$

Again, $\frac{d^2P}{dx^2} = \frac{d}{dx}\left(-\frac{32}{5}x + 12\right) = -\frac{32}{5} < 0$

$\therefore$ When $x = \frac{15}{8}$, tax is given by $\frac{1}{5}\left[20 \times \frac{15}{8} - 4 \times \left(\frac{15}{8}\right)^2\right] = Tk.\frac{75}{16}$

Hence the government will be able to collect Tk. $\frac{75}{16}$ as tax revenue. [Ans.]

Problem-19: The demand and cost functions of a company is given by:

$p = 20 - 4x$

$C = 4x$

(a) Find the optimum quantity, price and the profit on this level.
(b) What will be the new equilibrium after a tax of Tk. 0.50 is imposed?
(c) Determine the tax rate that will maximize tax revenue and determine that tax revenue.
(d) Find the total tax revenue if in addition 10% sales tax is also imposed.

Solution: (a) TR = price $\times$ quantity $= (20 - 4x)x = 20x - 4x^2$

$$MR = \frac{dR}{dx} = \frac{d}{dx}(20x - 4x^2) = 20 - 8x$$

$$\text{Cost, } C = 4x, MC = \frac{d}{dx}(4x) = 4$$

For optimum level, MR = MC

$$\Rightarrow 20 - 8x = 4, i.e., x = 2$$

(b) After Tax, $C = 4x + 0.5x = 4.50x$

$$MC = \frac{d}{dx}(4.50x) = 4.5$$

At the optimum level, $MC = MR$

$$\Rightarrow 20 - 8x = 4.5, i.e., x = \frac{31}{16} = 1.94$$

$$\therefore \text{Price, } p = 20 - 4 \times 1.94 = Tk.12.25$$

(c) Tax revenue is maximum where

$MR = MC$ (after tax t)

$$\Rightarrow 20 - 8x = 4 + t \Rightarrow x = \frac{16 - t}{8}$$

New price after tax is, $profit\ (p) = 20 - 4\left(\frac{16 - t}{8}\right) = \left(12 + \frac{t}{2}\right)$

Thus, profit after tax is:

$$\text{Profit } (P) = TR - TC$$

$$= (20x - 4x^2) - (4x + tx) = x(16 - 4x - t)$$

Substituting $x = \frac{16 - t}{8}$, we obtain

$$\text{Profit } = \frac{16 - t}{8}\left[16 - t - 4 \times \frac{16 - t}{8}\right] = \frac{(16 - t)^2}{16}$$

$$\text{Tax revenue, } T = tx = \frac{16t - t^2}{8}$$

T will be maximum where $\frac{dT}{dt} = 0$ and $\frac{d^2T}{dt^2} < 0$

$$\therefore \frac{dT}{dt} = \frac{d}{dt}\left(\frac{16t - t^2}{8}\right) = \frac{16 - 2t}{8} = 0 \Rightarrow t = 8$$

$$\text{Again } \frac{d^2T}{dt^2} = \frac{d}{dt}\left(\frac{16 - 2t}{8}\right) = -\frac{2}{8} = -\frac{1}{4} < 0.$$

Therefore, tax is maximum when $t = 8$.

(d) With sales tax of 10% the net $TR\ is: TR = 0.90(20 - 4x)x = 0.90(20x - 4x^2)$

$$\therefore MR = 0.90(20 - 8x)$$

At optimum level, $MR = MC$

$$\therefore 0.90(20-8x) = 4 \Rightarrow x = \frac{140}{72} = 1.94$$ **[Ans.]**

Problem-20: From past experience and estimation for the future, XYZ Company has decided that the cost of production of their sold product from an advanced process machine is:

$$C = 1064 + 5x + 0.04x^2$$

where C denotes total cost in Tk. and x denotes quantity produced and sold.
The marketing department has estimated that the price of the product is related to the quantity produced and sold by the equation:

$$p = 157 - 3x,$$

where p represents price per unit in Tk.
The government has proposed a tax of Tk. t per unit product but it is not expected that this will have any effect on the costs incurred in making p or on the demand price relationship. Find:

(a) the price and quantity that will maximize profit when there is no tax.
(b) the price and quantity that will maximize profit if the proposed tax is introduced.
(c) how much of the tax t per unit is passed on to the customer.
(d) the effect on the profit of the company if t is fixed at Tk. 4 per unit.

Solution: (a) Profit $(\pi) = \text{Re}venue(R) - Cost(C)$

Here Revenue $(\text{R}) = \text{Price}(\text{P}) \times \text{Quantity}(\text{x})$

$$= (157 - 3x)x = 157x - 3x^2$$

$$\text{Cost}(C) = 1064 + 5x + 0.04x^2$$

$$\therefore \pi = 157x - 3x^2 - (1064 + 5x + 0.04x^2)$$

$$= 152x - 3.04x^2 - 1064$$

Differentiating π w. r. to x, we have

$$\frac{d\pi}{dx} = \frac{d}{dx}(152x - 3.04x^2 - 1064) = 152 - 6.08x$$

Now $\frac{d\pi}{dx} = 0 \Rightarrow 152 - 6.08x = 0$

$$\Rightarrow x = \frac{152}{6.08} = 25 \text{ units}$$

$$\frac{d^2\pi}{dx^2} = \frac{d}{dx} \Rightarrow (152 - 6.08x) = -6.08 < 0$$

$\therefore$ Profit is maximum when 25 units are produced.

Now $p = 157 - 3 \times 25 = Tk.82$ per unit

and $\pi = 152 \times 25 - 3.04 \times (25)^2 - 1064 = Tk.836$ [Ans.]

