CHAPTER-13 INTEGRATION AND USE IN BUSINESS PROBLEMS

LEARNING OBJECTIVES

After studying this chapter, you will be able to understand:

- Basic concepts of indefinite and definite Integration.
- Technique of integration by using direct formula
- Calculation of Consumer's surplus and Producer's surplus by using definite integration
- Application of Integration in business problem

13.1 INTRODUCTION

The reverse process of differentiation is known as integration. to give a precise shape to the definition of integration, we observe:

If g(x) is a function of x such that $\frac{d}{dx}g(x)=f(x)$, then we define integral of $f(x)$ with respect to x to the function $g(x)$. This is put in the notational form as: $\int f(x)\,dx=g(x)$

The function $f(x)$ is called the integrand. Presence of dx is there just to remind us that integration is being done w. r. to x.

For example, the derivative of the function $y=x^6$ is $\frac{dy}{dx}=6x^5$ or, its differential is $dy=6x^5dx$. Hence, it is easily understood that the function having derivative $6x^5$ or, differential $6x^5dx$ is x^6.

In this case, the mathematical method, which determines the function $y=x^6$ from the derivative $\frac{dy}{dx}=6x^5$ is called integration. The function obtained by integration from the derivative of the function is called its integral. In the above example, the integral of $\frac{dy}{dx}=6x^5$ is $y=x^6$. Clearly, integration can be looked upon as anti-derivative i.e., it is inverse process of differentiation.

We note that, $\frac{dy}{dx}=6x^5$ if $y=x^6$

Again $\frac{dy}{dx}=6x^5$ if $y=x^6+4(\text{say})$ and $\frac{dy}{dx}=6x^5$ if $y=x^6+c$, where c is an arbitrary constant independent of x.

Hence, it follows that if the derivative of a function is $6x^5$ then, in general, the function is $x^6 + c$. Therefore, it is evident that the integral obtained by integration form the derivative of a function contains an additive arbitrary constant. For this reason the integral obtained as anti derivative is called an **Indefinite Integral.**

From the historical stand-point integration means summation. The origin of integration is in the effort of finding the sun of an infinite series whose each term is infinitesimally small. In fact, such infinite series originates from the endeavor of determining area of a bounded region and origin of integration is the effort of finding the value of such infinite series. In this case, the sum of the infinite series (when exists) is called the **Definite integral.**

From the above discussion it is clear that the process of integration has two viewpoints. We may consider it as the inverse of differentiation or else, we may consider it as the sum of an infinite series whose each term is infinitesimally small. However, these two viewpoints are not independent. By the fundamental theorem of integral calculus it can be shown that the two viewpoints are identical.

13.2 DEFINITION OF INTEGRATION

Let $f(x)$ be a given function of x. If we can determine another function $F(x)$ such that its derivative with respect to x is equal to $f(x)$ or, differential of F(x) is equal to $f(x)dx$, then $F(x)$ is defined as an **Indefinite Integral or Anti- derivative** of $f(x)$ with respect to x and is denoted by $\int f(x)\,dx = F(x)$. In this case, the process of finding $F(x)$ is called integration and $f(x)$ is called **Integrand.**

To signify the integration operation the integral sign $\int$ is written before the given function and the differential dx is written after the given function to indicate that x is the variable of integration. The symbol $\int f(x)dx$ is read as **"Integral of** $f(x)dx$**".**

Remarks: (i) The sign of integration (i.e., $\int$) is the initial letter of the word 'sum' (in elongated form), since integration is defined as a process of summation.

(ii) If $\frac{d}{dx}[F(x)] = f(x)$, then by definition, we have $\int f(x)\,dx = F(x)$.

13.3 CONSTANT OF INTEGRATION

If C is an arbitrary constant independent of x and $\frac{d}{dx}[F(x)] = f(x)$ then $\frac{d}{dx}[F(x)+c] = f(x)$.

Therefore, by definition we have, $\int f(x)\,dx = F(x)+c$.....(1)

Clearly, the general solution of $\int f(x)\,dx \text{ is } F(x)+c$; here $[F(x)+c]$ is called an **indefinite integral** of $\int f(x)\,dx$ and the arbitrary constant C (which is independent of x) is usually referred to as **the constant of integration**. From (1), it is obvious that giving different values to the arbitrary constant C we shall get different values of $\int f(x)\,dx$. This is the reason why $\int f(x)\,dx$ is referred to as an **indefinite integral.**

13.4 SOME STANDARD FORMULAS OF INTEGRATION

1. $\int dx = x + c$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
3. $\int \frac{1}{x} dx = \log x + c$
4. $\int e^x dx = e^x + c$
5. $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$
6. $\int a^x dx = \frac{a^x}{\log_e a} + c$
7. $\int \sin x dx = -\cos x + c$
8. $\int \cos x dx = \sin x + c$
9. $\int \tan x dx = -\log \cos x + c = \log \sec x + c$
10. $\int \cot x dx = -\log \cos ec x + c = \log \sin x + c$
11. $\int \sec x dx = \log(\sec x + \tan x) + c = \log \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) + c$
12. $\int \cos ec x dx = \log(\cos ec x - \cot x) + c = \log \tan \frac{x}{2} + c$
13. $\int \sec x \tan x dx = \sec x + c$
14. $\int \cos ec x \cot x dx = -\cos ec x + c$
15. $\int \sec^2 x dx = \tan x + c$
16. $\int \cos ec^2 x dx = -\cot x + c$
17. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
18. $\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + c$
19. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$
20. $\int \frac{-1}{1+x^2} dx = \cot^{-1} x + c$
21. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$
22. $\int \frac{-1}{x\sqrt{x^2-1}} dx = \cos ec^{-1} x + c$

13.5 FUNDAMENTAL RULES OF INTEGRATION

1. The integral of the product of a constant and a function is equal to the product of the constant and integral of the function. Symbolically we write $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a constant. For example: $\int 5x^3\,dx = 5.\frac{x^4}{4} + C$, where C is an integrating constant.

2. The integral of the sum or difference of functions is equal to the sum or difference of their integrals. Symbolically we write

$$\int [f_1(x) + f_2(x) + \ldots\ldots + f_n(x)]dx = \int f_1(x)dx + \int f_2(x)dx + \ldots\ldots + \int f_n(x)dx\,.$$

For example: $\int (x^3 + x^4 + x^2)dx = .\frac{x^4}{4} + \frac{x^5}{5} + \frac{x^3}{3} + C$, where C is an integrating constant.

13.6 METHOD OF SUBSTITUTION

This is a technique of reducing the integrand to one or other of the fundamental forms by the change of independent variable. Suppose we want to evaluate $\int f(x)dx$ by this method. Let, I= $\int f(x)\,dx$. Then by definition, $\frac{dI}{dx} = f(x)$.

Now we substitute, $x = \phi'(z) \therefore \frac{dx}{dz} = \phi'(z)$.

Therefore, $\frac{dI}{dz} = \frac{dI}{dx}.\frac{dx}{dz} = f(x)\,\phi'(z)$

or, $\frac{dI}{dz} = f\{\phi(z)\}\phi'(z)$ [$\because x = \phi(z)$]

Hence, by definition, $I = \int f\{\phi(z)\}\phi'(z)dz$

or, $f(x)dx = \int f\{\phi(z)\}\phi'(z)dz$.

Remarks:

1. We shall try to determine the substitution equation $x = \phi(z)$ in such a manner that the integrand $f\{\phi(z)\}\phi'(z)$ may reduce to a known standard form. Then by integration evaluate $\int f(x)dx$ as a function of z. Finally, using the relation $x = \phi(z)$ express $\int f(x)dx$ as a function of x.
2. Since $x = \phi(z)$, $\therefore dx = \phi'(z)dz$. Hence, to represent $\int f(x)dx$ as a function of z.

3. Sometimes it is found convenient to make the substitution in the form $F(x)dx = dz$ Taking differential we get, $F'(x)dx = dz.$ Now, using the relations $F(x) = z \, and \, F'(x)dx = dz$ transform $\int f(x)dx$ into the form $\int \phi(z)dz$.
4. Clearly, the substitution equation $x = \phi(z)$ or, $F(x) = z$ will be different for different integrals. Sometimes given integral can be evaluated by different substitutions.
5. There is no hard and fast rule for the evaluation of an integral by the substitution method. It solely depends on experience or observation. One should try to reduce the integrand in a known standard form in terms of the new variable. However, we can have some idea about the equation of transformation from the difficulty we face to obtain the integral and how that can be eliminated. In this connection, remember that all integrals cannot be obtained by this process.

SOME IMPROTANT FORMS

(a) $\int f(ax+b)dx$ Can be obtained by substitution method. Here the substitution equation is $ax + b = z.$

(b) $\int [f(x)]^n f'(x)dx$ Can be obtained by substitution method. Here the substitution is $f(x) = z$

Result: i. $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$, when $n \neq -1.$

ii. $\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$, when $n = -1.$

(c) $\int e^x [f(x) + f'(x)]dx$ can be obtained by substitution method. Here the substitution equation is $e^x f(x) = z.$

Result: $fe^x [f(x) + f'(x)]dx = e^x f(x) + c.$

13.7 SOME STANDARD INTEGRALS

Formulas:

1. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$
2. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$
3. $\int \frac{dx.}{\sqrt{x^2 - a^2}} = \ln\left\{\frac{1}{a}(x + \sqrt{x^2 - a^2})\right\} + c.$
4. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left\{\frac{1}{a}\left(x + \sqrt{x^2 + a^2}\right)\right\} + c$

5. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x} + c$

6. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \frac{x-a}{x+a} + c$

13.8 INTEGRATION BY PARTS

Let u and v be two functions of x. From differential calculus, we have $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Integrating both sides w. r. to x, we have

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx.$$

Transposing, we get $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$.....(1)

If $u = f(x)$ and $\frac{dv}{dx} = \Phi(x)$ then, $v = \int \Phi\ (x)dx$

From (i), we get

$$\int f(x)\Phi\ (x)\,dx = f(x)\left\{\int \Phi\ (x)\,dx\right\} - \int\left[\frac{d}{dx}f(x)\left\{\int \Phi\ (x)\,dx\right\}\right]dx$$

Therefore, the integral of the product of two functions is equal to:

First function ×integral of the second function –integral of (the derivative of the first × integral of the second function).

Formula: $\int uvdx = u\int vdx - \int\left[\frac{d}{dx}(u)\left\{\int vdx\right\}\right]dx$

For example: Integration the function: xe^x

Solution: $\int xe^x dx$

$$= x\int e^x dx - \int\left\{\frac{d}{dx}(x)\int e^x dx\right\}dx \quad \left[\because \int uvdx = \int\left\{\frac{d}{dx}(u)\int vdx\right\}dx\right]$$

$$= xe^x - \int(1.e^x)dx = xe^x - \int e^x dx$$

$$= xe^x - e^x + c = (x-1)e^x + c \quad \text{[Ans.]}$$

ILLUSTRATIONS

Illustration-01: Integrate the functions: (a) $11x^{6/5}$ (b) $7\sqrt[3]{x^4}$

Solution: (a) $\int(11x^{6/5})dx = 11\int(x^{6/5})dx = 11\times\frac{x^{\frac{6}{5}+1}}{\frac{6}{5}+1}+c \quad \left[\because \int(x^n)dx = \frac{x^{n+1}}{n+1}+c\right]$

$$= 11\times\frac{x^{\frac{11}{5}}}{\frac{11}{5}}+c = 11\times\frac{5}{11}x^{\frac{11}{5}}+c = 5x^{\frac{11}{5}}+c \quad \text{[Ans]}$$

(b) $\int(7\sqrt[3]{x^4})dx = 7\int\left(x^{\frac{4}{3}}\right)dx = 7\times\frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+c}+c \quad \left[\because \int(x^n)dx = \frac{x^{n+1}}{n+1}+c\right]$

$$= 7\times\frac{x^{\frac{7}{3}}}{\frac{7}{3}}+c = 7\frac{3}{7}x^{\frac{7}{3}}+c = 3x^{\frac{7}{3}}+c \quad \text{[Ans]}$$

Illustration-02: Find the values of (a) $\int\left(2^x+\frac{1}{2}e^{-x}+\frac{4}{x}-\frac{1}{\sqrt[3]{x}}\right)dx,$

(b) $\int(7x^2-3x+8-\frac{1}{\sqrt{x}}+\frac{1}{x}+\frac{1}{x^2})\,dx$

Solution: (a) $\int\left(2^x+\frac{1}{2}e^{-x}+\frac{4}{x}-\frac{1}{\sqrt[3]{x}}\right)dx = \int(2^x)dx+\int\left(\frac{1}{2}e^{-x}\right)dx+4\int\left(\frac{1}{x}\right)dx-\int\left(x^{\frac{1}{3}}\right)dx$

$$= 2^x\frac{1}{\log 2}+\frac{1}{2}.\frac{e^{-x}}{\frac{d}{dx}(-x)}+4\log x-\frac{x^{\frac{1}{3}+1}}{-\frac{1}{3}+1}+c = \frac{2^x}{\log 2}+\frac{1}{2}.\frac{e^{-x}}{-1}+4\log x-\frac{x^{\frac{2}{3}}}{\frac{2}{3}}+c$$

$$= \frac{2^x}{\log 2}-\frac{1}{2}e^{-x}+4\log x-\frac{3}{2}x^{\frac{2}{3}}+c \quad \text{[Ans]}$$

(b) $\int\left(7x^2-3x+8-\frac{1}{\sqrt{x}}+\frac{1}{x}+\frac{1}{x^2}\right)dx$

$$= \int(7x^2)\,dx-\int(3x)dx+\int(8)\,dx-\int\left(\frac{1}{\sqrt{x}}\right)dx+\int\left(\frac{1}{x}\right)dx+\int\left(\frac{1}{x^2}\right)dx$$

$$= 7.\frac{x^3}{3} - 3.\frac{x^2}{2} + 8x + \frac{x^{\frac{1}{2}+1}}{-\frac{1}{2}+1} + \log x + \frac{x^{-2+1}}{-2+1} + c$$

$$= \frac{7}{3}x^3 - \frac{3}{2}x^2 + 8x + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \log x + \frac{x^{-1}}{-1} + c = \frac{7}{3}x^3 - \frac{3}{2}x^2 + 8x + 2\sqrt{x} + \log x - \frac{1}{x} + c \quad \text{[Ans]}$$

Illustration-03: Integrate (a) $\frac{x^2 - 3x + \sqrt[3]{x} + 7}{\sqrt{x}}$ (b) $\frac{ax^3 + bx^2 + cx + d}{x}$

Solution: (a) Try yourself

(b) $$\int\left(\frac{ax^3 + bx^2 + cx + d}{x}\right)dx = \int\left(\frac{ax^3}{x} + \frac{bx^2}{x} + \frac{cx}{x} + \frac{d}{x}\right)dx$$

$$= \int\left(ax^2 + bx + c + \frac{1}{x}d\right)dx \quad = \int(ax^2)\,dx + \int(bx)\,dx + \int(c)\,dx + d\int\left(\frac{1}{x}\right)dx$$

$$= a.\frac{x^3}{3} + b.\frac{x^2}{2} + cx + d\ln x + c \quad = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d\ln x + c \quad \text{[Ans]}$$

Illustration-04: Evaluate: (a) $\int\left(8e^x - 4a^x + 3\cos x + \sqrt[4]{x}\right)dx$, (b) $\int\frac{\sin x}{1+\sin x}dx$

Solution: (a) $\int\left(8e^x - 4a^x + 3\cos x + \sqrt[4]{x}\right)dx = 8\int e^x dx - 4\int a^x dx + 3\int\cos x dx + \int\sqrt[4]{x}dx$

$$= 8e^x - \frac{4a^x}{\log a} + 3\sin x + \frac{4}{5}x^{\frac{5}{4}} + C \text{ [Ans.]}$$

(b) $$\int\frac{\sin x}{1+\sin x}dx = \int\left(\frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\right)dx = \int\frac{\sin x - \sin^2 x}{\cos^2 x}dx = \int\frac{\sin x}{\cos^2 x}dx - \int\frac{\sin^2 x}{\cos^2 x}dx$$

$$= \int\sec x\tan x dx - \int\tan^2 x dx = \int\sec x\tan x dx - \int(\sec^2 x - 1)dx = \sec x - \tan x + x + c \text{ [Ans.]}$$

Illustration-05: Integrate the following functions with respect to x:

(a) $\sqrt{x} - \frac{1}{\sqrt{x}}$, (b) $\frac{4x^6 + 3x^5 + 2x^4 + x^3 + x^2 + 1}{x^3}$

