Light and Quantum Physics

CHAPTER 47

47–1 Sources of Light

We have studied the propagation, reflection, refraction, diffraction, polarization, scattering, and interference of light. This chapter deals in part with the *production* of light and with the way that such studies led, in 1900, to the birth of modern quantum physics.

The most common light sources are heated solids and gases through which an electric discharge is passing. The tungsten filament of an incandescent lamp and the familiar neon sign are examples in each category. By analyzing the light from a source with a spectrometer, we can learn how strongly it radiates at various wavelengths. Figure 47–1, which is typical of spectra for heated solids, shows the results of such measurements for a heated tungsten ribbon at 2000° K.

The ordinate $\mathcal{R}_\lambda$ in Fig. 47–1 is called the *spectral radiancy*, defined so that the quantity $\mathcal{R}_\lambda\, d\lambda$ is the rate at which energy is radiated per unit area of surface for wavelengths lying in the interval λ to $\lambda + d\lambda$. Typical units for $\mathcal{R}_\lambda$ are watts/cm^2-μ; the corresponding units of $\mathcal{R}_\lambda\, d\lambda$ are watts/cm^2. In measuring $\mathcal{R}_\lambda$, all radiation emerging into the forward hemisphere is included.

Sometimes we wish to discuss the radiated energy without regard to its wavelength. An appropriate quantity here is the *radiancy* $\mathcal{R}$, defined as the rate per unit surface area at which energy is radiated into the forward hemisphere, appropriate units being watts/cm^2. It can be found by integrating the radiation present in all wavelength intervals:

$$\mathcal{R} = \int_0^\infty \mathcal{R}_\lambda\, d\lambda. \tag{47–1}$$

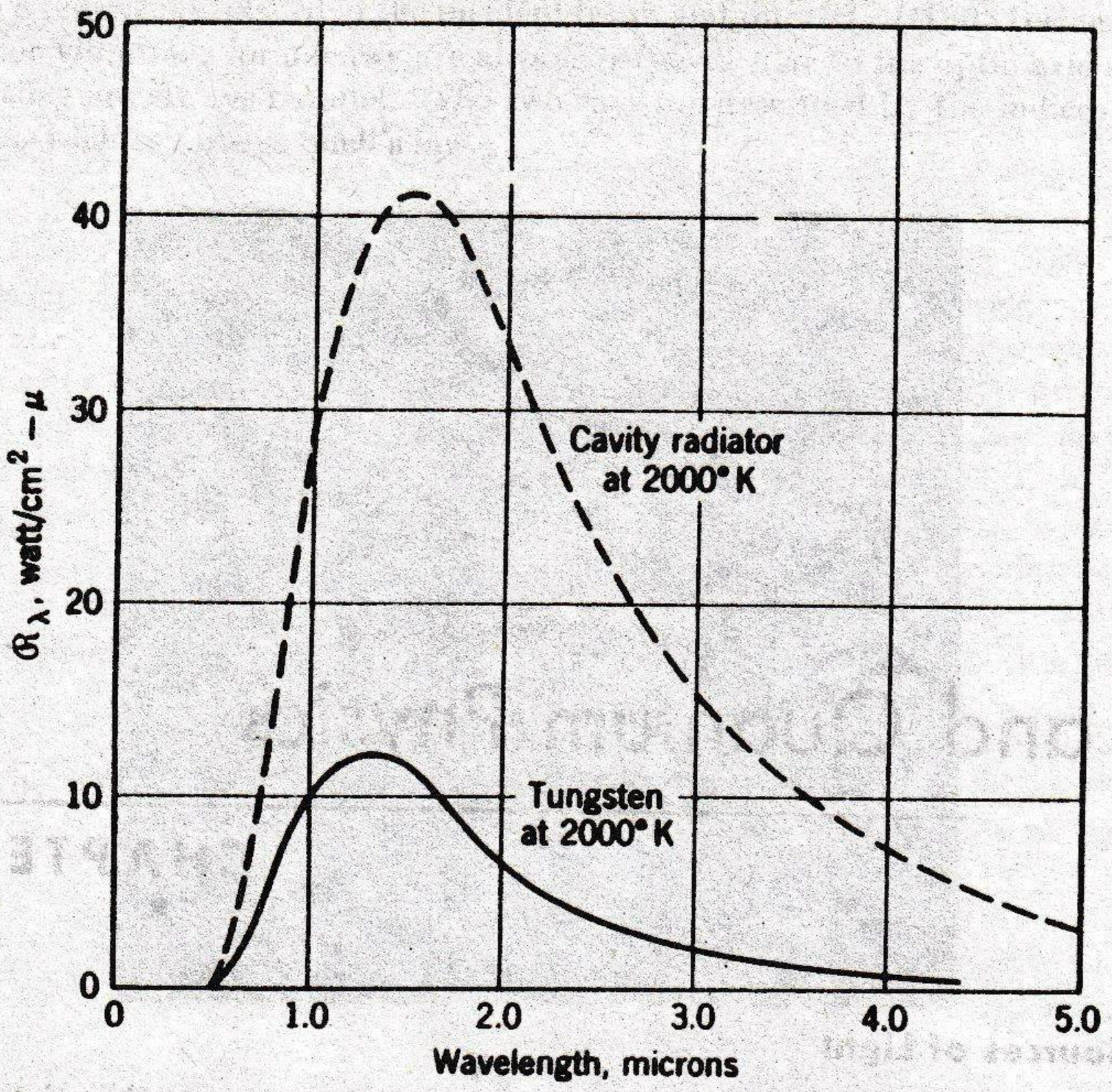


Fig. 47–1 The spectral radiancy of tungsten at 2000°K. The dashed curve refers to a cavity radiator at the same temperature. One micron (1 μ) = 10^{-6} meter = 10^4 A.