(b) When a tax (t) is introduced, $\pi = R - C$ becomes:

$$\pi = (157 - 3x)x - (1064 + 5x + 0.04x^2 + tx)$$
$$= 152x - 3.04x^2 - 1064 - tx$$

Differentiating π with respect to x, we have

$$\frac{d\pi}{dx} = 152 - 6.08x - t$$

Now $\frac{d\pi}{dx} = 0 \Rightarrow 152 - 6.08x - t = 0 \Rightarrow x = \frac{152 - t}{6.08}$

$$\frac{d^2\pi}{dx^2} = \frac{d}{dx}(152 - 6.08x - t) = -6.08 < 0$$

$\therefore$ For maximum profit, quantity $(x) = \frac{152}{6.08} - \frac{t}{6.08} = 25 - \frac{t}{6.08}$ units

Substituting for x in the price equation, we have

$$p = 157 - 3\left(25 - \frac{t}{6.08}\right) = 157 - 75 + \frac{3t}{6.08} = 82 + \frac{3t}{6.08}$$ [Ans.]

(c) The amount of tax passed on the customer is $\frac{3}{6.08} = .4939 = 49.34\ \%$ (approximately)

(d) When the tax per unit is Tk. 4, then $t = 4$.

$$\therefore x = 25 - \frac{4}{6.08} = 24.34 = 24 \text{ (in whole units).}$$

Now $\pi = 152x - 3.04x^2 - 1064 - tx$
$= 152 \times 24 - 3.04 \times (24)^2 - 1064 - 4 \times 24$
$= Tk.736.96$

The profit without tax in (a) above = Tk.836

Profit with tax of Tk. 4 = $Tk.736.96$

$\therefore$ Difference to profit = $Tk.(836 - 736.96) = Tk.99.04$

Problem-21: A monopolist's total cost is $TC = ax^2 + bx + c$ and the demand function is $p = \beta - \alpha x$, where x and p denote the units of output and price respectively and a, b, c, α and β are positive constants. If the government imposes tax at the rate of t per unit of output, show that the total tax is maximum when $t = (\beta - b)/2$.

Solution: After the imposition of tax t per unit, the total cost function TC is given by

$$TC = ax^2 + bx + c + tx$$

Revenue function, $R = px = (\beta - \alpha x)x = \beta x - \alpha x^2$

$\therefore$ Profit function, $P = R - C = (\beta x - \alpha x^2) - (ax^2 + bx + c + tx)$

For P to be maximum, $\frac{dP}{dx}=0$

$$\therefore \frac{dP}{dx}=\frac{d}{dx}\left\{\left(\beta x-\alpha x^2\right)-\left(ax^2+bx+c+tx\right)\right\}=\beta-2\alpha x-2ax-b-t$$

Put $\frac{dP}{dx}=0\Rightarrow \beta-2\alpha x-2ax-b-t=0\Rightarrow x=\frac{\beta-b-t}{2(\alpha+a)}$

$\frac{d^2P}{dx^2}=\frac{d}{dx}(\beta-2\alpha x-2ax-b-t)=-2\alpha-2a=-2(\alpha+a)<0$, since α and a are positive constants.

Therefore, the level of output that maximizes the profit is

$$x=\frac{\beta-b-t}{2(\alpha+a)}$$

The total tax revenue for this level of output is $T=tx=\frac{\beta t-bt-t^2}{2(\alpha+a)}$

For T to be maximum, First order condition: $\frac{dT}{dx}=0$

$$\therefore \frac{dT}{dx}=\frac{d}{dx}\left\{\frac{\beta t-bt-t^2}{2(\alpha+a)}\right\}=\frac{\beta-b-2t}{2(\alpha+a)}$$

Now $\frac{dT}{dx}=0\Rightarrow \frac{\beta-b-2t}{2(\alpha+a)}=0\Rightarrow \beta-b-2t=0\Rightarrow t\frac{1}{2}(\beta-b)$

$\frac{d^2T}{dx^2}=\frac{d}{dx}\left\{\frac{\beta-b-2t}{2(\alpha+a)}\right\}=-\frac{1}{(\alpha+a)}<0$, since α and a are positive constants.

Hence the tax rate t that maximizes the total tax revenue is $t=\frac{1}{2}(\beta-b)$. [Proved]

Problem-22: There are two Duopolists manufacturing equal and identical bicycles. The total cost of an output of x bicycles per month is Tk. $\left(\frac{x^2}{25}+3x+100\right)$ in each case. When the price is Tk p per bicycle the market demand is $x=75-3p$ bicycle per month. Find the total equilibrium output per month.

Solution: Let x_1 and x_2 denote the output per week of the two duopolists.

Then $x=x_1+x_2$ is the total output.

Demand function: $x=75-3p\Rightarrow 3p=75-x\Rightarrow p=25-\frac{x}{3}=25-\frac{(x_1+x_2)}{3}$

$\therefore$ Net revenue for the first firm

$$\pi_1 = R_1 - C_1 = \left[25x_1 - \frac{(x_1 + x_2)x_1}{3}\right] - \left[\frac{x_1^2}{25} + 3x_1 + 100\right]$$

$$\therefore \pi_1 = 22x_1 - \frac{28x_1^2}{75} - \frac{x_1x_2}{3} - 100$$

For maximum net revenue, $\frac{d\pi}{dx_1} = 0 = 22 - \frac{56}{75}x_1 - \frac{x_1}{3} \times \frac{dx_2}{dx_1} - \frac{x_2}{3}$

But $\frac{dx_2}{dx_1} = 0 \Rightarrow 0 = 22 - \frac{56x_1}{75} - \frac{x_2}{3}$ ...(1)

Similarly, we can show for the second firm:

$$0 = 22 - \frac{56x_2}{75} - \frac{x_1}{3} \quad ...(2)$$

The equilibrium outputs of the two firms in duopoly are the simultaneous solutions of (1) and (2). They are:

$$x_1 = \frac{51150}{2511} = x_2$$

i.e., $x_1 = x_2 = 20.37$ (approx).

$\therefore$ Total output per week is $2 \times 20.37 = 41$ (approx). [Ans.]

Problem-23: The total cost C for output x is given by $C = \frac{2}{3}x + \frac{35}{2}$.