Solution: (a) Let $I = \int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx = \int\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)dx = \int x^{\frac{1}{2}}dx - \int x^{-\frac{1}{2}}dx$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \quad = \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c \quad \text{[Ans.]}$$

(b) Let $I = \int\left(\frac{4x^6+3x^5+2x^4+x^3+x^2+1}{x^3}\right)dx$

$= \int\left(\frac{4x^6}{x^3}+\frac{3x^5}{x^3}+\frac{2x^4}{x^3}+\frac{x^3}{x^3}+\frac{x^2}{x^3}+\frac{1}{x^3}\right)dx \quad = \int\left(4x^3+3x^2+2x+1+\frac{1}{x}+\frac{1}{x^3}\right)dx$

$= \int(4x^3)\,dx+\int(3x^2)\,dx+\int(2x)\,dx+\int(1)\,dx+\int\left(\frac{1}{x}\right)dx+\int(x^{-3})\,dx$

$= 4.\frac{x^4}{4}+3.\frac{x^3}{3}+2.\frac{x^2}{2}+x+\ln x+\frac{x^{-2}}{-2}+c \quad = x^4+x^3+x^2+x+\log x-\frac{1}{2x^2}+c$ [Ans]

Illustration-06: Find the value of the integral: $\int\frac{x^4+x^2+1}{2(x^2+1)}dx,$

Solution: $\int\left(\frac{x^4+x^2+1}{2(x^2+1)}\right)dx = \int\left[\frac{x^4}{2(x^2+1)}+\frac{x^2+1}{2(x^2+1)}\right]dx = \int\left[\frac{x^4-1+1}{2(x^2+1)}+\frac{1}{2}\right]dx$

$= \int\left[\frac{x^4-1}{2(x^2+1)}+\frac{1}{2(x^2+1)}+\frac{1}{2}\right]dx = \int\left[\frac{(x^2+1)(x^2-1)}{2(x^2+1)}+\frac{1}{2(x^2+1)}+\frac{1}{2}\right]dx$

$= \int\left[\frac{x^2-1}{2}+\frac{1}{2}.\frac{1}{x^2+1}+\frac{1}{2}\right]dx \quad = \int\left[\frac{x^2-1}{2}+\frac{1}{2}.\frac{1}{x^2+1}+\frac{1}{2}\right]dx$

$= \int\left[\frac{x^2}{2}-\frac{1}{2}+\frac{1}{2}.\frac{1}{x^2+1}+\frac{1}{2}\right]dx \quad = \int\left[\frac{1}{2}x^2+\frac{1}{2}.\frac{1}{x^2+1}\right]dx$

$= \frac{1}{2}\int[x^2]\,dx+\frac{1}{2}\int\left[\frac{1}{x^2+1}\right]+dx \quad = \frac{1}{2}.\frac{x^3}{3}+\frac{1}{2}\tan^{-1}x+c$

$= \frac{1}{2}\left(\frac{x^3}{3}+\tan^{-1}x\right)+c$ [Ans.]

Illustration-07: Find the value of the integral $\int\frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}dx.$

Solution: $\int\left(\frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}\right)dx \quad = \int\left(\frac{\sin^3 x}{\sin^2 x\cos^2 x}+\frac{\cos^3 x}{\sin^2 x\cos^2 x}\right)dx$

$= \int\left(\frac{\sin x}{\cos^2 x}+\frac{\cos x}{\sin^2 x}\right)dx \quad = \int(\tan x\sec x+\cot x\operatorname{cosec}x)\,dx$

$= \int(\tan x\sec x)dx+\int(\cot x\operatorname{cosec}x)\,dx = \sec x-\operatorname{cosec}x+c$ [Ans.]

Illustration-08: Integrate $\frac{\cos 2x}{\cos^2 x \sin^2 x}$ with respect to x.

Solution: $\int\left(\frac{\cos 2x}{\cos^2 x \sin^2 x}\right)dx = \int\left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x}\right)dx$

$= \int\left(\frac{\cos^2 x}{\cos^2 x \sin^2 x} - \frac{\sin^2 x}{\cos^2 x \sin^2 x}\right)dx = \int\left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}\right)dx$

$= \int(\operatorname{cosec}^2 x - \sec^2 x)\,dx = \int(\operatorname{cosec}^2 x)dx - \int(\sec^2 x)\,dx$

$= -\cot x - \tan x + c = -(\cot x + \tan x) + c$ [Ans.]

By the method of substitution:

Illustration-09: Find the values of the integrals: (a) $\int e^{3+2\sin x}\cos x dx$,

(b) $\int(ax^2+2bx+c)^n(ax+b)dx$, (c) $\int\frac{1}{x(\log x)^2}dx$.

Solution:

(a) Let $I = \int e^{3+2\sin x}\cos x dx$

Put $z = 3 + 2\sin x$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(3+2\sin x)$

$\Rightarrow \frac{dz}{dx} = 2\cos x$

$\Rightarrow 2\cos x dx = dz$

$= \frac{1}{2}\int e^{3+2\sin x} 2\cos x dx$

$= \frac{1}{2}\int e^z dz$

$= \frac{1}{2}e^z + c$

$= \frac{1}{2}e^{3+2\sin x} + c$ [Ans.]

(b) Let $I = \int(ax^2+2bx+c)^n(ax+b)dx$

Put $z = ax^2 + 2bx + c$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(ax^2+2bx+c)$

$\Rightarrow \frac{dz}{dx} = 2ax + 2b$

$\Rightarrow dz = (2ax+2b)dx$

$\Rightarrow 2(ax+b)dx = dz$

$= \frac{1}{2}\int(ax^2+2bx+c)^n 2(ax+b)dx$

$= \frac{1}{2}\int z^n dz$

$= \frac{1}{2}\cdot\frac{z^{n+1}}{n+1} + c$

$= \frac{(ax^2+2bx+c)^{n+1}}{2(n+1)} + c$ [Ans.]

(c) Let $I = \int \frac{1}{x(\log x)^2} dx$

Put $z = \log x$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(\log x)$

$\Rightarrow \frac{dz}{dx} = \frac{1}{x}$

$\Rightarrow \frac{1}{x} dx = dz$

$= \int \frac{1}{(\log x)^2} \cdot \frac{1}{x} dx$

$= \int \frac{1}{z^2} dz$

$= \int z^{-2} dz$

$= \frac{z^{-2+1}}{-2+1} + c = \frac{z^{-1}}{-1} = -\frac{1}{z} + c = -\frac{1}{\log x} + c$ [*Ans.*]

Illustration-10: Integrate the following with respect to x:

(a) $(x^3+2)^{\frac{1}{2}} x^2$, (b) $\frac{x}{\sqrt{3x^2+1}}$, (c) $\frac{8x^2}{(x^3+2)^3}$, (d) $\frac{x^2}{(x^3+2)^{\frac{1}{4}}}$

Solution: (a) $I = \int (x^3+2)^{\frac{1}{2}} x^2 dx$

Put $z = x^3 + 2$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(x^3+2) \Rightarrow \frac{dz}{dx} = 3x^2 \Rightarrow dz = 3x^2 dx \Rightarrow x^2 dx = \frac{dz}{3}$

Now $I = \int z^{\frac{1}{2}} \frac{dz}{3} = \frac{1}{3} \int z^{\frac{1}{2}} dz = \frac{1}{3} \cdot \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{1}{3} \times \frac{2}{3} \times z^{\frac{3}{2}} + c = \frac{2}{9} z^{\frac{3}{2}} + c$

$= \frac{2}{9}(x^3+2)^{\frac{3}{2}} + c$ $[\because x^3 + 2 = z]$ **[Ans.]**

(b) $I = \int \frac{x dx}{\sqrt{3x^2+1}}$

Put $z = 3x^2 + 1$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(3x^2+1) = \frac{dz}{dx} 3 \times 2x \Rightarrow dz = 6x dx \Rightarrow x dx = \frac{dz}{6}$

Now $I = \int \frac{1}{\sqrt{z}} \frac{dz}{6} = \frac{1}{6} \int z^{\frac{-1}{2}} dz = \frac{1}{6} \times \frac{z^{\frac{-1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{1}{6} \times \frac{2}{1} \times z^{\frac{1}{2}} + c = \frac{1}{3} z^{\frac{1}{2}} + c$

$$=\frac{1}{3}\sqrt{z}+c=\frac{1}{3}\sqrt{3x^2+1}+c \qquad [\because 3x^2+1=z] \quad [\text{Ans.}]$$

(c) $I=\int\frac{8x^2}{\left(x^3+2\right)^3}dx$

Put $z=x^3+2$

$$\therefore \frac{dz}{dx}=\frac{d}{dx}\left(x^3+2\right)\Rightarrow \frac{dz}{dx}=3x^2 \Rightarrow dz=3x^2dx \Rightarrow x^2dx=\frac{dz}{3} \Rightarrow 8x^2dx=\frac{8}{3}dz$$

Now $I=\int\frac{1}{z^3}.\frac{8}{3}dz=\frac{8}{3}\int z^{-3}dz=\frac{8}{3}\times\frac{z^{-3+1}}{-3+1}+c=\frac{8}{3}\times\frac{1}{-2}z^{-2}+c$

$$=-\frac{4}{3}.\frac{1}{z^2}+c=-\frac{4}{3\left(x^3+2\right)^2}+c \quad [\because z=x^3+2] \quad [\text{Ans.}]$$

(d) $I=\int\frac{x^2dx}{\left(x^3+2\right)^{\frac{1}{4}}}$

Put $z=x^3+2$

$$\therefore \frac{dz}{dx}=\frac{d}{dx}\left(x^3+2\right)\Rightarrow \frac{dz}{dx}=3x^2 \Rightarrow dz=3x^2dx \Rightarrow x^2dx=\frac{dz}{3}$$

Now $I=\int\frac{1}{z^{\frac{1}{4}}}.\frac{dz}{3}=\frac{1}{3}\int z^{-\frac{1}{4}}dz=\frac{1}{3}\times\frac{z^{-\frac{1}{4}+1}}{-\frac{1}{4}+1}+c=\frac{1}{3}\times\frac{4}{3}z^{\frac{3}{4}}+c=\frac{4}{9}z^{\frac{3}{4}}+c$

$$=\frac{4}{9}\left(x^3+2\right)^{\frac{3}{4}}+c \quad [\therefore x^3+2=z] \qquad [\text{Ans.}]$$

Illustration-11: Find the value of the following integrals:

(a) $\int\frac{x^5dx}{1+x^{12}}$, (b) $\int\frac{e^x(1+x)}{\cos^2\left(xe^x\right)}dx$, (c) $\int\frac{\cot x}{\log\sin x}dx$, (d) $\int\frac{\sin 2x}{1+\sin^2 x}dx$.

Solution: (a) $I=\frac{x^5dx}{1+x^{12}}$

Put, $z=x^6$

$$\therefore \frac{dz}{dx}=\frac{d}{dx}\left(x^6\right)\Rightarrow \frac{dz}{dx}=6x^5 \Rightarrow dz=6x^5dx \Rightarrow x^5dx=\frac{dz}{6}$$

Now $I=\int\frac{x^5dx}{1+\left(x^6\right)^2}=\int\frac{1}{1+z^2}.\frac{dz}{6}=\frac{1}{6}\int\frac{dz}{1+z^2}=\frac{1}{6}\tan^{-1}+z+c$

$$\frac{1}{6}\tan^{-1}\left(x^6\right)+c \quad [\text{Ans.}]$$

(b) $I = \int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$

Put $z = xe^x$

$$\therefore \frac{dz}{dx} = \frac{d}{dx}(xe^x) \Rightarrow \frac{dz}{dx} = xe^x + e^x + 1 \Rightarrow dz = (xe^x + e^x \times 1)dx \Rightarrow e^x(1+x)dx = dz$$

Now $I = \int \frac{dz}{\cos^2 z} = \int \sec^2 z dz = \tan z + c$

$= \tan(xe^x) + c \quad [\because xe^x = z]$ [Ans.]

(c) $I = \int \frac{\cot x}{\log \sin x}dx$

put $\log \sin x = z$

$$\therefore \frac{dz}{dx} = \frac{d}{dx}(\log \sin x) \Rightarrow \frac{dz}{dx} = \frac{1}{\sin x}\cos x \Rightarrow \frac{dz}{dx} = \cot x \Rightarrow \cot x dx = dz$$

Now $I = \int \frac{1}{z}dz = \log z + c = \log(\log \sin x) + c$ [Ans.]

(d) Let $I = \int \frac{\sin 2x}{1+\sin^2 x}dx$

$= \int \frac{2\sin x \cos x}{1+\sin^2 x}dx$

$= \int \frac{1}{1+\sin^2 x}.2\sin x \cos x dx$

$= \int \frac{1}{z}dz = \log z + c$

$\log(1+\sin^2 x) + c$ [Ans.]

Put $z = 1+\sin^2 x$

$$\therefore \frac{dz}{dx} = \frac{d}{dx}(1+\sin^2 x)$$

$$\Rightarrow \frac{dz}{dx} = 2\sin x(\cos x)$$

$$\Rightarrow 2\sin x \cos x dx = dz$$

Illustration-12: Find the integral value of the following functions:

(a) $\frac{\sqrt{\tan^{-1} x}}{2(1+x^2)}$, (b) $\frac{3\cos x}{1+\sin^2 x}$.

Solution :

(a) Let $I = \int \frac{\sqrt{\tan^{-1} x}}{2(1+x^2)}dx$

$= \frac{1}{2}\int \sqrt{\tan^{-1} x}.\frac{1}{1+x^2}dx = \frac{1}{2}\int \sqrt{z}\,dz = \frac{1}{2}\int z^{\frac{1}{2}}dz$

Put $z = \tan^{-1} x$

$$\therefore \frac{dz}{dx} = \frac{d}{dx}(\tan^{-1} x)$$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{1}{1+x^2}dx = dz$$

$$=\frac{1}{2}\cdot\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c=\frac{1}{2}\cdot\frac{z^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{1}{3}\left(\tan^{-1}x\right)^{\frac{3}{2}}+c \qquad \textbf{[Ans.]}$$

(b) Let $I=\int\frac{3\cos x}{1+\sin^2 x}dx$

Put $z=\sin x$

$\therefore \frac{dz}{dx}=\frac{d}{dx}(\sin x)$

$\Rightarrow \frac{dz}{dx}=\cos x$

$\Rightarrow \cos x dx=dz$

$=3\int\frac{1}{1+\sin^2 x}\cos x dx$

$=3\int\frac{1}{1+z^2}dz$

$=3\tan^{-1}z+c$

$=3\tan^{-1}(\sin x)+c$ **[Ans.]**

Illustration-13: Integrate the following functions with respect to x:

(a) $\frac{3}{x\left[1+(\log x)^2\right]}$, (b) $\frac{4\cos x}{(1+\sin x)^2}$, (c) $\frac{\sin(2+3\log x)}{x}$.

Solution:

(a) Let $I=\int\frac{3}{x\left[1+(\log x)^2\right]}dx$

Put $z=\log x$

$\therefore \frac{dz}{dx}=\frac{d}{dx}(\log x)$

$\Rightarrow \frac{dz}{dx}=\frac{1}{x}$

$\Rightarrow \frac{1}{x}dx=dz$

$=3\int\frac{1}{1+(\log x)^2}\cdot\frac{1}{x}dx$

$=3\int\frac{1}{1+z^2}dz$

$=3\tan^{-1}z+c$

$=3\tan^{-1}(\log x)+c$ [Ans.]

(b) Let $I=\int\frac{4\cos x}{(1+\sin x)^2}dx$

Put $z=1+\sin x$

$\therefore \frac{dz}{dx}=\frac{d}{dx}(1+\sin x)$

$\Rightarrow \frac{dz}{dx}=\cos x$

$\Rightarrow \cos x dx=dz$

$=4\int\frac{1}{(1+\sin x)^2}\cos x dx$

$=4\int\frac{1}{z^2}dz$

$=4\int z^{-2}dz$

$=4.\frac{z^{-2+1}}{-2+1}+c=4.\frac{z^{-1}}{-1}+c=-\frac{4}{1+\sin x}+c$ [Ans.]

(c) Let $I = \int \frac{\sin(2+3\log x)}{x} dx$

Put $z = 2+3\log x$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(2+3\log x)$

$\Rightarrow \frac{dz}{dx} = 3.\frac{1}{x}$

$\Rightarrow \frac{3}{x} dx = dz$

$= \frac{1}{3}\int \sin(2+3\log x).\frac{3}{x} dx$

$= \frac{1}{3}\int \sin z dx$

$= \frac{1}{3}.(-\cos z)$

$= -\frac{1}{3}\cos(2+3\log x)+c$ **[Ans.]**

Illustration-14: Find the value of

(a) $\int \frac{\sin x}{3+7\cos x} dx$, (b) $\int \frac{\cos x}{9-2\sin x} dx$, (c) $\int \frac{\sec^2 x}{\sqrt{4-3\tan x}} dx$.