The radiancy $\mathcal{R}$ can be interpreted as the area under the plot of $\mathcal{R}_\lambda$ against λ. In Fig. 47–1 this area—and thus $\mathcal{R}$—is 23.5 watts/cm^2. The student should note the formal similarity between such curves and the Maxwell speed distribution curve of Section 24–2.

For every material there exists a family of spectral radiancy curves like that of Fig. 47–1, one curve for every temperature. If such families of curves are compared, no obvious regularities stand out. A quantitative understanding in terms of a basic theory presents serious difficulties. Fortunately, it is possible to work with an *idealized heated solid*, called a *cavity radiator*. Its light-emitting properties prove to be independent of any particular material and to vary in a simple way with temperature. In much the same way it proved convenient earlier to deal with an ideal gas rather than to analyze the properties of the infinite variety of real gases. The cavity radiator is the *ideal solid* as far as its light-emitting characteristics are concerned. We shall describe in the next two sections how the theoretical study of cavity radiation in 1900 by the German physicist Max Planck (1858–1947) laid the foundations of modern quantum physics.

47–2 Cavity Radiators

Let us construct a cavity in each of three metal blocks through the walls of which a small hole is drilled. Let the blocks be made of any suitable materials; for example, tungsten, tantalum, and molybdenum. Let each block be

raised to the same uniform temperature (say 2000°K) as determined by a suitable thermometer. Finally, let us observe the blocks by their emitted light in a dark room. Measurements of $\mathcal{R}$ and $\mathcal{R}_\lambda$ show the following:

1. The radiation from the cavity interior is always more intense than the radiation from the outside wall. Comparison of the two curves in Fig. 47–1 makes this clear for tungsten. For the three materials given, at 2000°K the ratio of the radiancy for the outside surface to that for the cavity is 0.259 (tungsten), 0.212 (molybdenum), and 0.232 (tantalum).

2. At a given temperature the radiancy of the hole is *identical for all three radiators*, in spite of the fact that the radiancies of the outer surfaces are different. At 2000°K the cavity radiancy (that is, the hole radiancy) is 90.0 watts/cm^2.

3. In contrast to the radiancy of the outer surfaces, the cavity radiancy $\mathcal{R}_c$ varies with temperature in a simple way, namely as

$$\mathcal{R}_c = \sigma T^4, \tag{47–2}$$

where σ is a universal constant (the Stefan-Boltzmann constant) whose measured value is 5.67×10^{-8} watt/(meter^2)(°K^4). The radiancy of the outer surfaces varies with temperature in a more complicated way and is different for different materials. It is often written as

$$\mathcal{R} = e\mathcal{R}_c = e\sigma T^4, \tag{47–3}$$

where e, the *emissivity*, depends on the material and the temperature.

4. $\mathcal{R}_\lambda$ for the cavity radiation varies with temperature in the way shown in Fig. 47–2. These curves depend only on the temperature and are quite independent of the material and of the shape and size of the cavity.

Figure 47–3 shows an actual cavity, consisting of a hollow thin-walled cylinder of tungsten heated by sending an electric current through it. The cylinder is mounted in an evacuated glass bulb, and a tiny hole is drilled