Find: (i) Cost when output is 4 units.

(ii) Average cost when output is 10 units.

(iii) Marginal cost when output is 3 units.

Solution: (i) Given $C = \frac{2}{3}x + \frac{35}{2}$

$$\therefore \textit{Cost for 4 unit outputs} = \frac{2}{3} \times 4 + \frac{35}{2} = Tk.20.16$$

(ii) Cost for 10 units output $= \frac{2}{3} \times 10 + \frac{35}{2} = \frac{145}{6} = Tk.24.16$

$$AC = \frac{145}{6} \times \frac{1}{10} = \frac{29}{12} = Tk.2.45$$

(iii) $MC = \frac{d}{dx}\left(\frac{2}{3}x + \frac{35}{2}\right) = \frac{2}{3} \times 1 + 0 = Tk.0.67$ [Ans.]

Problem-24: The average cost function (AC) for a commodity is given by $AC = x + 5 + \frac{36}{x}$ in terms of the output x. Find the outputs for which AC is increasing and the outputs for which

AC is decreasing, with increasing output. Also, find the total cost C and the marginal cost (MC) as function of x.

Solution: Slope of $AC = \frac{d}{dx}(AC) = \frac{d}{dx}\left(x+5+\frac{36}{x}\right) = 1-\frac{36}{x^2}$

AC is increasing if $1-\frac{36}{x^2} > 0$, i.e., $\frac{x^2-36}{x^2} > 0 \Rightarrow x^2-36 > 0 \Rightarrow x^2 > 36 \therefore x > 6$

and AC is decreasing if $1-\frac{36}{x^2} < 0$, i.e., if $x < 6$.

Now $AC = x+5+\frac{36}{x} = \frac{x^2+5x+36}{x}$

$\therefore Total\ \cos t, TC = x \times AC = x \times \frac{\left(x^2+5x+36\right)}{x} = x^2+5x+36$

Marginal Cost, $(MC) = \frac{dC}{dx} = \frac{d}{dx}\left(x^2+5x+36\right) = 2x+5$. [Ans]

Problem-25: The total cost function of a firm is given by $C = 0.04q^3 - 0.9q^2 + 10q + 10$

Find (a) Average cost (AC)
(b) Marginal cost (MC)
(c) Slope of AC.
(d) Slope of MC.
(e) Value of q at which average variable cost is minimum.

Solution: Given $C = 0.04q^3 - 0.9q^2 + 10q + 10$

(a) $AC = \frac{C}{q} = \frac{0.04q^3 - 0.9q^2 + 10q + 10}{q} = 0.04q^2 - 0.9q + 10 + \frac{10}{q}$

(b) Marginal cost $(MC) = \frac{dC}{dq} = \frac{d}{dq}\left(0.04q^3 - 0.9q^2 + 10q + 10\right) = 0.12q^2 - 1.8q + 10$

(c) Slope of $AC = \frac{d}{dq}(AC) = \frac{d}{dq}\left(0.04q^2 - 0.9q + 10 + \frac{10}{q}\right) = 0.08q - 0.9 - \frac{10}{q^2}$

(d) Slope of $MC = \frac{d}{dq}(MC) = \frac{d}{dq}\left(0.12q^2 - 1.8q + 10\right) = 0.24q - 1.8$

(e) When AVC is minimum, the slope of AVC curve is zero, i.e., $\frac{d}{dq}(AVC) = 0$

Total Cost, $C = 0.04q^3 - 0.9q^2 + 10q + 10$

Here Variable cost $= 0.04q^3 - 0.9q^2 + 10q$, $Fixed\ Cost = 10$.

Therefore, $AVC = \frac{Variable\ Cost}{q} = \frac{0.04q^3 - 0.9q^2 + 10q}{q} = 0.04q^2 - 0.9q + 10.$

$\therefore \frac{d}{dq}(AVC) = \frac{d}{dq}(0.04q^2 - 0.9q + 10) = 0.08q - 0.9$

Now $0.08q - 0.9 = 0 \Rightarrow q = \frac{0.9}{0.08} = 11.25$

$\frac{d^2}{dq^2}(AVC) = \frac{d}{dq}(0.08q - 0.9) = 0.08 > 0$

Therefore, when $q = 11.25$, average variable cost is minimum. [Ans.]

Problem-26: The cost function of a firm is: $C = 300x - 10x^2 + \frac{1}{3}x^3$, where C stands for and x for output.

Calculate (i) Output at which marginal cost is minimum.
(ii) Out put at which average cost is minimum.
(iii) Out put at which average cost is equal to marginal cost.

Solution: (i) $C = 300x - 10x^2 + \frac{1}{3}x^3$

$\therefore MC = \frac{dC}{dx} = \frac{d}{dx}\left(300x - 10x^2 + \frac{1}{3}x^3\right) = 300 - 10 \times 2x + \frac{1}{3} \times 3x^2 = 300 - 20x + x^2$

Differentiating MC w.r. to x, we have

$\frac{d(MC)}{dx} = \frac{d}{dx}(300 - 20x + x^2) = -20 + 2x$

Now $\frac{d(MC)}{dx} = 0 \Rightarrow -20 + 2x = 0 \Rightarrow x = 10$ is the necessary condition for marginal cost minimization.

To get the sufficient condition, we have

$\frac{d^2(MC)}{dx^2} = \frac{d}{dx}(-20 + 2x) = 2$, a positive quantity which means that marginal cost is minimum at $x = 10.$

$\therefore$ For 10 units output, MC is minimum. [Ans.]