Solution:

(a) $\int \frac{\sin x}{3+7\cos x} dx$

Put $z = 3+7\cos x$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(3+7\cos x)$

$\Rightarrow \frac{dz}{dx} = -7\sin x$

$\Rightarrow -7\sin x dx = dz$

$= \frac{1}{-7}\int \frac{1}{3+7\cos x}(-7\sin x) dx$

$= -\frac{1}{7}\int \frac{1}{z} dz = -\frac{1}{7}\log z + c$

$= \frac{-1}{7}\log(3+7\cos x)+c$ **[Ans.]**

(b) $\int \frac{\cos x}{9-2\sin x} dx$

Put $z = 9-2\sin x$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(9-2\sin x)$

$\Rightarrow \frac{dz}{dx} = -2\cos x$

$\Rightarrow -2\cos x dx = dz$

$= \frac{1}{-2}\int \frac{1}{9-2\sin x}(-2\cos x) dx$

$= -\frac{1}{2}\int \frac{1}{z} dz = -\frac{1}{2}\log z$

$= \frac{-1}{2}\log(9-2\sin x)$ **[Ans.]**

(c) $\int \frac{\sec^2 x}{\sqrt{4-3\tan x}}dx$

$= \frac{1}{-3}\int \frac{1}{\sqrt{4-3\tan x}}(-3\sec^2 x)dx$

$= -\frac{1}{3}\int \frac{1}{\sqrt{z}}dz = -\frac{1}{3}\int z^{-\frac{1}{2}}dz$

$= -\frac{1}{3}.\frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c = -\frac{1}{3}.\frac{z^{\frac{1}{2}}}{\frac{1}{2}}+c$

$= -\frac{1}{3}.\frac{2}{1}\sqrt{z}+c = -\frac{2}{3}\sqrt{4-3\tan x}+c$ **[Ans.]**

Put $z = 4-3\tan x$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(4-3\tan x)$

$\Rightarrow \frac{dz}{dx} = -3\sec^2 x$

$\Rightarrow -3\sec^2 x dx = dz$

Illustration-15: Find the integrals of the following functions with respect to x:

(a) $\left[\frac{\sec x}{(1-\tan x)^3}\right]^2$, (b) $\frac{\sec 3x\tan 3x}{2\sec 3x-5}$, (c) $\frac{\cos x\sin x}{5+\cos^2 x}$.

Solution:

(a) Let $I = \int\left[\frac{\sec x}{(1-\tan x)^3}\right]^2 dx$

$= \int \frac{\sec^2 x}{(1-\tan x)^6}dx$

$= -\int \frac{1}{(1-\tan x)^6}(-\sec^2 x)dx$

$= -\int \frac{1}{z^6}dz$

$= -\int z^{-6}dz = -\frac{z^{-6+1}}{-6+1}+c = \frac{1}{5z^5}+c$

$= \frac{1}{5(1-\tan x)^5}+c$ [Ans]

Put, $z = 1-\tan x$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(1-\tan x)$

$\Rightarrow \frac{dz}{dx} = -\sec^2 x$

$\Rightarrow -\sec^2 x dx = dz$

(b) Let $I = \int \frac{\sec 3x\tan 3x}{2\sec 3x-5}dx$

$= \frac{1}{6}\int \frac{1}{2\sec 3x-5}6\sec 3x\tan 3x dx$

$= \frac{1}{6}\int \frac{1}{z}dz$

Put, $z = 2\sec 3x-5$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(2\sec 3x-5)$

$\Rightarrow \frac{dz}{dx} = 2\sec 3x\tan 3x.3$

$$= \frac{1}{6}\log z + c$$

$$= \frac{1}{6}\log(2\sec 3x - 5) + c \quad \text{[Ans]}$$

(c) Let $I = \int \frac{\cos x \sin x}{5 + \cos^2 x} dx$

$$= \frac{1}{-2}\int \frac{1}{5 + \cos^2 x}(-2\cos x \sin x)dx$$

$$= -\frac{1}{2}\int \frac{1}{z} dz$$

$$= -\frac{1}{2}\log z + c = -\frac{1}{2}\log(5 + \cos^2 x) + c \text{ [Ans.]}$$

$\Rightarrow 6\sec 3x \tan 3x dx = dz$

Put, $z = 5 + \cos^2 x$

$\therefore \frac{dz}{dx} = \frac{d}{dx}(5 + \cos^2 x)$

$\Rightarrow \frac{dz}{dx} = 2\cos x(-\sin x)$

$\Rightarrow -2\cos x \sin x dx = dz$

SOME STANDARD INTEGRALS

Formulas:

1. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$
2. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
3. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left\{\frac{1}{a}(x + \sqrt{x^2 - a^2})\right\} + c.$
4. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left\{\frac{1}{a}\left(x + \sqrt{x^2 + a^2}\right)\right\} + c$
5. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\frac{a + x}{a - x} + c$
6. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\frac{x - a}{x + a} + c$

Illustration-16: Evaluate the following: (i) $\int \frac{dx}{3x^2 + 2x + 5}$, (ii) $\int \frac{dx}{2x^2 - x - 1}$.

Solution: (i) Let $I = \int \frac{dx}{3x^2 + 2x + 5}$

$$= \int \frac{dx}{3\left(x^2 + \frac{2}{3}x + \frac{5}{3}\right)} \quad = \frac{1}{3}\int \frac{dx}{\left(x + \frac{1}{3}\right)^2 + \frac{14}{9}} \quad = \frac{1}{3}\int \frac{dx}{\left(x + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{14}}{3}\right)^2}$$

$$= \frac{1}{3} \cdot \frac{1}{\frac{\sqrt{14}}{3}} \tan^{-1} \frac{x + \frac{1}{3}}{\frac{\sqrt{14}}{3}} + c \quad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \right]$$

$$= \frac{1}{\sqrt{14}} \tan^{-1} \frac{3x + 1}{\sqrt{14}} + c \quad \text{[Ans.]}$$

(ii) Let $I = \int \frac{dx}{2x^2 - x - 1} = \int \frac{dx}{2\left(x^2 - \frac{1}{2}x - \frac{1}{2}\right)} = \frac{1}{2} \int \frac{dx}{\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}} = \frac{1}{2} \int \frac{dx}{\left(x - \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$

$$= \frac{1}{2} \cdot \frac{1}{2\frac{3}{4}} \cdot \ln \frac{x - \frac{1}{4} - \frac{3}{4}}{x - \frac{1}{4} + \frac{3}{4}} \left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \frac{x - a}{x + a} \right] = \frac{1}{2} \cdot \frac{1}{\frac{3}{2}} \cdot \ln \frac{4x - 4}{4x + 2} = \frac{1}{2} \cdot \frac{2}{3} \cdot \ln \frac{4(x-1)}{2(2x+1)}$$

$$= \frac{1}{3} \ln \left\{ \frac{2(x-1)}{2x+1} \right\} \quad \text{[Ans]}$$

Illustration-17: Evaluate the integrals: (i) $\int \frac{3dx}{\sqrt{15 - 6x - x^2}}$, (ii) $\int \frac{dx}{\sqrt{2 + 3x - 2x^2}}$.

Solution: (i) Let $I = \int \frac{3dx}{\sqrt{15 - 6x - x^2}}$

$$= 3 \int \frac{dx}{\sqrt{24 - (x+3)^2}} = 3 \int \frac{dx}{\sqrt{(\sqrt{24})^2 - (x+3)^2}} = 3 \sin^{-1} \left(\frac{x+3}{\sqrt{24}} \right) \text{[Ans]}$$

$$\left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \right]$$

(ii) Let $I = \int \frac{dx}{\sqrt{2 + 3x - 2x^2}} = \int \frac{dx}{\sqrt{2\left(1 + \frac{3}{2}x - x^2\right)}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{1 - \left(x^2 - \frac{3}{2}x\right)}}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{25}{16} - \left(x - \frac{3}{4}\right)^2}} = \frac{1}{\sqrt{2}} \sin^{-1} \frac{x - \frac{3}{4}}{\frac{5}{4}} \quad \left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \right]$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{4x - 3}{5} \right) \quad \text{[Ans]}$$

INTEGRATION OF SPECIAL TRIGONOMETRIC FUNCTIONS

Illustration-18: Evaluate the following integrals:

(i) $\int \frac{dx}{5-4\cos x}$, (ii) $\int \frac{dx}{4+5\sin x}$ (iii) $\int \frac{dx}{2-3\sin 2x}$

Solution:

(i) Let $I = \int \frac{dx}{5-4\cos x}$

Put $z = \tan\frac{x}{2}$

$$= \int \frac{dx}{5-4.\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} = \int \frac{\frac{2}{1+z^2}dz}{5-4.\frac{1-z^2}{1+z^2}}$$

$$\Rightarrow \frac{dz}{dx} = \frac{d}{dx}\left(\tan\frac{x}{2}\right)$$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \int \frac{\frac{2}{1+z^2}dz}{\frac{5+5z^2-4+4z^2}{1+z^2}} = \int \frac{2}{9z^2+1}dz$$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) \Rightarrow \frac{dz}{dx} = \frac{1}{2}\left(1+z^2\right)$$

$$= \int \frac{2}{(3z)^2+1}dz = \frac{2}{3}\tan^{-1}(3z)$$

$$\Rightarrow \frac{dx}{dz} = \frac{2}{1+z^2} \Rightarrow dx = \frac{2}{1+z^2}dz$$

$$= \frac{2}{3}\tan^{-1}\left(3\tan\frac{x}{2}\right) \quad \text{[Ans]}$$

(ii) Let $I = \int \frac{dx}{4+5\sin x}$

Put $z = \tan\frac{x}{2} \therefore dz = \frac{1}{2}\sec^2\frac{x}{2}dx$

$$= \int \frac{dx}{4+5\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$\Rightarrow dz = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx$$

$$\Rightarrow dx = \frac{2}{1+z^2}dz$$

$$= \int \frac{\frac{2}{1+z^2}dz}{4+5.\frac{2z}{1+z^2}} = \int \frac{\frac{2}{1+z^2}dz}{\frac{4+4z^2+10z}{1+z^2}}$$

$$= \int \frac{2dz}{4z^2+10z+4} = \int \frac{2dz}{4\left(z^2+\frac{5}{2}z+1\right)} = \frac{1}{2}\int \frac{dz}{\left(z+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}.\frac{1}{2\frac{3}{4}}\log\left(\frac{z+\frac{5}{4}-\frac{3}{4}}{z+\frac{5}{4}+\frac{3}{4}}\right) = \frac{1}{3}\log\left(\frac{2+4z}{8+4z}\right) = \frac{1}{3}\log\left(\frac{1+2z}{4+2z}\right)$$

$$= \frac{1}{3}\log\left(\frac{1+2\tan\frac{x}{2}}{4+2\tan\frac{x}{2}}\right) \textbf{ (Ans.)}$$

(iii) Let $I = \int \frac{dx}{2-3\sin 2x}$

Put $z = \tan x$

$$= \int \frac{dx}{2-3.\frac{2\tan x}{1+\tan^2 x}}$$

$$\Rightarrow \frac{dz}{dx} = \frac{d}{dx}(\tan x) \Rightarrow \frac{dz}{dx} = \sec^2 x$$

$$= \int \frac{\frac{1}{1+z^2}dz}{2-3.\frac{2z}{1+z^2}} \quad \therefore \quad \Rightarrow \frac{dz}{dx} = \left(1+\tan^2 x\right) \Rightarrow \frac{dz}{dx} = \left(1+z^2\right)$$

$$\Rightarrow dx = \frac{1}{1+z^2}dz$$

$$= \int \frac{\frac{1}{1+z^2}dz}{\frac{2+2z^2-6z}{1+z^2}} = \int \frac{1}{2z^2-6z+2}dz = \frac{1}{2}\int \frac{1}{z^2-3z+1}dz = \frac{1}{2}\int \frac{dz}{\left(z-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= \frac{1}{2}.\frac{1}{2.\frac{\sqrt{5}}{2}}\log\left(\frac{z-\frac{3}{2}-\frac{\sqrt{5}}{2}}{z-\frac{3}{2}+\frac{\sqrt{5}}{2}}\right) = \frac{1}{2\sqrt{5}}\log\left(\frac{2z-3-\sqrt{5}}{2z-3+\sqrt{5}}\right)$$

$$= \frac{1}{2\sqrt{5}}\log\left(\frac{2\tan x-3-\sqrt{5}}{2\tan x-3+\sqrt{5}}\right) \textbf{ [Ans.]}$$

INTEGRATION BY PARTS

Let u and v be two functions of x. From differential calculus, we have $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$.

Integrating both sides w. r. to x, we have

$$uv=\int u\frac{dv}{dx}dx+\int v\frac{du}{dx}dx.$$

Transposing, we get $\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$.....(1)

If $u=f(x)$ and $\frac{dv}{dx}=\Phi(x)$ then, $v=\int\Phi\ (x)dx$

From (i), we get

$$\int f(x)\Phi\ (x)\,dx=f(x)\left\{\int\Phi\ (x)\,dx\right\}-\int\left[\frac{d}{dx}f(x)\left\{\int\Phi\ (x)\,dx\right\}\right]dx$$

Therefore, the integral of the product of two functions is equal to:
First function ×integral of the second function –integral of (the derivative of the first × integral of the second function).

Formula: $\int uvdx=u\int vdx-\int\left[\frac{d}{dx}(u)\left\{\int vdx\right\}\right]dx$

Illustration-19: Integration the functions: x^2e^x

Solution: $\int x^2e^x\,dx=x^2\int e^x\,dx-\int\left\{\frac{d}{dx}(x^2)\ \int e^x\,dx\right\}dx$

$$=x^2e^x-\int(2xe^x)\,dx=x^2e^x-2\left[x\int e^x\,dx-\int\left\{\frac{d}{dx}(x)\int e^x\,dx\right\}dx\right]$$

$$=x^2e^x-2\left(xe^x-\int e^x\,dx\right)=x^2e^x-2xe^x+2e^x+c=\left(x^2-2x+2\right)e^x+c\quad\text{[Ans]}$$

Illustration-20: Integration the functions: (i) $x\sin x$ (ii) $x^3\cos x$

Solution: (i) Let $I=\int x\sin x\,dx=x\int\sin x\,dx-\int\left\{\frac{d}{dx}(x)\int\sin x\,dx\right\}dx$

$$=x(-\cos x)-\int(1.(-\cos x))dx=-x\cos x+\int\cos x\,dx$$

$$=-x\cos x+\sin x=\sin x-x\cos x+c\quad\text{[Ans.]}$$

(ii) Let $I=\int x^3\cos x\,dx=x^3\int\cos x\,dx-\int\left\{\frac{d}{dx}(x^3)\int\cos x\,dx\right\}dx$

$$=x^3\sin x-\int3x^2\sin x\,dx=x^3\sin x-3\left[x^2\int\sin x-\int\left\{\frac{d}{dx}(x^2)\int\sin x\,dx\right\}dx\right]$$

$$=x^3\sin x-3\left[-x^2\cos x-\int\{2x(-\cos x)\}dx\right]$$

$$= x^3 \sin x - 3\left[-x^2 \cos x + 2\int x\cos x dx\right] = x^3 \sin x + 3x^2 \cos x - 6\int x\cos x dx$$

$$= x^3 \sin x + 3x^2 \cos x - 6\left[x\int \cos x dx - \int\left\{\frac{d}{dx}(x)\int \cos x dx\right\}dx\right]$$

$$= x^3 \sin x + 3x^2 \cos x - 6\left[x\sin x - \int \sin x dx\right]$$

$$= x^3 \sin x + 3x^2 \cos x - 6[x\sin x - (-\cos x)]$$

$$= x^3 \sin x + 3x^2 \cos x - 6x\sin x - 6\cos x$$

$$= x^3 \sin x - 6x\sin x + 3x^2 \cos x - 6\cos x$$

$$= (x^3 - 6x)\sin x + 3(x^2 - 2)\cos x + c \quad \text{[Ans]}$$

Illustration-21: Integration the function: (i) $x\log x$, (ii) $(\log x)^2$

Solution: (i) Let $I = \int x\log x dx = \log x\int x dx - \int\left\{\frac{d}{dx}(\log x)x dx\right\}dx$

$$= \log x\frac{x^2}{2} - \int\left\{\frac{1}{x}.\frac{x^2}{2}\right\}dx = \frac{x^2}{2}\log x - \frac{1}{2}\int x dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}.\frac{x^2}{2} \qquad = \frac{x^2}{4}2\log x - \frac{x^2}{4} \qquad = \frac{x^2}{4}(2\log x - 1) \qquad = \frac{x^2}{4}(\log x^2 - \log e)$$

$$= \frac{1}{4}x^2 \log\left(\frac{x^2}{e}\right) + c \quad \text{[Ans]}$$

(ii) Let $I = \int(\log x)^2 dx$

$$= \int(\log x)^2 (1)dx \qquad = (\log x)^2 \int 1dx - \int\left\{\frac{d}{dx}(\log x)^2 \int 1dx\right\}dx$$

$$= (\log x)^2 .x - \int 2\log x.\frac{1}{x}x dx \quad = x(\log x)^2 - 2\int \log x dx$$

$$= x(\log x)^2 - 2\int \log x(1)\, dx \qquad = x(\log x)^2 - 2\left[\log x\int 1dx - \int\left\{\frac{d}{dx}(\log x)1dx\right\}dx\right]$$

$$= x(\log x)^2 - 2\left[x\log x - \int\left\{\frac{1}{x}x\right\}dx\right] \quad = x(\log x)^2 - 2\left[x\log x - \int(1)dx\right]$$

$$= x(\log x)^2 - 2x\log x + 2x + c \quad \text{[Ans.]}$$

INTEGRATION USING PARTIAL FRACTION

Illustration-22: Integration the following function w. r. to x:

(i) $\dfrac{x}{(x-3)(x+1)}$, (ii) $\dfrac{1}{(x+1)(x+2)(x+3)}$

Solution: (i) Given $I=\int\dfrac{x}{(x-3)(x+1)}dx$

Let $\dfrac{x}{(x-3)(x+1)}=\dfrac{A}{(x-3)}+\dfrac{B}{(x+1)}$

$\Rightarrow(x-3)(x+1)\dfrac{x}{(x-3)(x+1)}=(x-3)(x+1)\left[\dfrac{A}{(x-3)}+\dfrac{B}{(x+1)}\right]$ [Multiplying both sides by $(x-3)(x+1)$]

$\Rightarrow x=(x+1)A+(x-3)B$.....(1)

Putting $x=3$ in (1), we get

$(1)\Rightarrow 3=(3+1)A+(3-3)B \quad \Rightarrow 3=4A \Rightarrow A=\dfrac{3}{4}$

Putting $x=-1$ in(1), we get

$(1)\Rightarrow -1=(-1+1)A+(-1-3)B \Rightarrow -1=-4B \Rightarrow B=\dfrac{1}{4}$

$\therefore I=\int\dfrac{x}{(x-3)(x+1)}dx$

$=\int\left[\dfrac{A}{(x-3)}+\dfrac{B}{(x+1)}\right]dx \qquad =A\int\dfrac{1}{(x-3)}dx+B\int\dfrac{1}{(x+1)}dx$

$=\dfrac{3}{4}\log(x-3)+\dfrac{1}{4}\log(x+1)+c \quad =\dfrac{1}{4}[3\log(x-3)+\log(x+1)]+c$

$=\dfrac{1}{4}[\log(x-3)^3+\log(x+1)]+c \quad =\dfrac{1}{4}\log(x-3)^3(x+1)+c$ [Ans]

(ii) Given $I=\int\dfrac{1}{(x+1)(x+2)(x+3)}dx$

let $\dfrac{1}{(x+1)(x+2)(x+3)}=\dfrac{A}{(x+1)}+\dfrac{B}{(x+2)}+\dfrac{C}{(x+3)}$

$\Rightarrow 1=(x+2)(x+3)A+(x+1)(x+3)B+(x+1)(x+2)C$.....(1)

[Multiplying both sides by $(x+1)(x+2)(x+3)$]

putting $x=-1$ in)(1), we get

$(1)\Rightarrow 1=(-1+2)(-1+3)A+(-1+1)(-1+3)B+(-1+1)(-1+2)C$

$$\Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

Putting $x = -2$ in (1), we get

$$(1) \Rightarrow 1 = (-2+2)(-2+3)A + (-2+1)(-2+3)B + (-2+1)(-2+2)C$$

$$\Rightarrow 1 = (-1)B \Rightarrow B = -1$$

Putting $x = -3$ in (1), we get

$$(1) \Rightarrow 1 = (-3+2)(-3+3)A + (-3+1)(-3+3)B + (-3+1)(-3+2)C$$

$$\Rightarrow 1 = 2C \qquad \Rightarrow C = \frac{1}{2}$$

$$\therefore I = \int \frac{1}{(x+1)(x+2)(x+3)}\,dx$$

$$= \int \left[\frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)}\right] dx \qquad = A\int \frac{1}{(x+1)}dx + B\int \frac{1}{(x+2)}dx + c\int \frac{1}{(x+3)}dx$$

$$= \frac{1}{2}\log(x+1) + (-1)\log(x+2) + \frac{1}{2}\log(x+3) + c$$

$$= \frac{1}{2}[\log(x+1) - 2\log(x+2) + \log(x+3)] + c \quad = \frac{1}{2}[\log(x+1) - \log(x+2)^2 + \log(x+3)] + c$$

$$= \frac{1}{2}\log\frac{(x+1)(x+3)}{(x+2)^2} + c \quad \text{[Ans]}$$

Illustration-23: Integrate the following functions w. r. to x:

(i) $\dfrac{(x-1)(x-2)}{(x+3)(x+4)(x+5)}$, (ii) $\dfrac{x+12}{x^2-13x+42}$, (iii) $\dfrac{2x-1}{(x+1)(x-1)(2x+1)}$.

Solution: (i) Given $I = \int \dfrac{(x-1)(x-2)}{(x+3)(x+4)(x+5)}\,dx$

Let $\dfrac{(x-1)(x-2)}{(x+3)(x+4)(x+5)} = \dfrac{A}{(x+3)} + \dfrac{B}{(x+4)} + \dfrac{C}{(x+5)}$

$$\Rightarrow (x-1)(x-2) = (x+4)(x+5)A + (x+3)(x+5)B + (x+3)(x+4)C \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

[*Multipling both sides by* $(x+3)(x+4)(x+5)$]

Putting $x = -3$ in (1), we get

$$(1) \Rightarrow (-3-1)(-3-2) = (-3+4)(-3+5)A \Rightarrow 20 = 2A \Rightarrow A = 10$$

Putting $x = -4$ in (1), we get

$$(1) \Rightarrow (-4-1)(-4-2) = (-4+3)(-4+5)B$$

$$\Rightarrow 30 = -B \Rightarrow -30 = -30$$

Putting $x = -5$ in (1), we get

$(1) \Rightarrow (-5-1)(-5-2) = (x^{-5}+3^{-5})(-5+3)(-5+4)(x+4)C \Rightarrow 42 = 2C \Rightarrow C = 21$

$$\therefore I = \int \frac{(x-1)(x-2)}{(x+3)(x+4)(x+5)}dx$$

$$= \int\left[\frac{A}{(x+3)}+\frac{B}{(x+4)}+\frac{C}{(x+5)}\right]dx \qquad = A\int\frac{1}{(x+3)}dx + B\int\frac{1}{(x+4)}dx + C\int\frac{1}{(x+5)}dx$$

$10\log(x+3)-30\log(x+4)+21\log(x+5)+c$ [Ans.]

(ii) Given $I = \dfrac{x+12}{x^2-13x+42} = \dfrac{x+12}{x^2-7x-6x+42} = \dfrac{x+12}{(x-7)(x-6)}$

Let $\dfrac{x+12}{(x-7)(x-6)} = \dfrac{A}{(x-7)}+\dfrac{B}{(x-6)}$

$\Rightarrow x+12 = (x-6)A+(x-7)B$.....(1) [Multiplying both sides by $(x-7)(x-6)$]

Putting $x=7$ in (1), we get

$(1) \Rightarrow 7+12 = (7-6)A+(7-7)B \Rightarrow 19 = A \Rightarrow A = 19$

Putting $x=6$ in (1), we get

$(1) \Rightarrow 6+12 = (6-6)A+(6-7)B \Rightarrow 18 = -B \Rightarrow B = -18$

$$\therefore I = \int\frac{x+12}{(x-7)(x-6)}dx = \int\left[\frac{A}{(x-7)}+\frac{B}{(x-6)}\right]dx$$

$$= A\int\frac{1}{(x-7)}dx + B\int\frac{1}{(x-6)}dx \qquad = 19\log(x-7)-18\log(x-6)+C \quad \text{[Ans.]}$$

(iii) Given, $I = \int\dfrac{2x-1}{(x+1)(x-1)(2x+1)}dx$

Let $\dfrac{2x-1}{(x+1)(x-1)(2x+1)} = \dfrac{A}{(x+1)}+\dfrac{B}{(x-1)}+\dfrac{C}{(2x+1)}$

$\Rightarrow 2x-1 = (x-1)(2x+1)A+(x+1)(2x+1)B+(x+1)(x-1)C$.....(1)

[Multiplying both sides by $(x+1)(x-1)(2x+1)$]

Putting $x=-1$ in (1), we get

$(1) \Rightarrow 2(-1)-1 = (-1-1)(-2+1)A \Rightarrow -3 = 2A \Rightarrow A = -\dfrac{3}{2}$

Putting $x=1$ in (1), we get

$(1) \Rightarrow 2(1)-1 = (1+1)(2+1)B \Rightarrow 1 = 6B \Rightarrow B = \dfrac{1}{6}$

Putting $x=-\dfrac{1}{2}$ in (1), we get

$$(1)\Rightarrow 2\left(-\frac{1}{2}\right)-1=\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}-1\right)C$$

$$\Rightarrow -2=\left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)C\Rightarrow -2=-\frac{3}{4}C \qquad \Rightarrow C=\frac{8}{3}$$

$$\therefore I=\int\frac{2x-1}{(x+1)(x-1)(2x+1)}dx \qquad =\int\left[\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(2x+1)}\right]dx$$

$$=A\int\frac{1}{(x+1)}dx+B\int\frac{1}{(x-1)}dx+C\int\frac{1}{(2x+1)}dx$$

$$=-\frac{3}{2}\log(x+1)+\frac{1}{6}\log(x-1)+\frac{8}{3}\log(2x+1)\frac{1}{2}+c$$

$$=-\frac{3}{2}\log(x+1)+\frac{1}{6}\log(x-1)+\frac{4}{3}\log(2x+1)+C \quad \text{[Ans.]}$$

Illustration-24: Integrate the following functions w. r. to x:

(i) $\dfrac{x^2-4}{(x^2-1)(3x+2)}$, (ii) $\dfrac{x^2}{(x-a)(x-b)}$

Solution: (i) Given, $I=\int\dfrac{x^2-4}{(x^2-1)(3x+2)}dx=\int\dfrac{x^2-4}{(x+1)(x-1)(3x+2)}dx$

Let $\dfrac{x^2-4}{(x+1)(x-1)(3x+2)}=\dfrac{A}{(x+1)}+\dfrac{B}{(x-1)}+\dfrac{C}{(3x+2)}$

$$\Rightarrow x^2-4=(x-1)(3x+2)A+(x+1)(3x+2)B+(x+1)(x-1)C.....(1)$$

[Multiplying both sides by $(x+1)(x-1)(3x+2)$]

Putting $x=-1$ in (1), we get

$$(1)\Rightarrow(1)^2-4=(-1-1)(-3+2)A \qquad \Rightarrow -3=2A\Rightarrow A=-\frac{3}{2}$$

Putting $x=1$ in (1), we get

$$(1)\Rightarrow(1)^2-4=(1+1)(3+2)B\Rightarrow -3=10B \quad \Rightarrow B=-\frac{3}{10}$$

Putting $x=-\dfrac{2}{3}$ in (1), we get

$$\left(-\frac{2}{3}\right)^2-4=\left(-\frac{2}{3}+1\right)\left(-\frac{2}{3}-1\right)C\Rightarrow\frac{4}{9}-4=\left(\frac{1}{3}\right)\left(-\frac{5}{3}\right)C\Rightarrow-\frac{32}{9}=-\frac{5}{9}C\Rightarrow C=\frac{32}{5}$$

$$\therefore I=\int\frac{x^2-4}{(x+1)(x-1)(3x+2)}dx$$

$$= \int\left[\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(3x+2)}\right]dx \quad = A\int\frac{1}{(x+1)}dx + B\int\frac{1}{(x-1)}dx + C\int\frac{1}{(3x+2)}dx$$

$$= -\frac{3}{2}\log(x+1) - \frac{3}{10}\log(x-1) + \frac{32}{5}\log(3x+2)\left(\frac{1}{3}\right) + c$$

$$= -\frac{3}{2}\log(x+1) - \frac{3}{10}\log(x-1) + \frac{32}{15}\log(3x+2) + c \quad \text{[Ans]}$$

(ii) Given $I = \int\frac{x^2}{(x-a)(x-b)}dx$

Let $\frac{x^2}{(x-a)(x-b)} = 1 + \frac{A}{(x-a)} + \frac{B}{(x-b)}$

$\Rightarrow x^2 = (x-a)(x-b) + (x-b)A + (x-a)B$.....(1) [Multiplying both sides by $(x-a)(x-b)$]

Putting $x = a$ in (1), we get

$(1) \Rightarrow a^2 = (a-a)(a-b) + (a-b)A + (a-a)B \Rightarrow a^2 = (a-b)A \Rightarrow A = \frac{a^2}{(a-b)}$

Putting $x = b$ in (1), we get

$(1) \Rightarrow b^2 = (b-a)(b-b) + (b-a)A + (b-a)B \Rightarrow b^2 = (b-a)B \Rightarrow B = \frac{b^2}{(b-a)}$

$\therefore I = \int\frac{x^2}{(x-a)(x-b)}dx$

$$= \int\left[1 + \frac{A}{(x-a)} + \frac{B}{(x-b)}\right]dx \quad = \int(1)dx + A\int\frac{1}{(x-a)}dx + B\int\frac{1}{(x-b)}dx$$

$$= x + \frac{a^2}{(a-b)}\log(x-a) + \frac{b^2}{(b-a)}\log(x-b) = x + \frac{a^2}{a-b}\log(x-a) - \frac{b^2}{a-b}\log(x-b) \text{ [Ans.]}$$

EXERCISE (Indefinite Integrals)

Fundamentals rules of integrals:-

1. Find the value of the integrals:
   (a) $\int 5x^3 dx$, (b) $\int(4x^3 + 3x^2 - 2x + 5)\,dx$, (c) $\int(x^2 - 1)^2 dx$, (d) $\int(3x^{-1} + 4x^2 - 3x + 8)\,dx$
2. Evaluate the followings:
   (a) $\int\frac{(x^2+1)^2}{x^3}dx$, (b) $\int\frac{(3x-2)^2}{\sqrt{x}}dx$.

3. Evaluate (a) $\dfrac{ax+bx^{-3}+cx^{-7}}{kx^{-2}}dx$, (b) $\int\dfrac{a+b\sin x}{\cos^2 x}dx$.

4. Integrate: (a) $\int\left(2^x-3e^{2x}+3^{-2x}\right)dx$, (b) $\int\left(\dfrac{2e^{4x}-3e^{2x}+4}{e^{3x}}\right)dx$.

5. Integrate the following functions with respect to x:

(a) $\dfrac{2\cos x}{5\sin^2 x}+\dfrac{1}{5\cos^2 x}$, (b) $\sqrt{1+\sin 2x}$.

6. Find a function whose derivative is: (a) $\dfrac{3x^2-4}{\sqrt{x}}$, (b) $\int\dfrac{x^2+1}{\sqrt{x}}dx$.

7. Integrate each of the following functions with respect to x:

(a) $\dfrac{e^{5x}-2e^{3x}+3}{e^x}$, (b) $\dfrac{e^{2x-1}-e^{1-2x}}{e^{x+2}}$.

8. Evaluate: (a) $\int\dfrac{x}{x+1}dx$, (b) $\int\dfrac{x^3}{x-1}dx$.

9. Evaluate each of the following integrals:

(a) $\int\left(4x^3-\dfrac{2}{x}+e^{2x}-1\right)dx$, (b) $\int\left(\sqrt{x}-\dfrac{1}{\sqrt[3]{x}}+2^x\right)dx$.