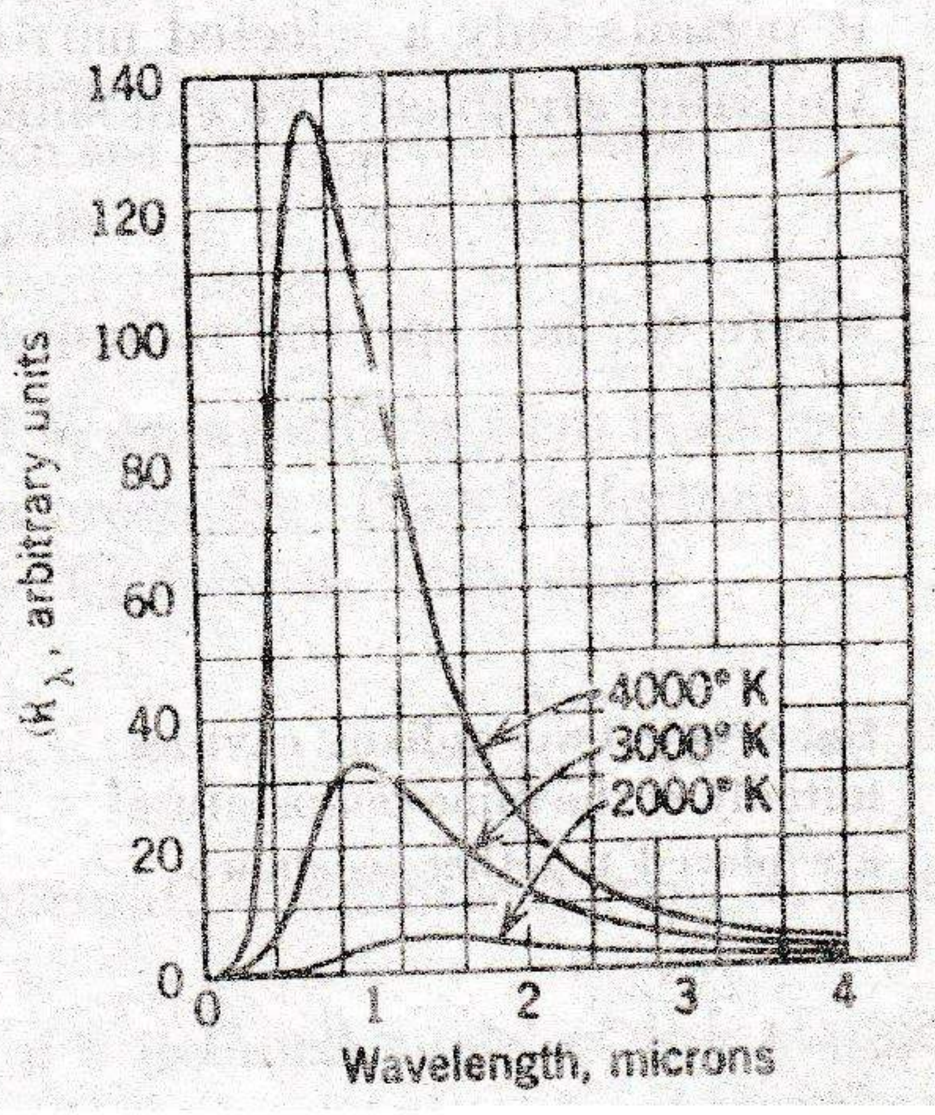


Fig. 47–2 The spectral radiancy for cavity radiation at three different temperatures.