(ii) Average Cost $(AC) = \frac{C}{x} = \frac{300x - 10x^2 + \frac{1}{3}x^3}{x} = 300 - 10x + \frac{1}{3}x^2$

Now to find the output at which average cost is minimum, we have to differentiate the AC w.r. to x,

$$\therefore \frac{d(AC)}{dx} = \frac{d}{dx}\left(300 - 10x + \frac{1}{3}x^2\right) = 0 - 10 + \frac{1}{3} \times 2x = -10 + \frac{2}{3}x.$$

$$\frac{d(AC)}{dx} = 0 \Rightarrow -10 + \frac{1}{3} \times 2x = 0 \Rightarrow \frac{2}{3}x = 10 \Rightarrow x = \frac{30}{2} = 15.$$

Also $\frac{d^2(AC)}{dx^2} = \frac{d}{dx}\left(-10 + \frac{2}{3}x\right) = \frac{2}{3}$, a positive quantity.

$\therefore$ Second condition is also satisfied.

Hence the output at which AC is minimum is given by $x = 15$. [Ans.]

(iii) Given AC = MC

$$\Rightarrow 300 - 10x + \frac{1}{3}x^2 = 300 - 20x + x^2 \Rightarrow \frac{2x^2}{3} = 10x$$

$$\Rightarrow \frac{2x^2}{3} - 10x = 0 \Rightarrow x\left(\frac{2x}{3} - 10\right) = 0 \Rightarrow x = 0, x = 15.$$

But $x = 0$ is not acceptable output.

Hence for $x = 15$, average cost is equal to marginal cost. [Ans]

Problem-27: The total monthly variable cost of x ton output by a firm producing metal is Tk. $\frac{1}{10}x^3 - 3x^2 + 5x$ and the fixed cost is Tk. 300 per month. Find the output for minimum average cost.

Solution: Total Cost (TC) = Total Variable Cost (TVC) + Fixed Cost (FC)

$$\therefore TC = \frac{1}{10}x^3 - 3x^2 + 5x + 300$$

$$AC = Average\ cost = \frac{TC}{x} = \frac{\frac{1}{10}x^3 - 3x^2 + 5x + 300}{x} = \frac{1}{10}x^2 - 3x + 5 + \frac{300}{x}$$

Now $\frac{d(AC)}{dx} = \frac{d}{dx}\left(\frac{1}{10}x^2 - 3x + 5 + \frac{300}{x}\right) = \frac{1}{5}x - 3 + 0 - \frac{300}{x^2} = \frac{1}{5}x - 3 - \frac{300}{x^2}$

$$\frac{d}{dx}(AC) = 0 \Rightarrow \frac{1}{5}x - 3 - \frac{300}{x^2} = 0 \Rightarrow \frac{x^3 - 15x^2 - 1500}{5x^2} = 0$$

$$\Rightarrow x^3 - 15x^2 - 1500 = 0 \Rightarrow x = 19.10$$

Again $\frac{d^2(AC)}{dx^2} = \frac{d}{dx}\left(\frac{1}{5}x - 3 - \frac{300}{x^2}\right) = \frac{1}{5} + \frac{600}{x^3} = \frac{1}{5} + \frac{600}{(19.10)^3} > 0$ *for* $x = 19.1$

Hence average cost is minimum when the output is 19.1 tons [Ans.]

Problem-28: If the demand function of the monopolist is $3q = 98 - 4p$ and average cost is $3q + 2$, where q is output and p is the price, find maximum profit of the monopolist.

Solution: Given Average Cost $(AC) = 3q + 2$

$\therefore$ Total Cost $(TC) = AC \times q$

$\Rightarrow TC = (3q+2) \times q$ $\qquad\qquad$ $\Rightarrow TC = 3q^2 + 2q$

$\therefore Marginal\ Cost\ (MC) = \frac{d}{dq}(TC)$

$\Rightarrow \mathrm{MC} = \frac{\mathrm{d}}{\mathrm{dq}}(3\mathrm{q}^2 + 2\mathrm{q})$ $\qquad\qquad$ $\Rightarrow MC = 6q + 2$

Here $3q = 98 - 4p$

$\Rightarrow 4p = 98 - 3q$ $\qquad\qquad$ $\Rightarrow p = \frac{98 - 3q}{4}$

$\Rightarrow p = \frac{98}{4} - \frac{3q}{4}$

We know, Total Revenue $(\mathrm{TR}) = \text{price} \times \text{quantity}$

$\Rightarrow TR = p \times q$ $\qquad\qquad$ $\Rightarrow TR = \left(\frac{98}{4} - \frac{3q}{4}\right) \times q$

$\Rightarrow \mathrm{TR} = \frac{98\mathrm{q}}{4} - \frac{3\mathrm{q}^2}{4}$

$\therefore Marginal\ Revenue\ (MR) = \frac{d}{dq}(TR)$

$\therefore MR = \frac{d}{dq}\left(\frac{98q}{4} - \frac{3q^2}{4}\right)$ $\qquad$ $\Rightarrow MR = \frac{98}{4} - \frac{6q}{4}$

Under monopoly business, profit will be maximum at $MC = MR$.

$\therefore MC = MR$

$\Rightarrow 6q + 2 = \frac{98}{4} - \frac{6q}{4}$ $\qquad\qquad$ $\Rightarrow 6q + 2 = \frac{98 - 6q}{4}$

$\Rightarrow 24q + 8 = 98 - 6q$ $\qquad\qquad$ $\Rightarrow 30q = 90$ $\qquad$ $\Rightarrow q = 3$

So the maximum profit of the monopolist will be obtained at $\mathrm{q} = 3$.

When $\mathrm{q} = 3$, then

Required maximum profit $(y) = TR - TC$

$= \left(\frac{98q}{4} - \frac{3q^2}{4}\right) - (3q^2 + 2q)$ $\quad = \frac{98q}{4} - \frac{3q^2}{4} - 3q^2 - 2q = \frac{98(3)}{4} - \frac{3(3)^2}{4} - 3(3)^2 - 2(3)$

$$= \frac{294}{4} - \frac{27}{4} - 27 - 6 \quad = 73.5 - 6.75 - 27 - 6 \quad = \text{Tk.}33.75 \text{ [Ans.]}$$

Problem-29: The marginal revenue (in thousand of taka) function for a particular commodity is $4 + r^{-0.03x}$, where x denotes the number of units sold. Determine the total revenue from the sale of 100 units. It is given that $e^{-3} = 0.05$ (approximate).