10. Find a function whose differential is $\dfrac{(1-x)^3}{x}$.

ANSWERS

1. (a) $\dfrac{5}{4}x^4+c$, (b) $x^4+x^3-x^3+5x+c$, (c) $\dfrac{x^5}{5}-\dfrac{2}{5}x^3+x+c$, (d) $3\log e^x+\dfrac{4x^3}{3}-\dfrac{3x^2}{2}+8x+c$

2. (a) $\dfrac{1}{2}x^2+2\log_e x-\dfrac{1}{2x^2}+c$ (b) $\dfrac{54}{7}x^{\frac{7}{2}}-\dfrac{108}{5}x^{\frac{5}{2}}+24x^{\frac{3}{2}}-16x^{\frac{1}{2}}+c$

3. (a) $\dfrac{ax^4}{4k}+\dfrac{b\log_e x}{k}-\dfrac{cx^{-4}}{4k}+c$ (b) $a\tan x+b\sec x+c$.

4. (a) $\dfrac{2^x}{\log_e 2}-\dfrac{3}{2}e^{2x}-\dfrac{1}{2}\dfrac{3^{-2x}}{\log_e 3}+c$, (b) $2e^x+3e^{-x}-\dfrac{4}{3}e^{-3x}+c$.

5. (a) $-\dfrac{2}{5}\operatorname{cosec} x+\dfrac{1}{5}\tan x+c$, (b) $\sin x-\cos x+c$

6. (a) $\sqrt{x}\left(\dfrac{6}{5}x^2-8\right)+c$, (b) $\dfrac{2}{5}x^{\frac{5}{2}}+2x^{\frac{1}{2}}+c$.

7. (a) $\dfrac{1}{4}e^{4x}-e^{2x}-3e^{-x}+c$, (b) $e^{x-3}+\dfrac{1}{3}e^{-3x-1}+c$.

8. (a) $x-\log(x+1)+c$, (b) $\frac{x^3}{3}+\frac{x^2}{2}+x+\log(x-1)+c$.

9. (a) $x^4-2\log x+\frac{1}{2}e^{2x}-x+c$, (b) $\frac{2}{3}x^{\frac{3}{2}}-\frac{3}{2}x^{\frac{2}{3}}+\frac{2^x}{\log_e 2}+c$.

10. $\log x-3x+\frac{3}{2}x^2-\frac{1}{3}x^3+c$.

Method of substitution:

1. Evaluate the following: (a) $\int\frac{1}{(3\tan x+1)\cos^2 x}dx$, (b) $\int e^x\sin(e^x)dx$, (c) $\int\frac{x}{(x^2+3)^2}dx$

2. find the integrals of the following functions with respect to x:
   (a) $\sin x\cos x$, (b) $\sec^n x\tan x$ (c) $\tan^3 x\sec^2 x$.

3. find the value of (a) $\int x(x^2+1)^3dx$, (b) $\int\frac{4x^2dx}{\sqrt{1-x^6}}$, (c) $\int 3x^2(x^3+5)^2dx$

4. Find the functions whose derivatives are: (a) $\sin x\cos^3 x$, (b) $e^{\tan x}\sec^2 x$, (c) $\frac{1}{\sqrt{x}}\sin\sqrt{x}$.

5. Find the value of the following integrals: (a) $\int\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx$ (b) $\int\frac{e^{m\tan^{-1}x}}{1+x^2}dx$ (c) $\int\frac{x^3dx}{(x^2+1)^3}$

ANSWERS

1. (a) $\frac{1}{3}\log(3\tan x+1)+c$, (b) $-\cos(e^x)+c$ (c) $-\frac{1}{2(x^2+3)}+c$

2. (a) $=\frac{-1}{4}\cos 2x+c$, (b) $\frac{\sec^n x}{n}+c$, (c) $\frac{\tan^4 x}{4}+c$.

3. (a) $\frac{1}{8}(x^2+1)^4+c$, (b) $\frac{4}{3}\sin^{-1}(x^3)+c$, (c) $\frac{1}{3}(x^3+5)^3+c$.

4. (a) $-\frac{1}{4}\cos^4 x+c$, (b) $e^{\tan x}+c$ (c) $-2\cos\sqrt{x}+c$.

5. (a) $e^{\sin^{-1}x}+c$, (b) $\frac{1}{m}e^{m\tan^{-1}x}+c$, (c) $-\frac{1}{4}\cdot\frac{2x^2+1}{(x^2+1)^2}+c$.

Standard Integrals:-

1. $\int\frac{dx}{2x^2-2x+1}$.
2. $\int\frac{dx}{\sqrt{7+5x-3x^2}}$.
3. $\int\frac{x^2dx}{x^4+1}$.
4. $\int\frac{5x-2}{1+2x+3x^2}dx$.
5. $\int\frac{dx}{5+4\cos x}$.
6. $\int\frac{dx}{4+5\sin^2 x}$.

ANSWERS

1. $\tan^{-1}(2x-1)+c$.

2. $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{6x-5}{\sqrt{109}}\right)+c$.

3. $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)-\frac{1}{4\sqrt{2}}\log\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}+c$.

4. $\frac{3}{4}\log(2x^2-2x+3)+\frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$.

5. $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$

6. $\frac{1}{6}\tan^{-1}\left(\frac{3}{2}\tan x\right)+c$

Integration by parts:-

Integrate the following functions w. r. to *x*:

1. $x^2\sin x$, 2. x^2e^{3x}. 3. $\log x$, 4. $x^n\log x$, 5. $x\tan^{-1}x$, 6. $e^{ax}\sin(bx+c)$, 7. $\frac{xe^x}{(x+1)^2}$,

8. $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}$, 9. $\frac{x^2\tan^{-1}x}{1+x^2}$, 10. $x\sec^{-1}x$, 11. $x\sec^2x$, 12. $\cot^{-1}x$ 13. $\frac{e^x(1+x\log x)}{x}$.

Integration using partial fraction:-

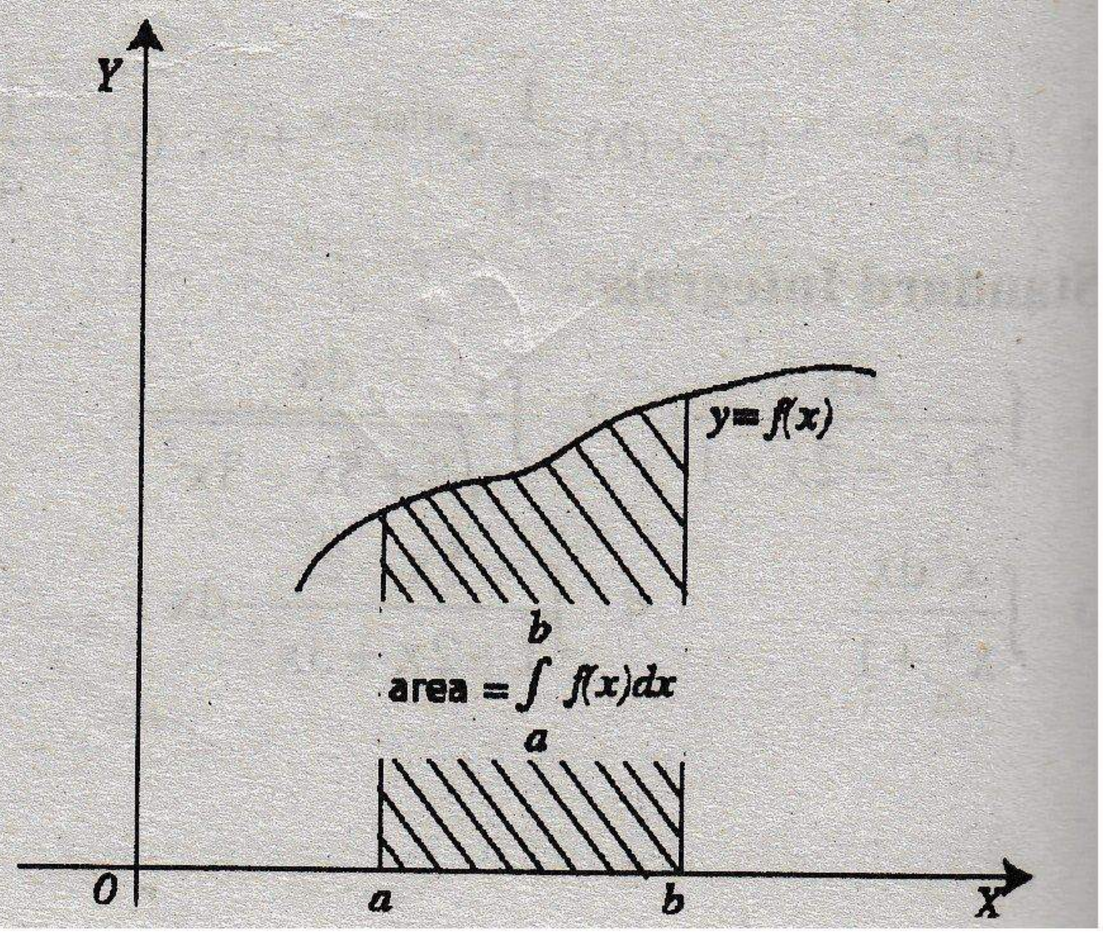
Integrate the following function w. r. to x:

1. $\frac{x}{(x-1)(2x+1)}$, 2. $\frac{1}{x-x^3}$, 3. $\frac{x^3}{(x-a)(x-b)(x-c)}$, 4. $\frac{\cos x}{(1+\sin x)(2+\sin x)}$

5. $\frac{1}{\sin x+\sin 2x}$ 6. $\frac{\sin 2x}{(\sin x+1)(\sin x+2)(\sin x+3)}$ 7. $\frac{1}{3\sin x+\sin 2x}$

13.9 DEFINITE INTEGRALS

The concept of definite integral and area under a curve are related to each other. In particular, suppose $y=f(x)$ is a continuous function defined on a closed interval, $I=[a,b]$ and $f(x)\geq 0$ (non-negative) for all x in I. Then for a, $b\in I$, the definite integral $\int_a^b f(x)dx$ is the area under the graph of y=f(x) and above the x-axis between the lines x=a and x=b, as shown in the figure. Definition: Let f(x) be a continuous function defined on the closed interval, I=[a,b] and F(x) be an anti-derivative



of f(x) in the same interval. Then for a, b ∈ I, the value $\int_a^b f(x)\,dx = F(b) - F(a)$ is called the definite integral from a to b of (x). In $\int_a^b f(x)dx$, the numbers a and b are termed as lower and upper limits of integration respectively.

Working Rule:

For evaluating the definite integral $\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$

(i) integrate $\int f(x)dx$ without considering lower and upper limits as usual.

(ii) To calculate $F(x)\Big|_a^b$, first replace x by the upper limit b obtaining F(b) and the replace x by the lower limit a obtaining F(a). Subtract F(a) from F(b). This gives the required value of definite integral.

The value of definite integral is unique and well defined, because in its evaluation, the constant of integration disappears.

13.10 PROPERTIES OF DEFINITE INTEGRAL

1. Reversing the order of the limits of integration changes the sign of the definite integral.
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$
2. If the upper limit of integration equals the lower limit of integration, the value of the definite integral is zero. $\int_a^a f(x)\,dx = F(a) - F(a) = 0$
3. The definite integral can be expressed as the sum of component sub integrals.
$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + + \int_b^c f(x)\,dx, \quad a \le b \le c.$$
4. the sum or difference of two definite integrals with identical limits of integration is equal to the definite integral of the sum or difference of the two functions.
$$\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \int_a^b [f(x) \pm g(x)]\,dx$$
5. The definite integral of a constant times a function is equal to the constant times the definite integral of the function. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
6. If f(x) is continuous on [a, b] then $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Illustration-25: Evaluate $\int_2^1 (3x-2)^2\,dx$

Solution: $\int_2^1 (3x-2)^2\,dx$

$$= \int_2^1 \left(9x^2-12x+4\right)dx = \left|\frac{9x^3}{3}-12.\frac{x^2}{2}+4x\right|_2^1 = \left|3x^3-6x^2+4x\right|_2^1$$

$$= \left(3.1^3-6.1^2+4.1\right)-\left(3.2^3-6.2^2+4.2\right)$$

$$= (3-6+4)-(24-24+8) = 1-8 = -7 \text{ (Ans.)}$$

Illustration-26: Evaluate $\int_1^4 \frac{dx}{\sqrt{x}}$

Solution: $\int_1^4 x^{-\frac{1}{2}}\,dx$

$$= \left|\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right|_1^4 = \left|\frac{\sqrt{x}}{\frac{1}{2}}\right|_1^4 = 2\left|\sqrt{x}\right|_1^4 = 2\left(\sqrt{4}-\sqrt{1}\right) = 2(2-1) = 2.1 = 2 \text{ (Ans.)}$$

Illustration-27: Evaluate $\int_{-2}^2 \left(2x^2-x^3\right)dx$

Solution: $\int_{-2}^2 \left(2x^2-x^3\right)dx$

$$= \left|2.\frac{x^3}{3}-\frac{x^4}{4}\right|_{-2}^2 = \left(2.\frac{8}{3}-\frac{16}{4}\right)-\left(2.\frac{-8}{3}-\frac{16}{4}\right)$$

$$= \frac{16}{3}-\frac{16}{4}+\frac{16}{3}+\frac{16}{4} = \frac{16}{3}+\frac{16}{3} = \frac{32}{3} \text{ (Ans.)}$$

Illustration-28: Evaluate $\int_1^2 e^{2x+2x^2}\,dx$

Solution: $\int_1^2 e^{2x+2x^2}\,dx = \left|\frac{e^{2x+2x^2}}{2+4x}\right|_1^2 = \frac{e^{2.2+2.4}}{10}-\frac{e^4}{6} = \frac{e^{12}}{10}-\frac{e^4}{6}$ (Ans.)

Illustration-29: Evaluate $\int_1^e \frac{dx}{x}$

Solution: $\int_1^e \frac{dx}{x} = |\ln x|_1^e = \ln e - \ln 1 = \ln e - 0 = \ln e = 1$ (Ans.)

Illustration-30: Evaluate (i) $\int_2^3 e^{-\frac{x}{2}} dx$ (ii) $\int_2^3 e^{2x} dx$

Solution: (i) $\int_2^3 e^{-\frac{x}{2}} dx = \left| \frac{e^{-\frac{x}{2}}}{-\frac{1}{2}} \right|_2^3 = -2\left(e^{-\frac{3}{2}} - e^{-\frac{2}{2}} \right) = -2\left(e^{-\frac{3}{2}} - e^{-1} \right)$ (Ans.)

(ii) $\int_2^3 e^{2x} dx = \left| \frac{e^{2x}}{2} \right|_2^3 = \frac{1}{2}\left(e^6 - e^4\right) = \frac{1}{2}e^4\left(e^2 - 1\right)$ (Ans.)

Illustration-31: Evaluate $\int_6^{14} \frac{dx}{x+2}$

Solution: $\int_6^{14} \frac{dx}{x+2}$

$= |\ln(x+2)|_6^{14} = \ln 16 - \ln 8 = \ln \frac{16}{8} = \ln 2$ (Ans.)

Illustration-32: Evaluate $\int_0^1 \log x \, dx$

Solution: $\int_0^1 \log x \, dx$

$= \int_0^1 (\log x)(1) dx = |\log x . x|_0^1 - \int_0^1 \frac{1}{x} . x . dx = |x \log x|_0^1 - |x|_0^1 = 1.\log 1 - 0 - (1 - 0)$

$= 0 - 0 - 1 + 0 = -1$ (Ans.) $(\because \log 1 = 0)$

Illustration-33: Evaluate $\int_6^1 x(x+1)dx$

Solution: $\int_6^1 x(x+1)dx = \int_0^1 (x^2+x)dx = \left|\frac{x^3}{3}+\frac{x^2}{2}\right|_0^1 = \left(\frac{1}{3}+\frac{1}{2}\right)-(0) = \frac{2+3}{6} = \frac{5}{6}$ (Ans.)