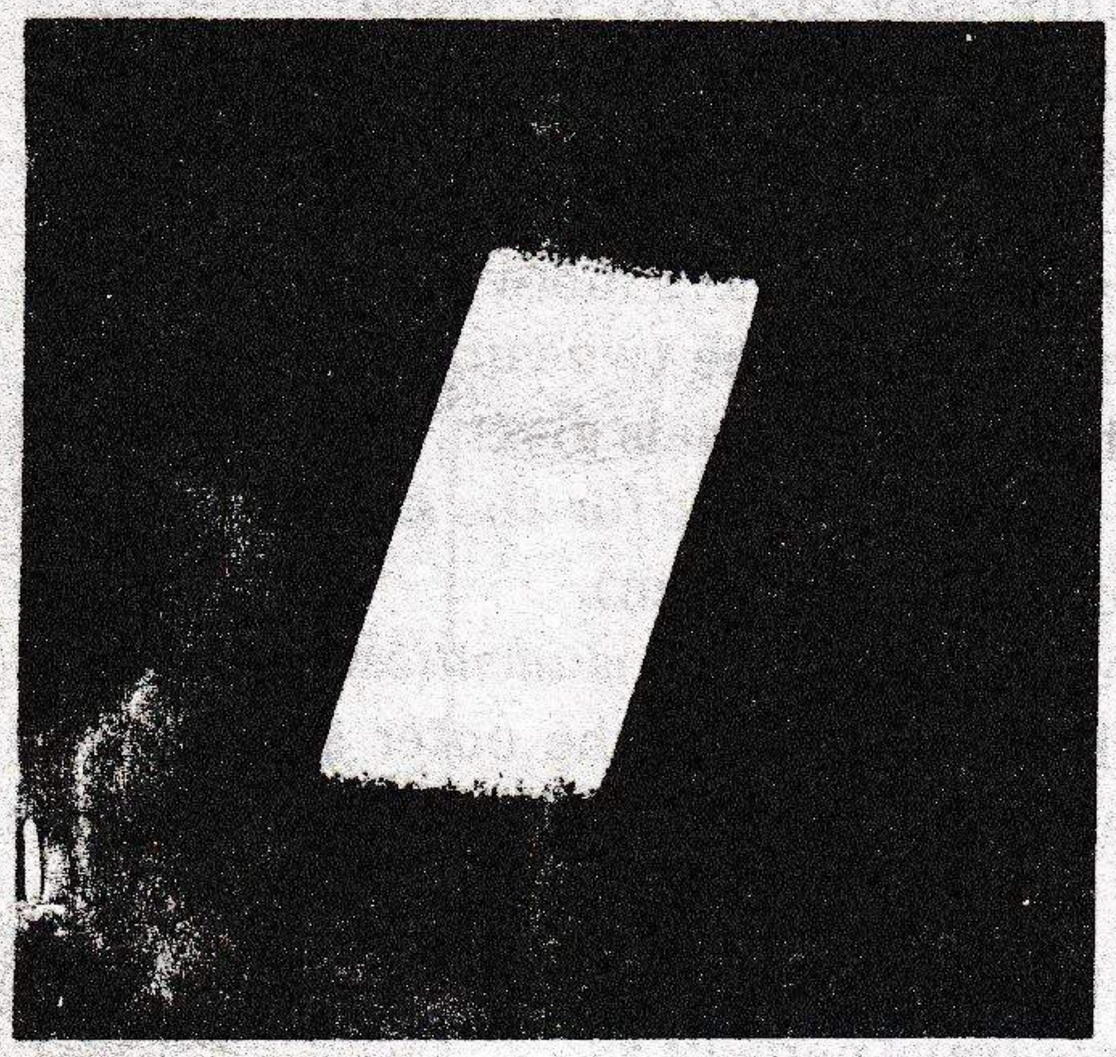


Fig. 47–3 Photograph of an incandescent tungsten tube with a small hole drilled in its wall. The radiation emerging from the hole is cavity radiation.

through the cylinder wall. It is clear from the photograph that the radiancy of the cavity interior is greater than that of the cavity walls.

Many of the facts just given about cavity radiation can be deduced from Fig. 47–4, which shows two cavities made of different materials, of arbitrary shapes, and with the same wall temperature T. Radiation, described by $\mathcal{R}_A$, goes from cavity A to cavity B and radiation described by $\mathcal{R}_B$ moves in the opposite direction. If these two rates of energy transfer are not equal, one end of the composite block will start to heat up and the other end will start to cool down, which is a violation of the second law of thermodynamics. (Why?) Thus we must have

$$\mathcal{R}_A = \mathcal{R}_B = \mathcal{R}_c, \tag{47–4}$$

where $\mathcal{R}_c$ describes the total radiation for *all* cavities.

Not only the total radiation but also the distribution of radiant energy with wavelength must be the same for each cavity in Fig. 47–4. This can be shown by placing a filter between the two cavity openings, so chosen that it permits only a selected narrow band of wavelengths to pass. Applying the same argument, we can show that we must have

$$\mathcal{R}_{\lambda A} = \mathcal{R}_{\lambda B} = \mathcal{R}_{\lambda c}, \tag{47–5}$$

where $\mathcal{R}_{\lambda c}$ is a spectral radiancy characteristic of all cavities.

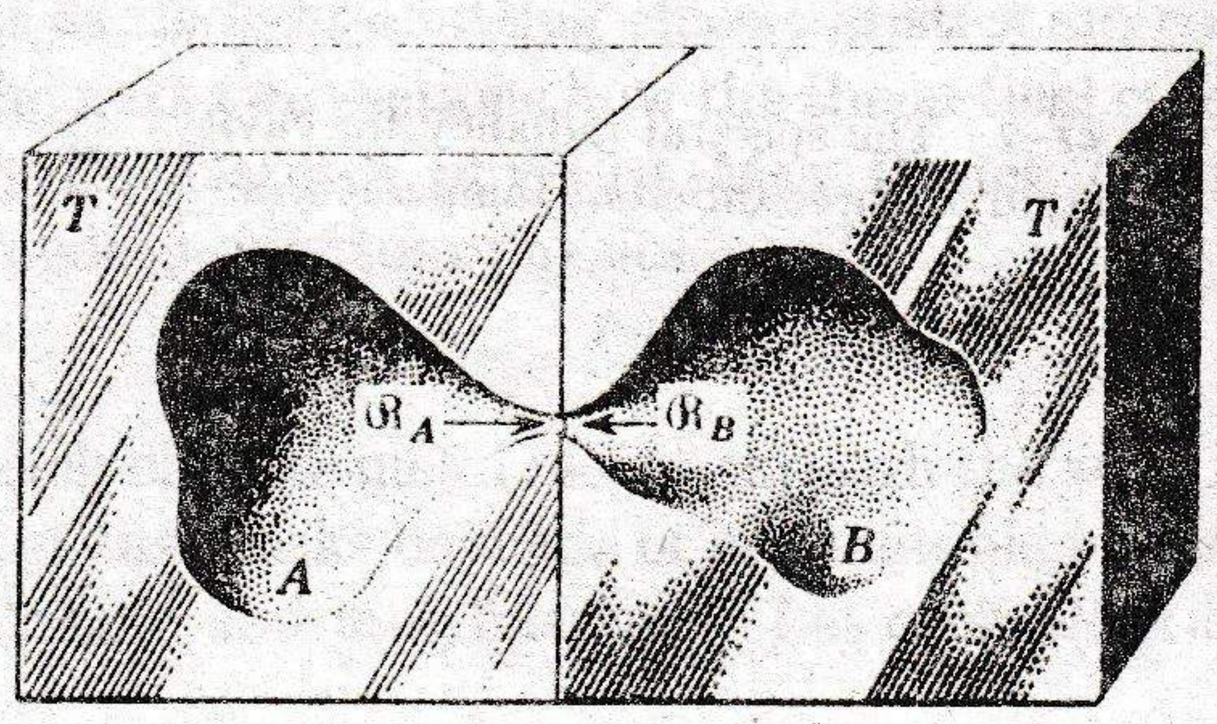


Fig. 47–4 Two radiant cavities initially at the same temperature are placed together as shown.