Solution:

If R denotes the revenue function, then we are given that

$$MR = 4 + e^{-0.03x}$$

$$\Rightarrow R = \int \left(4 + e^{-0.03x}\right) dx$$

$$= 4x - \frac{e^{-0.03x}}{.03} + k$$

where k is the constant of integration. However, $R = 0$ when $x = 0$

$$\therefore 0 = 0 - \frac{1}{.03} + k \Rightarrow k = 33.33$$

Hence the revenue from the sale of x units is given by

$$R(x) = 4x - \frac{1}{0.03} e^{-0.03x} + 33.33$$

Now , the revenue from the sale of 100 units is

$$R(100) = 400 - \frac{e^{-3}}{0.03} + 33.33$$

$$= 400 - \frac{0.05}{0.03} + 33.33$$

$$= 400 - 1.66 + 33.33$$

$$= 431.67 \text{ thousands of Taka} = \text{Tk.}4,31,670$$

Problem-30: If the marginal revenue function for a commodity is $MR = 9 - 4x^2$, find the demand function.

Solution: Since MR is the derivative of the revenue function R, we have

$$R = \int MR\, dx$$

$$= \int (9 - 4x^2)\, dx = 9x - \frac{4}{3}x^3 + k \cdots\cdots (i)$$

where k is the constant of integration. However, $R = 0$ when $x = 0$.

Therefore,

$$0 = 0 - 0 + k \Rightarrow k = 0$$

Form (i) we get, $R = 9x - \frac{4}{3}x^3$

But $R = px$, where p is the price per unit

Hence $px = 9x - \frac{4}{3}x^3$

$\Rightarrow p = 9 - \frac{4}{3}x^2$, which is the required demand function.

Problem-31: The marginal cost $C'(x)$ and the marginal revenue $R'(x)$ are given by $C'(x) = 20 + \frac{x}{20}$ and $R'(x) = 30$. The fixed cost is Tk.200. Determine the maximum profit and the number of items produced for this profit.

Solution: Let P denote the profit function, then

$$\frac{dP}{dx} = \frac{dR}{dx} - \frac{dC}{dx} = 30 - 20 - \frac{x}{20} = 10 - \frac{x}{20}$$

Integrating both sides with respect to x, we get

$$P = \int \left(10 - \frac{x}{20}\right) dx = 10x - \frac{x^2}{40} + k \cdots\cdots (i)$$

where k is the integrating constant. However, we are given that fixed cost is 200. That is

$P = -200$ when $x = 0$.

From (i): $-200 = k$

Thus $P = 10x - \frac{x^2}{40} - 200$ is the profit function.

For P to be maximum, it is necessary that

$$\frac{dP}{dx} = 0, \text{ i.e. } 10 - \frac{x}{20} = 0 \Rightarrow x = 200$$

Also $\frac{d^2P}{dx^2} = -\frac{1}{20} < 0$. Thus the output $x = 200$ certainly gives the maximum profit. The maximum profit is given by: $P_{max} = 10(200) - \frac{(200)^2}{40} - 200 = 800$

Hence maximum profit is Tk.800 which is obtained when 200 items are produced.

Problem-32: A company suffers a loss of Tk.121.50 if one of its products does not sell at all. Marginal revenue and marginal cost functions for the product are given by $MR = 30 - 6x$ and $MC = -24 + 3x$. Determine the total profit function and the profit maximizing level of output.

Solution: If P denotes the profit function, then

$$\frac{dP}{dx} = MR - MC = (30 - 6x) - (-24 + 3x) = 54 - 9x$$

Integrating both sides with respect to x, we have

$$P = \int (54 - 9x)dx = 54x - \frac{9}{2}x^2 + k$$

where k is an integrating constant. Since the company suffers a loss of Tk. 121.50 if the product does not sell at all, i.e. $P = -121.50$ when $x = 0$. Therefore

$$-121.50 = 54(0) - \frac{9}{2}(0)^2 + k \Rightarrow k = -121.50$$

Thus the total profit function is given by

$$P = 54x - \frac{9}{2}x^2 - 121.50$$

For P to be maximum, it is necessary that: $\frac{dP}{dx} = 0$ *i.e.*, $54 - 9x = 0 \Rightarrow x = 6$

Also, $\frac{d^2P}{dx^2} = -9$, which is negative.

This shows that profit is maximum when $x = 6$.

Ans. Prifit function: $P = 54x - \frac{9}{2}x^2 - 121.50$

And profit maximizing output is: $x = 6$.

BRIEF REVIEW

Definition

Derivative of a function: If the limit $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists, then $f'(x)$ is called the differential coefficient or derivative of $f(x)$. The process or technique of finding the first derivative of a function is called differentiation. The first derivative of the function $y=f(x)$ is denoted by $\frac{dy}{dx}$ or $D(y)$ or $f'(x)$ or $y'(x)$ or y_1.

The second derivative of the function $y=f(x)$ is denoted by $\frac{d^2y}{dx^2}$ or $D^2(y)$ or $f''(x)$ or $y''(x)$ or y_2 and so on.

$\frac{dy}{dx}$: It means the differentiation of y with respect to x. It measures the rate of change of the dependent variable y with respect to the change of independent variable x.

Condition for the Minimum: For the given function $y=f(x)$, find $\frac{dy}{dx}$. Then put $\frac{dy}{dx}=0$ and find the value or values of x. Let these are: a, b, c,.... Now if at $x=a$, if $\frac{d^2y}{dx^2}>0$, the function $f(x)$ has minimum value at $x=a$ and $y_{min}=f(a)$.