Illustration-34: Evaluate $\int_2^4 (3^x + \ln x)dx$

Solution: $\int_2^4 (3^x + \ln x)dx$

$$= \int_2^4 3^x dx + \int_2^4 \ln x\, dx = \left|\frac{3^x}{\ln 3}\right|_2^4 + |\ln x.x|_2^4 - \int_2^4 \left(\frac{1}{x}.x\right)dx$$

$$= \frac{1}{\ln 3}(3^4 - 3^2) + (4\ln 4 - 2\ln 2) - |x|_2^4$$

$$= \frac{81-9}{\ln 3} + (8\ln 2 - 2\ln 2) - (4-2) = \frac{72}{\ln 3} + 6\ln 2 - 2 \quad \text{(Ans.)}$$

Illustration-35: Evaluate $\int_0^{\ln 2} \frac{e^x}{1+e^x}dx$

Solution: $\int_0^{\ln 2} \frac{e^x}{1+e^x}dx$

Let, $1+e^x = z$

$e^x\, dx = dz$

x	0	ln2
z	2	3

$$= \int_2^3 \frac{dz}{z} = |\ln z|_2^3 = \ln 3 - \ln 2 = \ln\frac{3}{2} \quad \text{(Ans.)}$$

Illustration-36: Evaluate $\int_2^5 \frac{2x}{1+x^2}dx$

Solution: $\int_2^5 \frac{2x}{1+x^2}dx$

Let $1+x^2 = z$

$\therefore 2xdx = dz$

x	2	5
z	5	26

$$= \int_5^{26} \frac{dz}{z} = |\ln z|_5^{26} = \ln 26 - \ln 5 = \ln\frac{26}{5} \quad \text{(Ans.)}$$

Illustration-37: Evaluate $\int_0^8 6x\sqrt{100+x^2}\,dx$

Solution: $\int_0^8 6x\sqrt{100+x^2}\,dx$

Let $100+x^2=z$

$\therefore 2xdx=dz$

$\therefore xdx=\frac{dz}{2}$

x	0	8
z	100	6400

$$=\int_{100}^{6400} 6.\sqrt{z}.\frac{dz}{2}=3\left|\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\right|_{100}^{6400}$$

$$=3.\frac{2}{3}\left\{(6400)^{\frac{3}{2}}-(100)^{\frac{3}{2}}\right\}=2\left\{(80)^3-(10)^3\right\}=2(51{,}200-1000)$$

$$=2(50200)=100{,}400 \quad \text{(Ans.)}$$

Illustration-38: Evaluate $\int_{-3}^{-2}\frac{1}{x}dx$

Solution: $\int_{-3}^{-2}\frac{1}{x}dx$

$$=\left|\log|x|\right|_{-3}^{-2}=\log|-2|-\log|-3|=\log 2-\log 3=\log\frac{2}{3} \quad \text{(Ans.)}$$

Illustration-39: Evaluate $\int_2^5\frac{xdx}{x^2+1}$

Solution: $\int_2^5\frac{xdx}{x^2+1}$

$$=\frac{1}{2}\int_2^5\frac{2xdx}{x^2+1}=\frac{1}{2}\left|\log|x^2+1|\right|_2^5 \quad \left[\because \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c\right]$$

$$=\frac{1}{2}(\log 26-\log 5)=\frac{1}{2}\log\frac{26}{5}=\log\sqrt{\frac{26}{5}} \quad \text{(Ans.)}$$

Illustration-40: Evaluate $\int_0^2 xe^x dx$

Solution: $\int_0^2 xe^x dx$

$$= x\int_0^2 e^x dx - \int_0^2\left[\frac{d}{dx}(x)\int e^x dx\right]dx = \left|xe^x\right|_0^2 - \int_0^2 1.e^x dx = \left|xe^x\right|_0^2 - \left|e^x\right|_0^2$$

$$= \left(2e^2 - 0\right) - \left(e^2 - e^0\right) = 2e^2 - e^2 + 1 = e^2 + 1 \quad \text{(Ans.)}$$

Illustration-41: Evaluate $\int_2^3 \frac{xe^x}{(x+1)^2}dx$

Solution: $\int_2^3 \frac{xe^x}{(x+1)^2}dx$

$$= \int_2^3 \frac{(x+1-1)}{(x+1)^2}e^x dx = \int_2^3\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]e^x dx = \left|\frac{1}{x+1}e^x\right|_2^3 = \frac{e^3}{4} - \frac{e^2}{3} \quad \text{(Ans.)}$$

$$\left[\because \int\{f(x) - f'(x)\}e^x dx = e^x f(x) + c\right]$$

Illustration-42: Evaluate $\int_1^{e^2} \frac{dx}{x(1+\log x)^2}$

Solution: $\int_1^{e^2} \frac{dx}{x(1+\log x)^2}$

Suppose $1 + \log x = z$

$$\therefore 0 + \frac{1}{x}\frac{dx}{dz} = 1$$

$$\therefore \frac{1}{x}dx = dz \text{ and } 1 + \log x = z$$

$x = 1 \Rightarrow z = 1 + \log 1 = 1$

$x = e^2 \Rightarrow z = 1 + \log e^2 = 1 + 2\ln e = 3$

$$= \int_1^3 \frac{dz}{z^2} = \int_1^3 z^{-2}dz = \left|\frac{z^{-1}}{-1}\right|_1^3 = \left|-\frac{1}{t}\right|_1^3$$

$$= -\frac{1}{3} + 1 = \frac{2}{3}$$

Illustration-43: Evaluate $\int_3^4 \frac{dx}{x^2 - 4}$

Solution: $\int_3^4 \frac{dx}{x^2 - 4}$

$$= \int_3^4 \frac{dx}{x^2 - (2)^2} = \frac{1}{2(2)}\left|\log\left|\frac{x-2}{x+2}\right|\right|_3^4 = \frac{1}{4}\left(\log\left|\frac{4-2}{4+2}\right| - \log\left|\frac{3-2}{3+2}\right|\right)$$

$$= \frac{1}{4}\left|\log\frac{1}{3} - \log\frac{1}{5}\right| = \frac{1}{4}\log\frac{5}{3} \quad \text{(Ans.)}$$

Illustration-44: Compute the value of the following integrals: $\int_3^{11}\sqrt{2x+3}\,dx$

Solution: Let $I=\int_3^{11}\sqrt{2x+3}\,dx=\frac{1}{2}\int_3^{11}\sqrt{2x+3}.2dx$

Let, $z=2x+3$

$$\Rightarrow\frac{dz}{dx}=2\Rightarrow dz=2dx$$

x	3	11
z	9	25

$$=\frac{1}{2}\int_9^{25}\sqrt{z}\,dz=\frac{1}{2}\left[\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\right]_9^{25}$$

$$=\frac{1}{2}.\frac{2}{3}\left[z^{\frac{3}{2}}\right]_9^{25}=\frac{1}{3}\left[(25)^{\frac{3}{2}}-(9)^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[\left(5^2\right)^{\frac{3}{2}}-\left(3^2\right)^{\frac{3}{2}}\right]=\frac{1}{3}[125-27]=\frac{98}{3}$$

Illustration-45: $\int_3^4\frac{dx}{25-x^2}$

Solution: Let $I=\int_3^4\frac{dx}{25-x^2}=\int_3^4\frac{dx}{5^2-x^2}$

$$=\frac{1}{2\times5}\left[\log\frac{5+x}{5-x}\right]_3^4\quad\left[\because\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\frac{a+x}{a-x}\right]\quad=\frac{1}{10}\left[\log\frac{5+4}{5-4}-\log\frac{5+3}{5-3}\right]$$

$$=\frac{1}{10}[\log9-\log4]=\frac{1}{10}\log\left(\frac{9}{4}\right)=\frac{1}{10}\log\left(\frac{3}{2}\right)^2=\frac{1}{10}.2\log\left(\frac{3}{2}\right)=\frac{1}{5}\log\frac{3}{2}\quad\textbf{[Ans]}$$

13.11 BUSINESS APPLICATION OF INTEGRATION

We know that for any function $f(x)$, the marginal variation denoted by Mf can be obtained by differentiating the function $f(x)$ w. r. to x i. e. $Mf = \frac{df}{dx} = f'(x)$

(a) Find the cost function when marginal cost is given.

If the total cost of producing x units of an output is $C(x)$, then the average cost (AC) per unit and marginal cost (MC) are: $AC = \frac{C(x)}{x}$ and $MC = \frac{dC}{dx} = C'(x)$.

Note that, marginal cost is the derivative of the total cost function $f(x)$ with respect to x. Thus, total cost is the integral of the marginal cost function with respect to x, i.e. $C(x) = \int MC dx + k$

The value of constant of integration k can be obtained in terms a fixed cost or initial overhead, i.e. the cost when x=0

Problem-1: The marginal cost function of a product is given as: MC=1.075-0.004x, where x is the output. Find the total and the average cost function of the firm if the cost producing 5 units is Tk. 81.5.

Solution: Given MC=1.075-0.004x

Integrating w. r. to x, we get

$C = \int (1.075 - 0.004x)dx = 1.075x - 0.004 \times \frac{x^2}{2} + k = 1.075x - 0.002x^2 + k$, k is a constant.

When $x = 5$, the fixed cost is Tk. 81.50. Therefore we have $81.50 = 1.075 \times 5 - 0.002 \times 5^2 + k = 76.175$

Hence the total cost is given by $C = 1.075x - 0.002x^2 + 76.175$ and average cost, $AC = \frac{C}{x} = \frac{1.075x - 0.002x^2 + 76.175}{x} = 1.075 - 0.002x + \frac{76.175}{x}$ [Ans]

(b) **find the total revenue function and demand function when the marginal revenue function is given.**

If p is the price per unit and x is the number of units of an output, then demand function is given by $p = f(x)$. The total revenue R is given by $R = p \times x = xf(x)$

The marginal revenue (MR) with respect to demand function is given by $\text{MR} = \frac{dr}{dx} R'(x)$.

Hence, the total revenue R is the integral of MR with respect to x. i.e. $R = \int MR dx + c$ where c is the constant of integration whose value can be calculated in terms of initial demand, i.e. R=0 if the output x is zero.

Further, the average revenue (AR) is given by $AR = \frac{R(x)}{x}$

Problem-2: The marginal revenue function of a firm is $MR = 28q - e^{0.2q}$, find the total revenue function $R(q)$. What initial condition you would take to determine the constant of integration.

Solution: Given $MR = 28q - e^{0.2q}$

$$\therefore R = \int MR\,dq = \int \left(28q - e^{0.2q}\right) dq = 14q^2 - \frac{1}{0.2} e^{0.2q} + k,\ \text{k is a integration constant.}$$

$$= 14\text{q}^2 - 5\text{e}^{0.2\text{q}} + \text{k}$$

if $\text{q} = 0$, then $\text{R}(0) = 0$. Therefore, we have

$$0 = 14 \times 0 - 5 \times \text{e}^0 + \text{k} \Rightarrow \text{k} = 5$$

Hence, the total revenue function $\text{R}(\text{q})$ is given by $\text{R} = 14\text{q}^2 - 5\text{e}^{0.2\text{q}} + 5$. [Ans]

(c) **Find the consumption function when the marginal propensity to consume (MPC) is given.**

For the consumption function $\text{C} = \text{f}(\text{x})$ where C is the total national consumption and x is the total national income of a person, the marginal propensity to consume (MPC) is given by

$$\text{MPC} = \frac{\text{dC}}{\text{dx}}$$

That is, MPC is the derivative of the consumption function with respect to x. hence, the total national consumption C is given by the integral of MPC with respect to x, i.e.

$\text{C} = \int \text{MPC}\,\text{dx} + \text{k}$, where k is the constant of integration whose value can be calculated if the value C is known for some x.

Further, if $x = c + S$, where S is the savings, then the marginal propensity to save is

$$\text{MPS} = \frac{\text{dS}}{\text{dx}} = 1 - \frac{\text{dC}}{\text{dx}}$$

Problem-3: If the marginal propensity to consume is given by:

$MPC = 1.50 + 0.20\,y^{-2}$, where y is the income, find the total consumption function.

Given that $C(10) = 4.80$

Solution: we have $MPC = \frac{dC}{dy} = 1.50 + 0.20\,y^{-2}$ then the total consumption function is given by $C = \int MPC\,dy = \int \left(1.50 + 0.20\,y^{-2}\right) dy = 1.50\,y - \frac{0.20}{y} + k$ when

$C(10) = 4.80 \Rightarrow C = 4.80$ for $y = 10$, we have

$$4.80 = 1.50 \times 10 - \frac{0.20}{10} + k \Rightarrow k = -10.18$$

Hence, the total consumption function is, $C = 1.50\,y - \frac{0.20}{y} - 10.18$ [Ans]

Problem-4: If the marginal propensity to save (MPS) is $1.5 + 0.2x^{-2}$, when x is the income. Find the consumption function, given that the consumption is 4.8 when income is 10.

Solution: Given $\text{MPS} = 1.5 + 0.2x^{-2} = \frac{dp}{dx}$

$$\therefore p = \int MPS\, dx = \int (1.5 + 0.2x^{-2})\, dx = 1.5x + 0.2\left(\frac{x^{-2+1}}{-2+1}\right) + k = 1.5x - \frac{0.2}{x} + k$$

Now $P = 4.8$ when x=10 $\therefore 4.80 = 1.5 \times 10 - \frac{0.20}{10} + k \Rightarrow k = -10.18$

Hence the consumption function is : $P = 1.5x - \frac{0.2}{x} - 10.18$

Problem-5: The marginal cost function of a product is given by

$\frac{dC}{dq} = 100 - 10q + 0.1q^2$, where q is the output. Obtain the total and the average cost function of the firm under the assumption that its fixed cost is Tk. 500.

Solution: Given $\frac{dC}{dq} = 100 - 10q + 0.1q^2 = MC$

Integrating both sides w. r. to q, we have

$$C = \int MC\, dq = \int (100 - 10q + 0.1q^2)\, dq$$

$$= 100q - 10 \times \frac{q^2}{2} + 0.1 \times \frac{q^3}{3} + k = 100q - 5q^2 + \frac{q^3}{30} + k \quad \ldots\ldots\ldots\ldots (1)$$

Now the fixed cost is 500, i.e., when $q = 0$, $c = 500$.

$$\therefore 500 = 100 \times 0 - 5 \times 0 + \frac{1}{30} \times 0 + k = k \Rightarrow 500$$

Hence total cost function is $C = 100q - 5q^2 + \frac{q^3}{30} + 500$

Average cost function is $AC = \frac{C}{q} = \frac{1}{q}\left(100q - 5q^2 + \frac{q^3}{30} + 500\right)$

$$= 100 - 5q + \frac{q^2}{30} + \frac{500}{q} \quad \text{[Ans.]}$$

Problem-6: The marginal cost function of manufacturing x shoes is $6+10x-6x^2$. The total cost of producing a pair of shoes is Tk.12. Find the total and average cost function.

Solution: Given $MC = 6 + 10x - 6x^2 = \frac{dC}{dx}$

$$\therefore C=\int\frac{dC}{dx}dx=\int\left(6+10x-6x^2\right)dx=6x+10\times\frac{x^2}{2}-6\times\frac{x^3}{3}+k,$$ where k is a constant. Now when output, x=2 then cost C=12.

$$\therefore 12=6(2)+10\times\frac{(2)^2}{2}-6\times\frac{(2)^3}{3}+k\Rightarrow k=12-12-20+16=-4$$

$\therefore$ The total cost function is: $C=6x+5x^2-2x^3-4$

Further the average cost function AC is given by

$$AC=\frac{C}{x}=6+5x-2x^2-\frac{4}{x}$$ [Ans]

Problem-7: The marginal cost function of a firm is given by $\boldsymbol{MC=3000e^{0.3x}+50}$, when x is quantity produced. If fixed cost is Tk. 80,000, find the total cost function of the firm.

Solution: The total cost function of the firm is given by

$$TC=\int(MC)dx+k,$$ where k is constant of integration.

$$\therefore TC=\int\left(3000e^{0.3x}+50\right)dx+k=3000\times\frac{e^{0.3x}}{0.3}+50x+k=10000e^{0.3x}+50x+k$$

when $x=0$, TC=80,000, therefore, we have, $\boldsymbol{80{,}000=10{,}000}+k\Rightarrow k=\boldsymbol{70{,}000}$

$$\therefore TC=10000e^{0.3x}+50x+70{,}000$$

Problem-8: The marginal cost is given by $MC=4+5x^2+\frac{3}{2}e^{-x}$, where x is the quantity produced. Find the total cost of production when x=2, if fixed cost is Tk. 6.

Solution: we have, $MC=4+5x^2+\frac{3}{2}e^{-x}$

$$TC=\int MCdx=\int\left(4+5x^2+\frac{3}{2}e^{-x}\right)dx$$

$$=4x+\frac{5x^2}{3}-\frac{3}{2}e^{-x}+k,$$ where k is a integration constant.

Given that when output x=0, total cost (TC)=6

$$\therefore 6=-\frac{3}{2}+k\Rightarrow k=\frac{15}{2}$$

$$\therefore TC=4x+\frac{5x^3}{3}-\frac{3}{2}e^{-x}+\frac{15}{2}$$

$$TC\big|_{x=2}=4\times2+\frac{5\times8}{3}-\frac{3}{2}e^{-2}+\frac{15}{2}=8+\frac{40}{3}-\frac{3}{2}\times0.1353+\frac{15}{2}=Tk.28.63$$ [Ans]

Problem 9: If the marginal revenue function for output x is given by $MR = \frac{6}{(x+2)^2} + 5$, find the total revenue function and deduce the demand function.