47–3 Planck's Radiation Formula

A theoretical explanation for the cavity radiation was the outstanding unsolved problem in physics during the years before the turn of the present century. A number of capable physicists advanced theories based on classical physics, which, however, had only limited success. Figure 47–5, for example, shows the theory of Wien; the fit to the experimental points is reasonably good, within the experimental error of the data, but definitely not exact. Wien's formula is

$$\mathcal{R}_\lambda = \frac{c_1}{\lambda^5} \frac{1}{e^{c_2/\lambda T}},$$

where c_1 and c_2 are constants that must be determined empirically by fitting the theoretical formula to the experimental data.

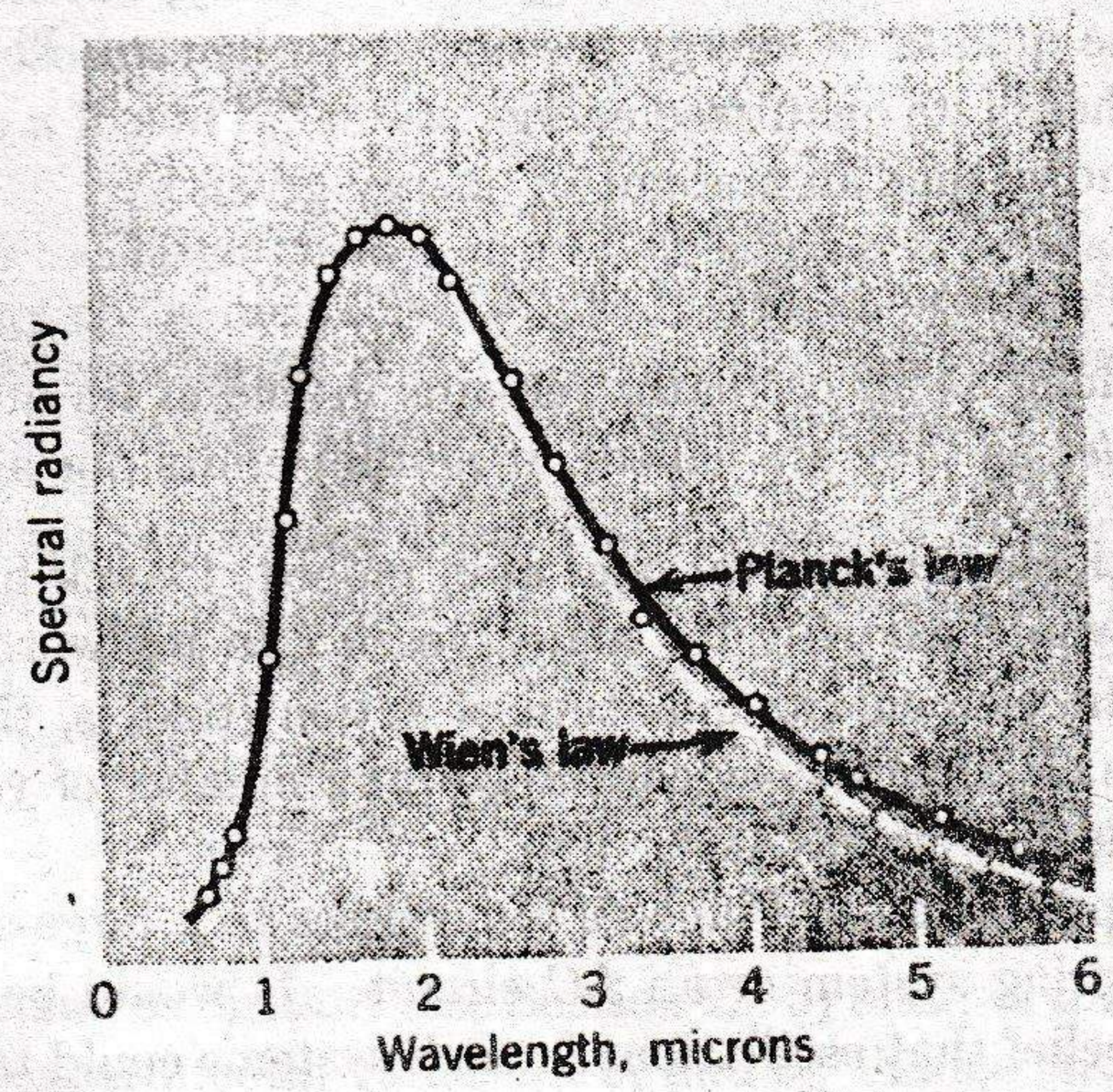


Fig. 47–5 The circles show the experimental spectral radiancy data of Coblentz for cavity radiation. The theoretical formulas of Wien and Planck are also shown, Planck's providing an excellent fit to the data.