Condition for the Maximum: For the given function $y=f(x)$, find $\frac{dy}{dx}$. Then put $\frac{dy}{dx}=0$ and find the value or values of x. Let these are: a, b, c,.... Now if at $x=a$, if $\frac{d^2y}{dx^2}<0$, the function $f(x)$ has maximum value at $x=a$ and $y_{max}=f(a)$.

Point of inflection: Inflection points can occur only where the second derivative equals zero and third derivative not equal to zero. The sign of the first derivative is immaterial for an inflection point. For an inflection point at a, the conditions are

(i) $\frac{dy}{dx}=0$ (ii) $\frac{d^2y}{dx^2}=0$ at $x=a$. (iii) $\frac{d^3y}{dx^3}\neq 0$ at $x=a$.

Quiz Questions

Multiple Choice Questions

1. Which is the slope of the straight line $y = 2x + 5$

a) -2 b) 5 c) 1/2 d) 2

2. The term derivative is a generalized expression for measuring-

a) Slope of a function b) The rate of change of a function
c) Both a & b d) None of them.

3. The n-th derivative of a function $y = f(x)$ is denoted by

a) $\frac{d^n y}{dx^n}$ b) Dy^n c) $D^n y$ d) Both a & c

4. When $y = 7x^2 + 9x + 2$ *find of* $\frac{d^3 y}{dx^3}$

a) 0 b) 2 c) 23 d) 7

5. Which one is the derivative of e^{mx} : i) e^{mx} ii) me^{mx} iii) $m^2 e^{mx}$ iv) mx

6. Determine $\frac{dx}{dy}$ *of the function* $y = \log \sin x$

a) $\frac{1}{\cot x}$ b) $(\tan x + \cot x)$ c) $\cot x$ d) 1

7. A function $y = f(x)$ is said to be an increasing function of x if.

a) $\left[\frac{dy}{dx}\right]_{x=a} > 0$ b) $\left[\frac{dy}{dx}\right]_{x=a} < 0$ c) $\left[\frac{dy}{dx}\right]_{x=a} = 0$ d) $\left[\frac{dy}{dx}\right]_{x=a} \neq 0$

8. Inflection point can occur if (at point *a*):

a) $\frac{d^2 y}{dx^2} = 0$ b) $\frac{d^2 y}{dx^2} \neq 0$ c) $\frac{d^2 y}{dx^2} = 1$ d) $\frac{d^2 y}{dx^2} = -1$

9. Which one is the necessary condition for maxima and minima of a function $y = f(x)$

a) $\frac{dy}{dx} \neq 0$ b) $\frac{dy}{dx} = 0$ c) $\frac{dy}{dx} = 1$ d) $\frac{dy}{dx^2} \neq 0$

10. Which one is the condition for maxima value of a function $y = f(x)$.

a) $\frac{dy}{dx} \neq 0$ b) $\frac{dy}{dx} = 0$ and $\frac{d^2 y}{dx^2} < 0$ c) $\frac{dy}{dx} = 1$ d) $\frac{dy}{dx^2} \neq 0$

11. Which one is the condition for minima value of a function $y = f(x)$.

a) $\frac{dy}{dx} \neq 0$ b) $\frac{dy}{dx} = 0$ and $\frac{d^2 y}{dx^2} > 0$ c) $\frac{dy}{dx} = 1$ d) $\frac{dy}{dx^2} \neq 0$

Which one of the following statement is true/false?

a. The derivative of $\frac{dy}{dx}$ is the second derivative $\frac{d^2y}{dx^2}$.

b. $\frac{d}{dx}(\tan x) = \cot^2 x$

c. Calculus may be classified into two categories.

d. Revenue(R) = price(p) × quantity(x)

e. $\frac{dy}{dx}$ = slope or gradient (m) in case of a straight line

f. $\frac{dy}{dx}$ = Marginal Cost (MC)

g. Profit = Revenue – Cost

h. Loss = Cost - Revenue

Brief questions

1. What do you mean by derivative of a function $y = f(x)$?

2. What is meant by $\frac{dy}{dx}$?

3. If $y = 3x^3$, then $\frac{dy}{dx} = ?$

4. If $y = mx + c$, then $\frac{dy}{dx} = ?$

5. If $y = e^{2x}$, then $\frac{dy}{dx} = ?$

6. If $y = 37$, then $\frac{dy}{dx} = ?$

7. If $y = \frac{1}{x}$, then $\frac{dy}{dx} = ?$

8. If $y = x\sqrt{x}$, then $\frac{dy}{dx} = ?$

9. If $y = \log x + 7x^2 + 3$, then $\frac{dy}{dx} = ?$

10. If $y = a^x$, then $\frac{dy}{dx} = ?$

Conceptual, Analytical & Numerical Questions

1. What do you mean by derivative of a function $y = f(x)$?

1. What is meant by $\frac{dy}{dx}$?
2. Write down the working rule for the determination of extreme values of a function?
3. What is the Condition for the Minimum value of a function at a point?
4. What is the Condition for the Maximum value of a function at a point?
5. What do you mean by point of inflection?
6. When a function is said to be increasing at a point?
7. When a function is said to be decreasing at a point?
8. Define increasing and decreasing function with examples.

Numerical questions

1. Find $\frac{dy}{dx}$ for the following cases:

(a) $y = \frac{(1-x)^2}{x^2}$, (b) $y = (3x^2+1)(x^3+2x)$, (c) $y = \frac{(x+1)(2x-1)}{(x-3)}$

(d) $\frac{x^{1/2}+1}{x^{1/2}}$, (e) $y = \sqrt{3x^2-7}$, (f) $y = (3x^3-5x^2+8)^3$

2. Find the differential coefficient of the following function:

(a) $\tan(5x+7)$, (b) $\sec^3(4x)$, (c) $\frac{1-\sin x}{1+\cos x}$, (d) $\log(\sin x)$

(e) $\log(x\cos x)$, (f) $\tan^{-1}\sqrt{x}$, (g) $\log\sin x^2$.