Solution: Given $MR = \frac{6}{(x+2)^2} + 5$

Total revenue function is given by $R = \int MR\,dx = \int \left\{ \frac{6}{(x+2)^2} + 5 \right\} dx$

$$= \int \frac{6}{(x+2)^2} dx + 5\int dx = -\frac{6}{(x+2)} + 5x + k$$

Since total revenue is zero at $x = 0$, we get $0 = -\frac{6}{2} + k \Rightarrow k = 3$

$$\therefore R = 3 - \frac{6}{x+2} + 5x$$

Also we know $R = p \times x \Rightarrow p = \frac{R}{x} = \frac{3 - \frac{6}{x+2} + 5x}{x} = \frac{3}{x} - \frac{6}{x(x+2)} + 5 = \frac{3x+6-6}{x(x+2)} + 5 = \frac{3}{x+2} + 5$

Hence $\mathrm{p} = \frac{3}{\mathrm{x}+2} + 5$ is the required demand function. [Ans]

(d) Relation between Profit, Marginal cost and Revenue

Suppose the marginal cost and the marginal revenue functions of a firm are given. We have to find the maximum profit of this firm. Now question is, how to calculate the maximum profits? By equating marginal cost to marginal revenue, we can find the output that maximizes total profits. To calculate total profits at this output, we have

$$\frac{dP}{dx} = \frac{dR}{dx} - \frac{dC}{dx} = 0 \Rightarrow \frac{dR}{dx} = \frac{dC}{dx} \Rightarrow MR = MC$$

Where P, R, C and x represent the total profit, total revenue, total cost and output respectively.

Again, $\frac{dP}{dx} = \frac{dR}{dx} - \frac{dC}{dx}$

Integrating, we have $P = \int \frac{dR}{dx} dx - \int \frac{dC}{dx} dx + k = R - C + k$

Where the constant of integration k can be found from the additional information given. Profit is maximized when marginal revenue equals marginal cost, given the assumption of pure competition. Total profit is the integral of marginal revenue minus marginal cost from zero quantity to quantity for which profit is maximized.

Problem-10: The marginal cost of production of a firm is given as $C'(q) = 5 + 0.13q$. Further, the marginal revenue is $R'(q) = 18$. Also it is given that $C(0) =$ Tk. 120. Compute the total profits.

Solution: For maximum profits, marginal cost=marginal revenue.

$$\text{i.e., } C'(q) = R'(q) \Rightarrow 5 + 0.13q = 18 \Rightarrow q = \frac{13}{0.13} = 100$$

Also $P(q) = \int R'(q)\,dq - \int C'(q)\,dq = R(q) - C(q)$

Now $R(q) = \int R'(q)\,dq = \int 18\,dq = 18q + k_1$, k_1 is a constant.

Put $k_1 = 0$, as under pure competition, total revenue =Price×quantity.

$\therefore R(q) = 18q$ and $C(q) = \int C'(q)\,dq = \int (5 + 0.13q)\,dq = 5q + \frac{0.13}{2} + q^2 + k_2$ k_2 is a constant. From the information C(0)= Tk. 120, we get

$$C(0) = 5 \times 0 + \frac{0.13}{2} \times 0 + k_2 = 120 \Rightarrow k_2 = 120$$

$$\therefore C(q) = 5q + \frac{0.13}{2} q^2 + 120 = 5q + 0.06q^2 + 120$$

Hence profit, $P(q) = R(q) - C(q) = 18q - 5q - 0.065q^2 - 120 = 13q - 0.065q^2 - 120$

$\therefore$ when q=100, total profits is $P(100) = 13 - 100 - 0.065(100)^2 - 120$

$= 1300 - 650 - 120 = Tk530$ [Ans]

Problem-11: The sunrise co. Ltd. Has approximated the marginal revenue function for one of its products by $MR = 20x - 2x^2$. The marginal cost function is approximated by $MC = 81 - 16x + x^2$. Determine the profit maximizing output and the total profit at the optimal output.

Solution: For Profit-maximizing output , MR = MC i.e.,

$$20x - 2x^2 = 81 - 16x + x^2$$

$$\therefore \Rightarrow -81 + 36x - 3x^2 = 0 \Rightarrow x^2 - 12x + 27 = 0 \Rightarrow (x-3)(x-9) = 0 \Rightarrow x = 3, 9$$

Now $MP = MR - MC \Rightarrow \frac{dP}{dx} = MR - MC = 20x - 2x^2 - 81 + 16x - x^2 = -3x^2 + 36x - 81$

For maximum profit, $\frac{d^2p}{dx^2} < 0$

To verify this, we first find $\frac{d^{2}p}{dx^2} = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}\left(-3x^2 + 36x - 81\right) = -6x + 36$

Now when $x = 3$, $\frac{d^2p}{dx^2} = -6 \times 3 + 36 = 18 > 0$ and

when $x=9$, $\frac{d^2P}{dx^2}=-6\times9+36=-18<0$.

Therefore, at $x=9$, profit is maximum.

Total profit, $TP=\int_0^9 MPdx=\int_0^9\left(-3x^2+36x-81\right)dx=\left[-x^3+18x^2-81x\right]_0^9$

$=\left[-9^3+18\times9^2-81\times9\right]=0$ i.e no profit [Ans]

Problem 12: Pritom Co. Ltd. Suffers a loss of Tk.121.50 if one of its special products does not sell. Marginal revenue is approximated by $MR=30-6x$ and marginal cost by $MC=-24+3x$. Determine the total profit function, the break-even pints, and the total profit between breakeven points.

Solution: For total profit, we first determine marginal profit

$$MP=MR-MC=(30-6x)-(-24+3x)=54-9x$$

Total profit function, $TP=\int MPdx=\int(54-9x)dx=54x-\frac{9x^2}{2}+k$

Since a loss of Tk 121.50 occurs when there is no sale, k must equal –121.5. Consequently, total profit function equals:

$$P(x)=-121.50+54x-\frac{9}{2}x^2$$

Solving for break-even points, set $P(x)=0\Rightarrow0=-121.5+54x-\frac{9}{2}x^2$

$$\Rightarrow\frac{-243+108x-9x^2}{2}=0\Rightarrow9x^2-108x+243=0\Rightarrow x^2-12x+27=0$$

$$\Rightarrow(x-3)(x-9)=0\Rightarrow x=3,9$$

Integrating the profit function between break-even points will give total profit between break-even points.

$$\therefore TP=\int_3^9\left(-121.5+54x-\frac{9}{2}x^2\right)dx=\left[\left(-121.50x+54\times\frac{x^2}{2}-\frac{9}{6}x^3\right)\right]_3^9$$

$$=\left[-121.50\times9+54\times\frac{9^2}{2}-\frac{9}{6}\times9^3\right]-\left[-121.50\times3+54\times\frac{3^2}{2}-\frac{3}{2}\times3^3\right]$$

$=Tk.162.$ [Ans]

CONSUMER'S SURPLUS

Suppose $P = f(x) = D(x)$ be the demand function for a commodity, where P is the price which the consumer is willing to pay, and x is the quantity of the commodity. Obviously this function is the decreasing function because as P increases, the quantity x is which the consumer is willing tobuy decreases. If p_0 is the market price and x_0 is the corresponding market demand for the commodity, then the consumer expenditure will be X_0P_0. Now Consumer who would be willing to pay more than P_0 (it is also the price
at which producer is willing to sell) will gain. The total expenditure of the consumer for buying the quantity x is $\int_0^{x_0} g(x)dx$.

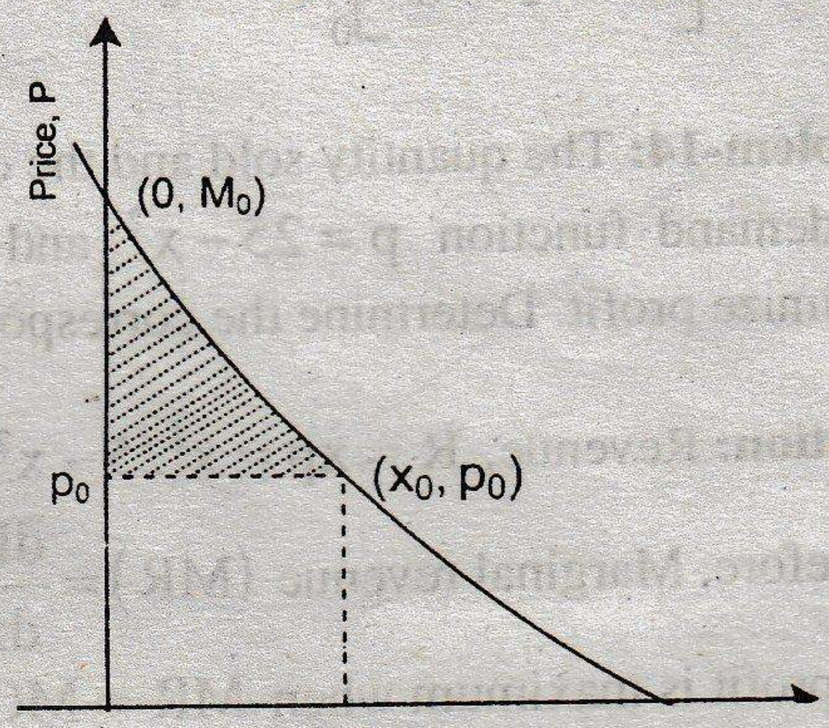


Fig: Consumer's Surplus

The net gain to consumer is represented by the area below the demand curve and above the line $p = p_0$. This area is called the consumer's surplus. In other words, consumer's surplus is the amount which a consumer is willing to pay for a commodity rather than go without it, minus what he would have to pay actually for it at the market price. CS is evaluated as:

$$\text{Consumer's surplus} = \int_0^{x_0} f(x)dx - p_0x_0 = \int_0^{x_0} D(x)dx - p_0x_0$$

Remarks:

1. under pure competition, the price P_0 is determined by equating the demand and supply functions and from this relation the demand x_0 is calculated.
2. under monopoly, the price P_0 is determined by the relation $MR = MC$. From this price value P_0, we obtain the corresponding value of x_0 and then the consumer's surplus is calculated in the usual way.

Problem-13: The demand law for a commodity is $P = 20 - q - q^2$. Find the consumer's surplus when the demand is 3.

Solution: Here $p = f(q) = 20 - q - q^2$

When the demand $q_0 = 3$, the price $p_0 = 20 - 3 - (3)^2 = 8$

$$\therefore \text{ Consumer's surplus} = \int_0^{q_0} (q)dq - p_0q_0 = \int_0^3 (20 - q - q^2)\,dq - (8 \times 3)$$

$$= \left[20q - \frac{q^2}{2} - \frac{q^3}{3} \right]_0^3 - 24 = \left[20 \times 3 - \frac{(3)^2}{2} - \frac{(3)^3}{3} \right] - 24 = \frac{45}{2} \quad [\text{Ans}]$$

Problem-14: The quantity sold and the corresponding price, under a monopoly are determined by the demand function $p = 25 - x^2$ and the marginal cost $MC = 15 + x$ in such a way as to maximize profit. Determine the corresponding consumer's surplus.

Solution: Revenue, $R = xp = x(25 - x^2) = 25x - x^3$.

Therefore, Marginal revenue $(MR) = \frac{dR}{dx} = \frac{d}{dx}(25x - x^2) = 25 - 3x^2$

The profit is maximum when $MR = MC$.

i.e. $25 - 3x^2 = 15 + x \Rightarrow 3x^2 + x - 10 = 0 \Rightarrow x = -2, \frac{5}{3}$.

But quantity cannot be negative, hence $x \neq 2 \therefore x = \frac{5}{3}$.

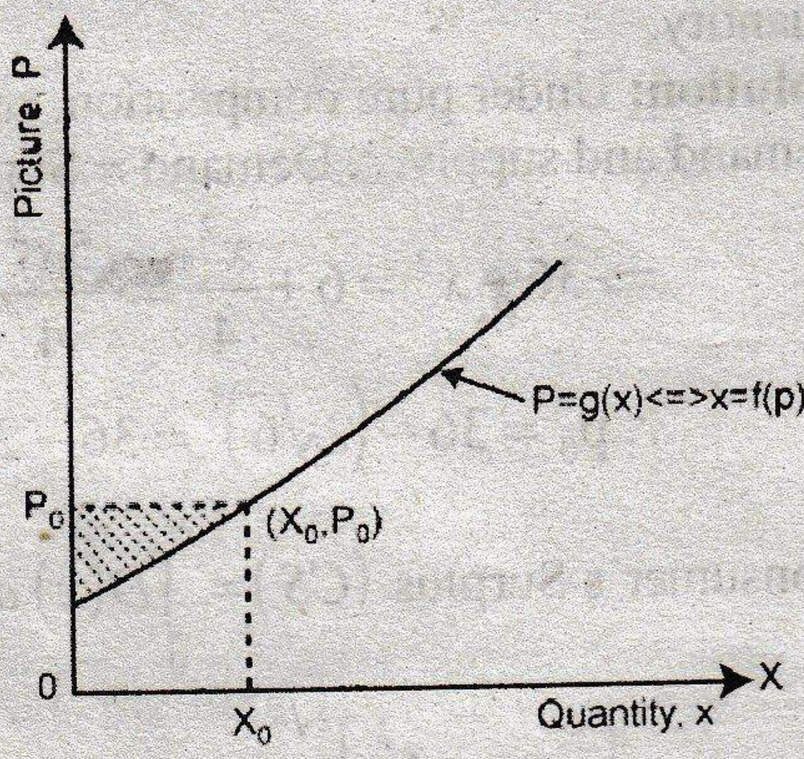
Therefore, when $x_0 = \frac{5}{3}$, then $p_0 = 25 - \left(\frac{5}{3}\right)^2 = 25 - \frac{25}{9} = \frac{200}{9}$

$$\therefore \text{Consumer's Surplus} = \int_0^{5/3} (25 - x^2)\,dx - p_0 x_0 = \int_0^{5/3} (25 - x^2)\,dx - \frac{5}{3} \times \frac{200}{9}$$

$$= \left[25x - \frac{x^3}{3} \right]_0^{5/3} - \frac{1000}{27} = \left[25\left(\frac{5}{2}\right) - \frac{1}{3}\left(\frac{5}{3}\right)^3 \right] - \frac{1000}{27} = \frac{250}{81} [Ans.]$$

PRODUCER'S SURPLUS

Suppose $p = g(x) = s(x)$ be the supply function for a commodity. If p_0 is the market price and x_0 is the corresponding market supply for the commodity, then the producer's revenue will be $x_0 p_0$. Now producer who would be willing to supply the commodity below the market price p_0 (it is the price at which consumers is willing to buy) will earn the revenue as shown in figure. The total revenue of the producer for supplying the quantity x is $\int_0^{x_0} g(x)dx$.



The net revenue to producer is represented by the area above the supply curve and below the line $p = p_0$. This area is called the producer's surplus and PS is evaluated as:

$$\text{Producer's surplus} = p_0 x_0 - \int_0^{x_0} f(x)\,dx = x_0 p_0 - \int_0^{x_0} S(x)\,dx$$

Problem-15: If the supply law is $p = (x+2)^2$ and the price is fixed as $p_0 = 25$, find the producer's surplus.

Solution: Given $p = (x+2)^2$ and $p_0 = 25$.

Now $p = (x+2)^2 \Rightarrow 25 = (x_0+2)^2 \Rightarrow x_0 + 2 = 5 \Rightarrow x_0 = 3$.

$$\text{Producer's surplus} = x_0 p_0 - \int_0^{x_0} (x+2)^2\,dx = 3\times 25 - \int_0^3 (x+2)^2\,dx = 75 - \left[\frac{(x+2)^3}{3}\right]_0^3$$

$$= 75 - \frac{125}{3} + \frac{8}{3} = 36 \text{ [Ans.]}$$

Problem-16: If the supply curve is $p = \sqrt{10+x}$ and the quantity sold in market is 6 units, find the producer's surplus.

Solution: Given $p = \sqrt{10+x}$

If $x_0 = 6$, then $p_0 = \sqrt{10+6} = 4$

$\therefore x_0 = 6$ and $p_0 = 4$

Hence producer's surplus, $PS = x_0 p_0 - \int_0^{x_0} p(x)dx = 6\times 4 - \int_6^6 \sqrt{10+x}\,dx$

$$= 24 - \left[\frac{(10+x)^{3/2}}{3/2}\right]_0^6 = 24 - \frac{2}{3}\left[(16)^{3/2} - (10)^{3/2}\right] = 2.42 \quad \text{[Ans.]}$$

Problem-17: Determine consumer surplus and producer surplus under pure competition for the demand function $p = 36 - x^2$ and supply function $p = 6 + \frac{x^2}{4}$, where p is the price and x is quantity.