In 1900 Max Planck pointed out that if Wien's formula were modified in a simple way it would prove to fit the data precisely. Planck's formula, announced to the Berlin Physical Society on October 19, 1900, was

$$\mathcal{R}_\lambda = \frac{c_1}{\lambda^5} \frac{1}{e^{c_2/\lambda T} - 1}. \tag{47–6}$$

This formula, though interesting and important, was still empirical at that stage and did not constitute a theory.

Planck sought such a theory in terms of a detailed model of the atomic processes taking place at the cavity walls. He assumed that the atoms that

make up these walls behave like tiny electromagnetic oscillators, each with a characteristic frequency of oscillation. The oscillators emit electromagnetic energy into the cavity and absorb electromagnetic energy from it. Thus it should be possible to deduce the characteristics of the cavity radiation from those of the oscillators with which it is in equilibrium.

Planck was led to make two radical assumptions about the atomic oscillators. As eventually formulated, these assumptions are the following:

1. An oscillator cannot have *any* energy but only energies given by *

$$E = nh\nu, \tag{47-7}$$

where ν is the oscillator frequency, h is a constant (now called *Planck's constant*), and n is a number (now called a *quantum number*) that can take on only integral values. Equation 47–7 asserts that the oscillator energy is *quantized*.

2. The oscillators do not radiate energy continuously, but only in "jumps," or *quanta*. These quanta of energy are emitted when an oscillator changes from one to another of its quantized energy states. Thus, if n changes by one unit, Eq. 47–7 shows that an amount of energy given by

$$\Delta E = \Delta nh\nu = h\nu \tag{47-8}$$

is radiated. As long as an oscillator remains in one of its quantized states (or *stationary states* as they are called), it neither emits nor absorbs energy.

These assumptions were radical ones and, indeed, Planck himself resisted accepting them wholeheartedly for many years. In his words, "My futile attempts to fit the elementary quantum of action [that is, the quantity h] somehow into the classical theory continued for a number of years, and they cost me a great deal of effort."

Consider the application of Planck's hypotheses to a large-scale oscillator such as a mass-spring system or an LC circuit. It would be a stoutly defended common belief that oscillations in such systems could take place with *any* value of total energy and not with only certain discrete values. In the decay of such oscillations (by friction in the mass-spring system or by resistance and radiation in the LC circuit), it would seem that the mechanical or electromagnetic energy would decrease in a perfectly continuous way and not by "jumps." There is no basis in everyday experience, however, to dismiss Planck's assumptions as violations of "common sense," for Planck's constant proves to have a very small value, namely

$$h = 6.625 \times 10^{-34} \text{ joule-sec.}$$

The following example makes this clear.

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* Later developments show that the correct formula for a harmonic oscillator is

$$E = (n + \tfrac{1}{2})h\nu.$$

This change makes no difference to Planck's conclusions, however.

▶ **Example 1.** A mass-spring system has a mass $m = 1.0$ kg and a spring constant $k = 20$ nt/meter and is oscillating with an amplitude of 1.0 cm. (*a*) If its energy is quantized according to Eq. 47–7, what is the quantum number n? (*b*) If n changes by unity, what fractional change in energy occurs?

(*a*) From Eq. 15–8 the frequency is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{20 \text{ nt/meter}}{1.0 \text{ kg}}} = 0.71 \text{ cycles/sec}.$$

From Eq. 8–8 the mechanical energy is

$$E = \tfrac{1}{2}kx_{\max}^2 = \tfrac{1}{2}(20 \text{ nt/meter})(10^{-2} \text{ meter})^2 = 1.0 \times 10^{-3} \text{ joule}.$$

From Eq. 47–7 the quantum number is

$$n = \frac{E}{h\nu} = \frac{1.0 \times 10^{-3} \text{ joule}}{(6.6 \times 10^{-34} \text{ joule-sec})(0.71 \text{ cycles/sec})} = 2.1 \times 10^{30}.$$

(*b*) If n changes by unity, the fractional change in energy is given by dividing Eq. 47–8 by Eq. 47–7, or

$$\frac{\Delta E}{E} = \frac{h\nu}{nh\nu} = \frac{1}{n} = \sim 10^{-30}.$$

Thus for large-scale oscillators the quantum numbers are enormous and the quantized nature of the energy of the oscillations will not be apparent. Similarly, we are not aware in large-scale experiments of the discrete nature of mass and the quantized nature of charge, that is, of the existence of atoms and electrons. ◀

On the basis of his two assumptions, Planck was able to derive his radiation law (Eq. 47–6) entirely from theory, receiving a Nobel prize for this accomplishment in 1918. His theoretical expressions for the hitherto empirical constants c_1 and c_2 were

$$c_1 = 2\pi c^2 h \qquad \text{and} \qquad c_2 = \frac{hc}{k},$$

where k is Boltzmann's constant (see Section 23–5) and c is the speed of light. By inserting the experimental values for c_1 and c_2, Planck was able to derive the values of both h and k. Planck described his theory to the Berlin Physical Society on December 14, 1900. Quantum physics dates from that day. Planck's ideas soon received re-enforcement from Einstein, who, in 1905, applied the concepts of energy quantization to a new area of physics, the photoelectric effect.