3. Differentiate the following functions w. r. to x:

(a) $\sin(x^2+2x-5)^7$, (b) $(x^4-4x^3+9)\tan xe^x$.

4. Find the derivatives of the followings with respect to x:

(a) $\frac{1}{\sqrt{x}}$, (b) $\frac{-3}{\sqrt[3]{x^7}}$, (c) $\left(x-\frac{1}{x}\right)^2$, (d) $\frac{5x^4-6x^2-7x+8}{5x-6}$, (d) $\sqrt{ax^2+bx+c}$.

5. Find the derivative of the followings functions w. r. to x:

(a) $(1-x^2)\tan x$, (b) $\dfrac{\tan x^2}{ax+b}$, (c) $\dfrac{\log\cos x}{\tan(\log x)}$, (d) $\dfrac{1-\cos 2x}{\sin 2x}$.

6. Find the derivative of the following functions w. r. to x:

(a) $\log\sin\sqrt{x}$, (b) $\sqrt{\sin\sqrt{x}}$, (c) $e^{\log\tan^{-1}x}$, (d) $b\tan^{-1}\left(\dfrac{x}{a}\tan^{-1}\dfrac{x}{a}\right)$,

(e) $e^{\tan^{-1}x}\log(\sec^2 x^3)$. (f) $\log\sec^{-1}x^2+\log(\log\cos x^2)$

7. If $y=\dfrac{1}{3}\log\dfrac{x+1}{\sqrt{(x^2-x+1)}}+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}$, *prove that* $\dfrac{dy}{dx}=\dfrac{1}{1+x^3}$.

8. Find $\dfrac{dy}{dx}$ in the following cases:

(a) $x=\dfrac{3at}{(1+t^3)}, y=\dfrac{3at^2}{(1+t^3)}$. (b) $x=\log t+\sin t, y=e^t+\cos t$.

9. (a) If $y=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, prove that $\dfrac{dy}{dx}=\dfrac{2}{1+x^2}$.

(b) If $y=\sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$, *prove that* $\dfrac{dy}{dx}=\dfrac{2}{1+x^2}$.

(c) $y=\tan^{-1}\left(\dfrac{a\sin x+b\cos x}{a\cos x-b\sin x}\right)$, *find* $\dfrac{dy}{dx}$.

10. Differentiate the following functions w. r. to x:

(a) $\log\left[e^{3x}\left(\dfrac{5x-3}{4x+2}\right)^{\frac{1}{3}}\right]$, (b) $x^{\tan x}+(\sin x)^{\cos x}$, (c) $\dfrac{(3x+5)^{\frac{2}{7}}e^{3x}}{(x-9)^{\frac{1}{2}}}$

11. (a) If $y=x^{x^x}$, then find $\dfrac{dy}{dx}$.

(b) If $x^y=e^{x-y}$, prove that $\dfrac{dy}{dx}=\dfrac{\log x}{(1+\log x)^2}$

12. Differentiate the following w. r. to x:

(a) $\dfrac{x^{1/2}(5-2x)^{2/3}}{(4-3x)^{3/4}(7-4x)^{4/5}}$, (b) $(\cot x)^{\sin x}+(\tan x)^{\cos x}$

13. Find $\frac{dy}{dx}$ for the following functions:

(a) $x^3 + y^3 = 3axy$, (b) $x^2 - y^2 + 3x = 5y$, (c) $x^3 - xy^2 + 3y^2 + 2 = 0$

(d) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

14. (a) If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

(b) If $y = x^{x \cdot x \cdot \ldots \cdot \infty}$, prove that $x\frac{dy}{dx} = \frac{y^2}{1 - y\log x}$

(c) If $y = a\sin nx + b\cos nx$, *then prove that* $\frac{d^2y}{dx^2} + n^2 y = 0.$

(d) If $y = \frac{1}{2}\left(\sin^{-1} x\right)^2$, then prove that $\left(1 - x^2\right)y_2 - xy_1 = 1.$

(e) If $y = 2x + \frac{4}{x}$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

(f) If $y = a^{\sin^{-1} x}$, then prove that $\left(1 - x^2\right)y_2 - xy_1 = a^2 y.$

(g) If $y = \sin\left(m\sin^{-1} x\right)$, then prove that $\left(1 - x^2\right)y_2 - xy_1 + m^2 y = 0.$

15. Find $\frac{d^2y}{dx^2}$ for the function $y = (2+x)\sin^{-1} x.$

16. (a) If $y = \sin(\sin x)$, prove that $y_2 + y_1 \tan x + y\cos^2 x = 0.$

(b) If $y = e^x \cos x$, then prove that $y_4 + 4y = 0.$

(c) If $y = x\sin(\log x) + x\log x$, then show that $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x$

(d) If $y = \sin^{-1} x$, show that $\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0.$

(e) If $y = e^{\tan^{-1} x}$, prove that $\left(1 + x^2\right)y_2 + (2x - 1)y_1 = 0.$

17. If $y = \log\left(x + \sqrt{1+x^2}\right)$ then show that $\left(1 + x^2\right)y_2 + (2x - 1)y_1 = 0.$

18. If $y = \cos\left(m\sin^{-1} x\right)$, then show that $\left(1 - x^2\right)y_2 + m^2 y = xy_1.$

19. Show that the curve $y = x^3(3 - x)$ has a point of inflexion at the point (1,2).

20. Investigate the maxima and minima of the function $2x^3+3x^2-36x+10$.

21. Find the extreme values of the function x^3e^{-x}.

22. Find the maxima and minima of the functions:

(a) $\frac{2}{3}x^3+\frac{1}{2}x^2-6x+8$, (b) $x^4+2x^3-3x^2-4x+4$.

23. Find the maximum and minimum values of the function $(x-2)^6(x-3)^5$.

24. (a) Show that $x^5-5x^4+5x^3-1$, has neither a maximum value when $x=1$, minimum value when x=3, and point of inflexion when $x=0$.

(b) Show that x^3-3x^2+3x+7 has neither a maximum nor a minimum value.