Solution: Under pure competition, market equilibrium conditions can be obtained by equating the demand and supply. $\therefore$ Demand = Supply

$$\Rightarrow 36 - x^2 = 6 + \frac{x^2}{4} \Rightarrow \frac{5x^2}{4} = 30 \Rightarrow x^2 = \frac{30 \times 4}{5} = 24 \Rightarrow x = 2\sqrt{6} = x_0$$

$$\therefore p_0 = 36 - \left(2\sqrt{6}\right)^2 = 36 - 24 = 12$$

$$\text{Consumer's Surplus } (CS) = \int_0^{x_0} D(x)\,dx - p_0 x_0 = \int_0^{2\sqrt{6}} \left(36 - x^2\right) dx - 2\sqrt{6} \times 12$$

$$= \left[36x - \frac{x^3}{3}\right]_0^{2\sqrt{6}} - 24\sqrt{6} = 72\sqrt{6} - 16\sqrt{6} - 24\sqrt{6} = 32\sqrt{6}$$

$$\text{Producer's Surplus} = p_0 x_0 - \int_0^{x_0} S(x)\,dx = 2\sqrt{6} \times 12 - \int_0^{2\sqrt{6}} \left(6 + \frac{x^2}{4}\right) dx$$

$$= 24\sqrt{6} - \left[6x + \frac{x^3}{12}\right]_0^{2\sqrt{6}} = 24\sqrt{6} - 12\sqrt{6} - 4\sqrt{6} = 8\sqrt{6} \text{ [Ans.]}$$

Problem-18: Find the consumer surplus and producer surplus under pure competition for demand function $p = \frac{8}{x+1} - 2$ and supply function $p = \frac{1}{2}(x+3)$, where p is price and x is quantity.

Solution: Under pure competition, market equilibrium conditions can be obtained by equating the demand and supply.

$$\therefore \frac{8}{x+1} - 2 = \frac{1}{2}(x+3) \Rightarrow 16 - 4(x+1) = (x+3)(x+1)$$

$$\Rightarrow 16 - 4x - 4 = x^2 + 4x + 3 \Rightarrow x^2 + 8x - 9 = 0 \Rightarrow (x+9)(x-1) = 0 \therefore x = 1 \text{ or } x = -9$$

But $x \neq -9$ as quantity cannot be negative. $\therefore x = 1$

When $x = 1, p = \frac{1}{2}(x+3) = \frac{1}{2}(1+3) = 2.$

$$\text{Consumer surplus} = \int_0^{x_0} D(x)\,dx - p_0 x_0 = \int_0^{1} \left(\frac{8}{x+1} - 2\right) dx - 1 \times 2$$

$$= [8\log(x+1) - 2x]_0^1 - 2 = 8\log 2 - 2 - 2 = 8\log 2 - 4.$$

$$\text{Producer surplus} = p_0 x_0 - \int_0^{x_0} S(x)\,dx = 1 \times 2 - \int_0^{1} \frac{1}{2}(x+3)\,dx$$

$$= 2 - \left[\frac{1}{2}\left(\frac{x^2}{2} + 3x\right)\right]_0^1 = 2 - \frac{1}{2}\left(\frac{1}{2} + 3\right) = 2 - \frac{7}{4} = \frac{1}{4}$$

Problem-19: The demand and supply function under perfect competition are $y = 16 - x^2$ and $y = 2x^2 + 4$ respectively. Find the market price, consumer's surplus and producer's surplus.

Solution: Demand function: $y = 16 - x^2$ ...(1)

Supply function: $y = 2x^2 + 4$ ...(2)

Subtracting (1) from (2), we have

$$0 = 12 - 3x^2 \Rightarrow x = 2 = x_0$$

When $x = 2, y = 16 - (2)^2 = 12 = y_0$

Thus when the quantity demanded or supplied is 2 units, the price is 12 units.

$$\text{Consumer's surplus} = \int_0^2 (16 - x^2)\, dx - 2 \times 12$$

$$= \left[16x - \frac{x^3}{3}\right] - 24 = 32 - \frac{8}{3} - 24 = \frac{16}{3} - 5.33$$

$$\text{Producer's surplus} = 2 \times 12 - \int_0^2 (2x^2 + 4)\, dx$$

$$= 24 - \left[\frac{2x^2}{3} + 4x\right]_0^2 = 24 - \left[\frac{16}{3} + 8\right] = \frac{32}{3} = 10.67 \text{ [Ans.]}$$

Problem-20: Demand and supply functions are $D(x) = (12 - 2x)^2$ and $S(x) = 56 + 4x$ respectively. Determine CS under monopoly (so as to maximize the profit) and the supply function is identified with the marginal cost function.

Solution: Total revenue $= TR = x \times D(x)$

$$= (144 - 48x + 4x^2)x \quad = 144x - 48x^2 + 4x^3$$

$$\therefore MR = \frac{d}{dx}(144x - 48x^2 + 4x^3) = 144 - 96x + 12x^2$$

Since the supply price is identified with MC, we have $MC = 56 + 4x$

In order to find CS under monopoly, i.e., to maximize profit, we must have, $MR = MC$

$$\Rightarrow 144 - 96x + 12x^2 = 56 + 4x \Rightarrow 12x^2 - 100x + 88 = 0$$

$$\Rightarrow 3x^2 - 25x + 22 = 0 \Rightarrow x = 1 = x_0 \text{ or } x = \frac{22}{3} = x_0$$

when $x_0 = 1, D(x_0) = p_0 = (12 - 2 \times 1)^2 = 100$

$$\therefore CS = \int_0^1 (12-2x)^2 = x_0 p_0 = \int_0^1 (144-48x+4x^2)dx - 1\times 100$$

$$= \left[144x - 48.\frac{x^2}{2} + 4.\frac{x^3}{3}\right]_0^1 - 100 = 144 - 24 + \frac{4}{3} - 100 = \frac{64}{3} \text{ units.}$$

Again when $x_0 = \frac{22}{3}, p_0 = \left(12 - \frac{44}{3}\right)^2 = \frac{64}{9}$

$$\therefore CS = \int_0^{22/3} \left(144-48x+4x^2\right)dx - \frac{22}{3}\times\frac{64}{9} = \left[144x - 48\times\frac{x^2}{2} + 4\times\frac{x^3}{3}\right]_0^{22/3} - \frac{22}{3}\times\frac{64}{9} = \frac{19{,}360}{81} \text{ units.}$$

BRIEF REVIEW

Definition

Integration: Integration is the inverse process of differentiation. Mathematically, let $f(x)$ be a given function of x such that $\frac{d}{dx}(F(x)) = f(x)$, then $F(x)$ is called an *Indefinite Integral* of $f(x)$ with respect to x and is denoted by $\int f(x)\,dx = F(x) + C$. In this case, the process of finding $F(x)$ is called integration and $f(x)$ is called *Integrand* and C is called constant of integration. For example: $\frac{d}{dx}(x^3) = 3x^2$ but $\int 3x^2 dx = x^3 + C$.

Constant of integration: We have $\frac{d}{dx}[F(x)] = f(x)$ and $\frac{d}{dx}[F(x)+c] = f(x)$.

Therefore, by definition we have, $\int f(x)\,dx = F(x) + C$.....(1)

Here the arbitrary constant C (which is independent of x) is usually referred to as the *constant of integration.*

Types of Integration: Integration may be divided into two categories such as

(i) Indefinite integration, Example: $\int x^2 dx = \frac{x^3}{3} + C$, which is indefinite value;

and (ii) definite integration; Example: $\int_2^3 x^2 dx = \left|\frac{x^3}{3}\right|_2^3 = \frac{1}{3}(3^3 - 2^3) = \frac{19}{3}$, which is definite value.

Definite Integration: Let $\int f(x)\ dx = F(x)+C$, then $\int_a^b f(x)dx = \left|F(x)+C\right|_a^b$

$=\{F(b)+C\}-\{F(a)+C\}=F(b)-F(a)$, which is definite. Therefore the integral $\int_a^b f(x)dx$ is called definite integral of $f(x)$ from a to b and here a is called lower limit and b is called upper limit of integration.

Quiz Questions

Multiple Choice Questions

1. Which is the correct formula of $\int x^n\ dx$?

(a) $\frac{x^n-1}{n-1}+c$ (b) $n\frac{x^n+1}{n+1}+c$ (c) $\frac{x^n+1}{n+1}+c$ (d) $\frac{x^n-1}{n+1}+c$

2. Choose the correct integration of $\int e^{2x}\ dx$?

(a) $e^{2x}+c$ (b) $\frac{1}{2x}e^{2x}+c$ (c) $2e^{2x}+c$ (d) $\frac{1}{2}e^{2x}+c$

3. Find out the right integration of $\int 3^x\ dx$?

(a) $\frac{1}{3}.3^x+c$ (b) $\frac{3^x}{\log e^3}+c$ (c) $\frac{3^x}{\log e^x}+c$ (d) $\frac{3^x}{\log 3^x}+c$

4. What is the correct integration of $\int\frac{dx}{\sqrt{3^2-x^2}}$

(a) $Sin^{-1}\frac{x}{9}+c$ (b) $Sin^{-1}\frac{x}{3}+c$ (c) $Sin^{-1}x+c$ (d) $Sin^{-1}3x+c$

5. Which is right of $\int\frac{dx}{x^2+4^2}$

(a) $\frac{1}{4}\tan^{-1}\frac{x}{4}+c$ (b) $\log\left\{\frac{1}{4}\left(x+\sqrt{x^2+4}\right)\right\}+c$ (c) $\frac{1}{8}\log\frac{4+x}{4-x}+c$ (d) $\tan^{-1}\frac{x}{4}+c$

6. What is the appropriate formula of $\int uvdx$?

(a) $v\int udx-\int\left[\frac{d}{dx}(u)\left\{\int vdx\right\}\right]dx$ (b) $u\int v\,dx-\int\left[\frac{d}{dx}(v)\left\{\int udx\right\}\right]dx$

(c) $v\int vdx-\int\left[\frac{d}{dx}(u)\left\{\int vdx\right\}\right]$ (d) $u\int vdx-\int\left[\frac{d}{dx}(v)\left\{\int udx\right\}\right]dx$

7. Find out the correct integration of $\int x\,Sinx\,dx$.

(a) $Cosx - Sinx + c$ (b) $Sinx - x\cos x + c$ (c) $Sinx + x\cos x + c$ (d) $Cosx + x\sin x + c$

8. What is the value of $\int_0^2 \frac{1}{x^3} dx$? (a) $\frac{1}{8}$ (b) $-\frac{1}{4}$ (c) $-\frac{1}{8}$ (d) $\frac{1}{4}$

9. Choose the value of $12\int_1^3 x^3\,dx$ (a) 78 (b) 280 (c) 120 (d) 240

10. Which is the correct value of $\int_0^3 (x^2 - 2x + 3)dx$ (a) 9 (b)-9 (c) 0 (d) 27

Which one of the following statement is true/false?

a. Integrations are of three kinds.

b. Anti-derivative is called integration.

c. $\int a^x dx = \frac{a^x}{\ln a} + C$

d. $\int_a^b f(x)dx = \int_b^a f(x)dx$

e. $\int e^{2x} dx = 2e^{2x} + C$

f. The definite integral $\int_a^b f(x)dx$ represents area of $f(x)$ from a to b.

Brief Questions

1. Mention the types of integration.

2. What is the value of $\int_0^4 dx$?

3. What is the value of $\int 2^x dx$?

4. What is the value of $\int_0^2 e^{2x} dx$?

5. $\int (u.v)dx = ?$, where u and v are both functions of x.

6. What is the value of $\int x\sqrt{x}\,dx$?

7. What is the value of $\int (3x + 2 + e^x)\,dx = ?$

8. Evaluate: $\int \log x\,dx$.

Conceptual, Analytical & Numerical Questions

1. Distinguish between differentiation and integration.
2. Distinguish between definite and indefinite integration.
3. What do you mean by constant of integration?
4. What is indefinite integration?
5. What is definite integration?
6. Why definite integral called definite ?
7. If MC = $2x+5$, then what will be the total cost?

Numerical questions

1. Evaluate (a) $\int_0^a \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}$ (b) $\int_0^{\pi} \frac{dx}{1+\frac{1}{2}\cos x}$, (c) $\int_2^3 e^{2x}dx$, (d) $\int_0^{\frac{\pi}{2}} \sqrt{1+\sin x}\,dx$

2. Find the value of : (a) $\int_0^{\frac{\pi}{3}} \frac{\cos x dx}{3+4\sin x}$, (b) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}}$

3. Evaluation the following integrals: (a) $\int_0^{\frac{\pi}{2}} \frac{dx}{3+\cos 2x}$, (b) $\int_0^{\frac{\pi}{4}} \frac{dx}{2+\tan x}$

   (c) $\int_2^5 \frac{\sqrt{x}}{\sqrt{7-x}+\sqrt{x}}dx$

4. Evaluate (i) $\int_0^2 \frac{\sqrt{x}dx}{\sqrt{x}+\sqrt{2-x}}$ (ii) $\int_0^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$

5. Prove that (a) $\int_3^8 \frac{dx}{x-3} = \log 5$, (b) $\int_0^{\frac{\pi}{2}} \frac{\cos x dx}{(1+\sin x)(2+\sin x)} = \log\frac{4}{3}$,

   (c) $\int_0^{\pi} \frac{dx}{5+3\cos x} = \frac{\pi}{4}$, (d) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x} = \frac{\pi}{4}$

6. If marginal cost of a firm is given by $MC(q) = 2+5e^q$ .find the total cost if $C(0) = 100$ Also find average cost. What will be the marginal average and total cost for $q = 60$ units?

7. The marginal cost function of a firm is $100-10x+0.1x^2$, where x is the output. Obtain the total cost function of the firm under the assumption that its fixed cost is Tk. 500.

8. The marginal cost of production of consumer goods is found to be $MC = 200-40x+3x^2$, where x is the number of units produced. The fixed cost of production is Tk. 18,000. Find the cost function. If the manufacturer fixes the price per unit at Tk. 6800.
   a. Find the revenue function.
   b. Find the profit function.

c. Find the sales volume that gives maximum profit?
d. What is the profit at this sales volume?

9. A company determines that the marginal cost of producing x units of a particular commodity during a one-day operation is $MC = 16x - 1591$. The selling price of commodity is fixed at Tk. 9. per unit and the fixed cost is Tk. 1800 per day.
What is the maximum profit that can be obtained in a one day operation?

10. If the marginal cost function is given by $MC = \frac{3}{\sqrt{3q+4}}$ and fixed cost is Tk. 2, find the average cost for 4 units of units of output.

11. The marginal revenue function of a service-providing firm is given by $7 - 4x - x^2$. Find the total revenue function and the demand function of that organization.

12. If the marginal revenue and the marginal cost for an output x of a commodity are given are as $MR = 5 - 4x + 3x^2$ and $MC = 3 + 2x$ respectively, and if the fixed cost is zero, find the profit function and the profit when the output is $x = 4$.

13. If the marginal cost function is $MC = x^2 - 16x + 20$ and marginal revenue function is $MR = 20 - 2x$, determine the profit-maximizing output and the corresponding total profit. Cost is in units of Tk. 1000 and x is in units of output.

14. The demand function and supply function of a firm are given by: $D(x) = 25 - 5x + \frac{x^2}{4}$ and $S(x) = 5x - \frac{x^2}{4}$. Determine the consumer's and the producer's surplus. Assume a monopoly situation.

15. Find the consumer's surplus (at equilibrium price) if the demand function is $q = \frac{25}{4} - \frac{p}{8}$ and supply function is $p = 5 + q$.

16. Find consumer's surplus and producer's surplus defined by the demand curve $D(x) = 20 - 5x$ and supply curve $S(x) = 4x + 8$.

17. The quality sold and the corresponding price under monopoly are determined by the demand law $p = 16 - x^2$ and by the $MC = 6 + x$ in such a way as to maximize the profit. Determine corresponding consumer surplus.

18. The demand law and marginal cost of a firm is given by $p = 45 - x^2$ and $MC = 6 + \frac{x^2}{4}$ respectively. Determine consumer surplus.

19. Assume that the demand and average cost curves of steel are:
$p = 2.34 - 1.34x$ and $AC = \frac{1}{x} - 0.83 + 0.85x$, x is the quantity of steel demanded.
Show that consumer's surplus under monopoly and perfect competition is 0.351 and 0.129 respectively.

20. Find the consumer's surplus if the demand curve is $D(x) = 50 - 0.025x^2$. It is known that the market quantity is 20 units.