Before discussing this effect, it is important to realize that although Planck had quantized the energies of the oscillators in the cavity walls he *still treated the radiation within the cavity* as an electromagnetic wave. Einstein's analysis of the photoelectric effect first pointed out the inadequacy of the wave picture of light in certain situations.

47–4 Photoelectric Effect

Figure 47–6 shows an apparatus used to study the photoelectric effect. Monochromatic light, falling on metal plate A, will liberate *photoelectrons*, which can be detected as a current if they are attracted to metal cup B by

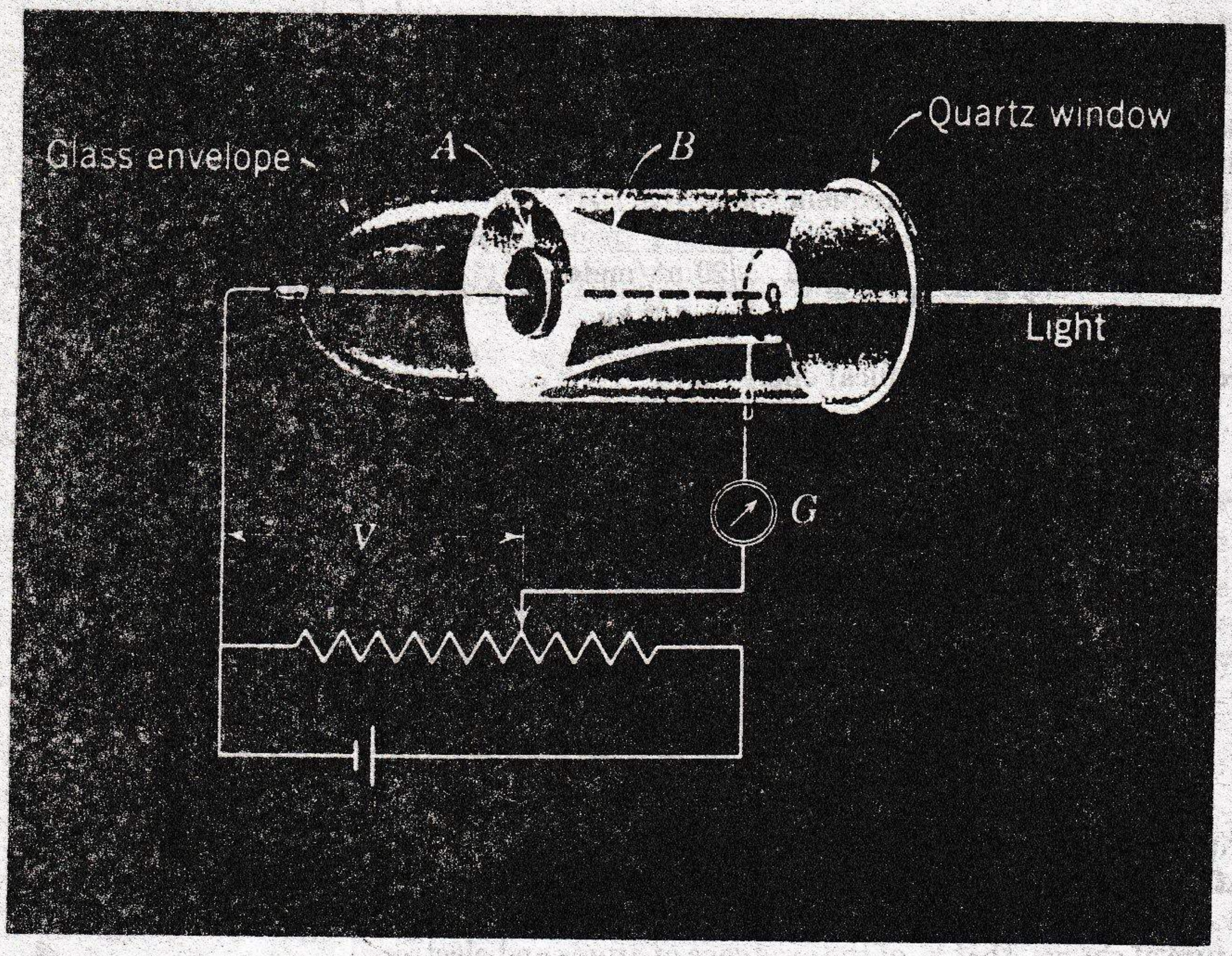


Fig. 47–6 An apparatus used to study the photoelectric effect. V cannot only be varied continuously but can be reversed in sign by a switching arrangement not shown.

means of a potential difference V applied between A and B. Galvanometer G serves to measure this *photoelectric current*.

Figure 47–7 (curve a) is a plot of the photoelectric current in an apparatus like that of Fig. 47–6, as a function of the potential difference V. If V is made large enough, the photoelectric current reaches a certain limiting value at which *all* photoelectrons ejected from plate A are collected by cup B.

If V is reversed in sign, the photoelectric current does not immediately drop to zero, which proves that the electrons are emitted from A with a finite velocity. Some will reach cup B in spite of the fact that the electric field opposes their motion. However, if this reversed potential difference is made large enough, a value V_0 (the *stopping potential*) is reached at which the photoelectric current does drop to zero. This potential difference V_0, multiplied by electron charge, measures the kinetic energy K_{max} of the *fastest* ejected photoelectron. In other words,

$$K_{max} = eV_0. \tag{47–9}$$

Here K_{max} turns out to be independent of the intensity of the light as shown by curve b in Fig. 47–7, in which the light intensity has been reduced to one-half.