25. Find the maximum and the minimum values of the function $x^5-5x^4+5x^3-1$. Discuss its nature at $x=0$.

26. State whether $y=x^2-6x+13$ has a maximum or a minimum value. Find the value.

27. For a unit demand function of $p=24-8x$ where x is the number of units and p is the price in Taka, find the sales function. If the average cost per unit is Tk. 8, find

a. The profit function.

b. The number of units that maximize the profit function.

28. If the total cost function of a firm is $C=\frac{1}{3}x^3-5x^2+30x+10$, where C is the total cost and x is the output, and price under perfect competition is given as Tk. 6. Find for what value of x the profit will be maximized. Examine both first and second order conditions.

29. If the demand function for a commodity is given by $p=12e^{\frac{q}{4}}$ where p is the price per unit and q is the number of units demanded. Determine the price and the quantity for which the revenue is maximum.

30. State the conditions for a maximum profit. Find the profit maximizing output level if $p=200-10x$ and $AC=10+\frac{x}{25}$.

31. A firm produced an output of x tons of a certain product at a total cost given by $C=x^3-4x^2+7x$. Find the output at which the average cost is the least and the corresponding value of the average cost. What is the total cost at this level?

32. If the cost function is $C(x)=4x+9$ and the revenue function is $R(x)=9x-x^2$, where x is the number of units produced, R and C are measured in Taka, find the maximum profit.
33. Suppose the cost function is given by $C(x)=x^2+5$ and the price function is $p=12-2x$, where p is the price in rupees and x is the number of units produced. Find the maximum profit.
34. A company has produced x items for which the total cost is C and the total revenue R given by equation: $R=3x$ and $C=100+0.015x^2$. Find how many items should be produced to maximize the profit. What is this profit?
35. A sofa – set manufacturer can manufacture x sofa sets per week at a total cost of Tk. $\left(\frac{1}{2}x^2+3x+100\right)$. How many set per week should be manufactured for maximum monopoly revenue when the demand law of his product is $x=10\sqrt{25-p}$ set per week. Also find the net revenue with this output.
36. The cost function $C(x)$ for producing x units of a commodity is given by

$$C(x)=\frac{1}{3}x^3-5x^2+75x+10$$

At what level of output the marginal cost attain its minimum? What is the marginal cost at this level of production.

37. If q the number of workers employed, the average cost of production is given by $C=\frac{3}{2(q-4)}+24q.$

Shows that $q=4.25$ will make the expression minimum. In the interest of the management, will you then advise to employ four or five worker? Give reasons for your answer.

38. The following expressions define a firm's total revenue and total cost functions:

Total revenue $=18x-x^2+24$

Total cost $=\frac{1}{2}x^2-2.5x^2+50$

a. Use calculus methods to find optimum production level.
b. State the firm's profits at the optimum production level.

39. A tractor company can manufacture at most 1,000 heavy – duty tractors per year. Also, from past demand data, the company knows that the number of heavy – duty tractors it can sell

depends only on the price p of each unit. The company also knows that the cost to produce the units is a function of the number of units x sold. Assume that the price function is $p = 29{,}000 - 3x$ and the cost function $C = 2{,}00{,}0000 + 20{,}000x + 5x^2$. How many units should be produced to maximize profit?

40. A manufacturer estimates that he can sell 500 articles per week if his unit price is Tk. 20.00 and that his weekly sales will rise by 50 units with each Tk. 0.50 reduction in price. The cost of producing and selling x articles a week is $C(x) = 6200 + 6.10x + 0.0003x^2$. Find

    a. The price function

    b. The level of weekly production for maximum profit.

    c. The price per article at the maximum level of production.

41. A trucking company has an average engine overhaul cost of Tk. 1,000 and routine maintenance cost (in Taka) of $C = 0.40x + 10^{-5}x^2$, where x is the interval in kilometers between engine overhauls.

    a. Show that the total engine maintenance cost Tk. (per km) is given by

    $$\frac{1000}{x} + 0.04 + 100^5 x$$

    b. Find the rate of change of the total maintenance cost with respect to the engine overhaul to interval $\frac{dC}{dx}$

42. Given $p = 20 - q, C = 2 + 8q + q^2$. Where p = price, q = output, R = revenue, M = profit.

    Find: (a) q which maximize profit and corresponding values of p, R and M.

    (b) q which maximizes sales (total revenue) and corresponding values of p, R and M.

    (c) q which maximizes sales subject to the constraint M<8 and corresponding values of p, R and M.

43. A monopolist has the following demand and cost functions.

    $p = 30 - q$

    $C = 160 + 8q$

    The Government levies a tax at the rate of 2 per unit sold. Find profit maximizing price and quantity after tax levy.

44. A firm has the following functions:

    $p = 100 - 0.01q$

    $\pi = 50q + 30{,}000$

A tax of Tk. 10 per unit is levied. What will be the profit – maximizing price and quantity before the tax and after the tax ? Which does the monopolist find it better to increase the price by less than the increase in tax?

45. If the relevant position of the demand function is $p = 100 - 0.01q$, when q is weekly production and p is weekly price and cost function is $c = 50q + 30{,}000$.
    a. Find maximum profit, output, price, and total profit.
    b. Suppose government decides to levy a tax of Tk. 10 per unit of product sold, what will happen to price, quantity sold and total profit.
46. Given the demand function $p = (10 - x)^2$ and the cost function $G = 55x - 8x^2$. Find the maximum profit. What would be the effect of an imposition of a tax of Tk. 9 per unit quantity on price?
47. The demand and cost functions of a firm are given by
    $q = 10{,}000 - 100p$ and
    $c = 59q + 30{,}000$.
    where q = quantity demanded, p = price/unit, c = total cost
    a. Determine the optimum level of q that the firm should sell.
    b. Assuming that the above firm has to pay a sales tax at the rate of Tk. 10 per unit, find out the optimum sales.