Figure 47–8 shows the stopping potential V_0 as a function of the frequency of the incident light for sodium. Note that there is a definite cutoff frequency

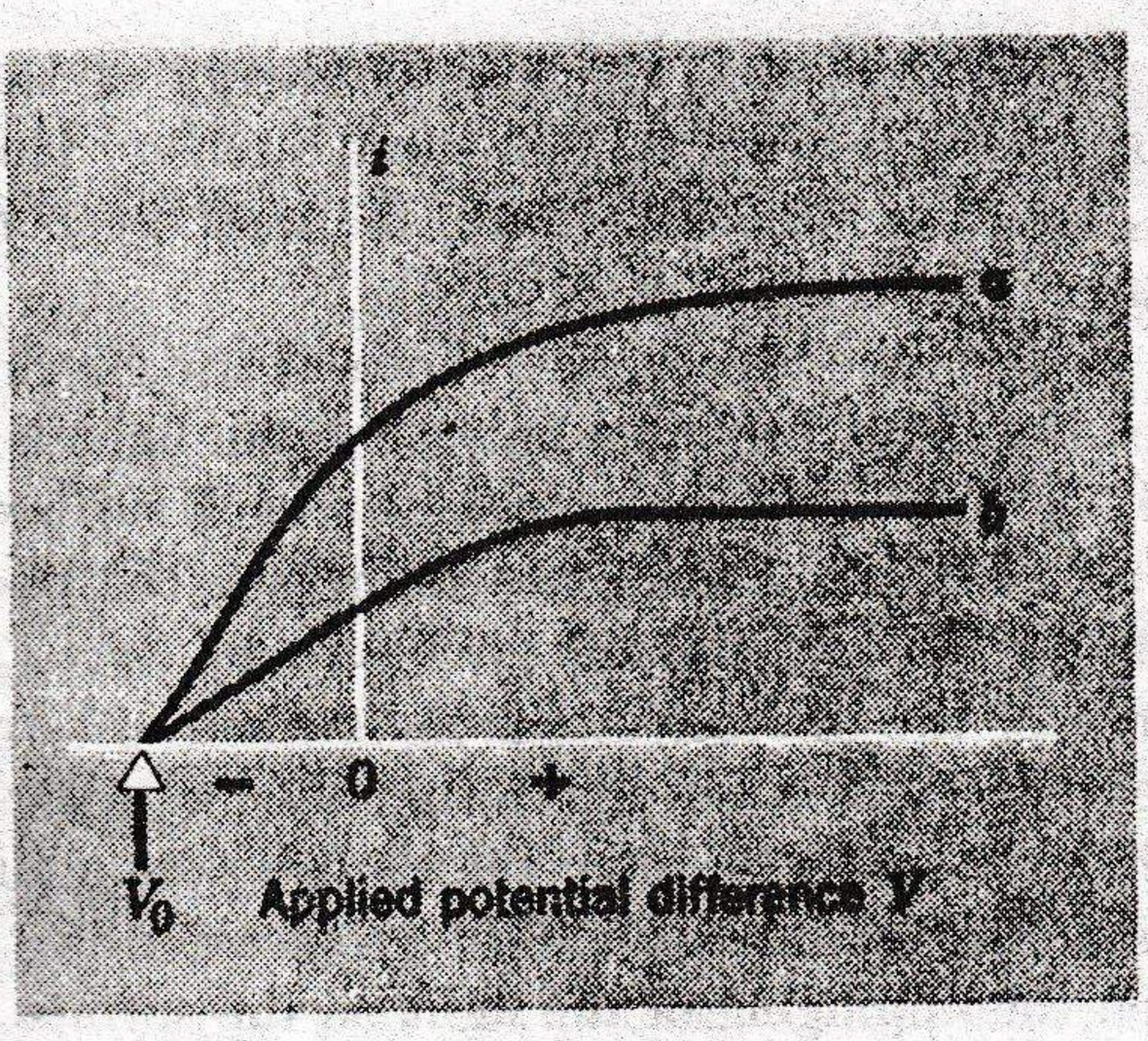


Fig. 47–7 Some data taken with the apparatus of Fig. 47–6. The applied potential difference V is called positive when the cup B in Fig. 47–6 is positive with respect to the photoelectric surface A. In curve b the incident light intensity has been reduced to one-half that of curve a.

ν_0, below which no photoelectric effect occurs. These data were taken by R. A. Millikan (1868–1953), whose painstaking work on the photoelectric effect won him the Nobel prize in 1923. Because the photoelectric effect is largely a surface phenomenon, it is necessary to avoid oxide films, grease, or other surface contaminants. Millikan devised a technique to cut shavings from the metal surface under vacuum conditions, a "machine shop *in vacuo*" as he called it.

Three major features of the photoelectric effect cannot be explained in terms of the wave theory of light:

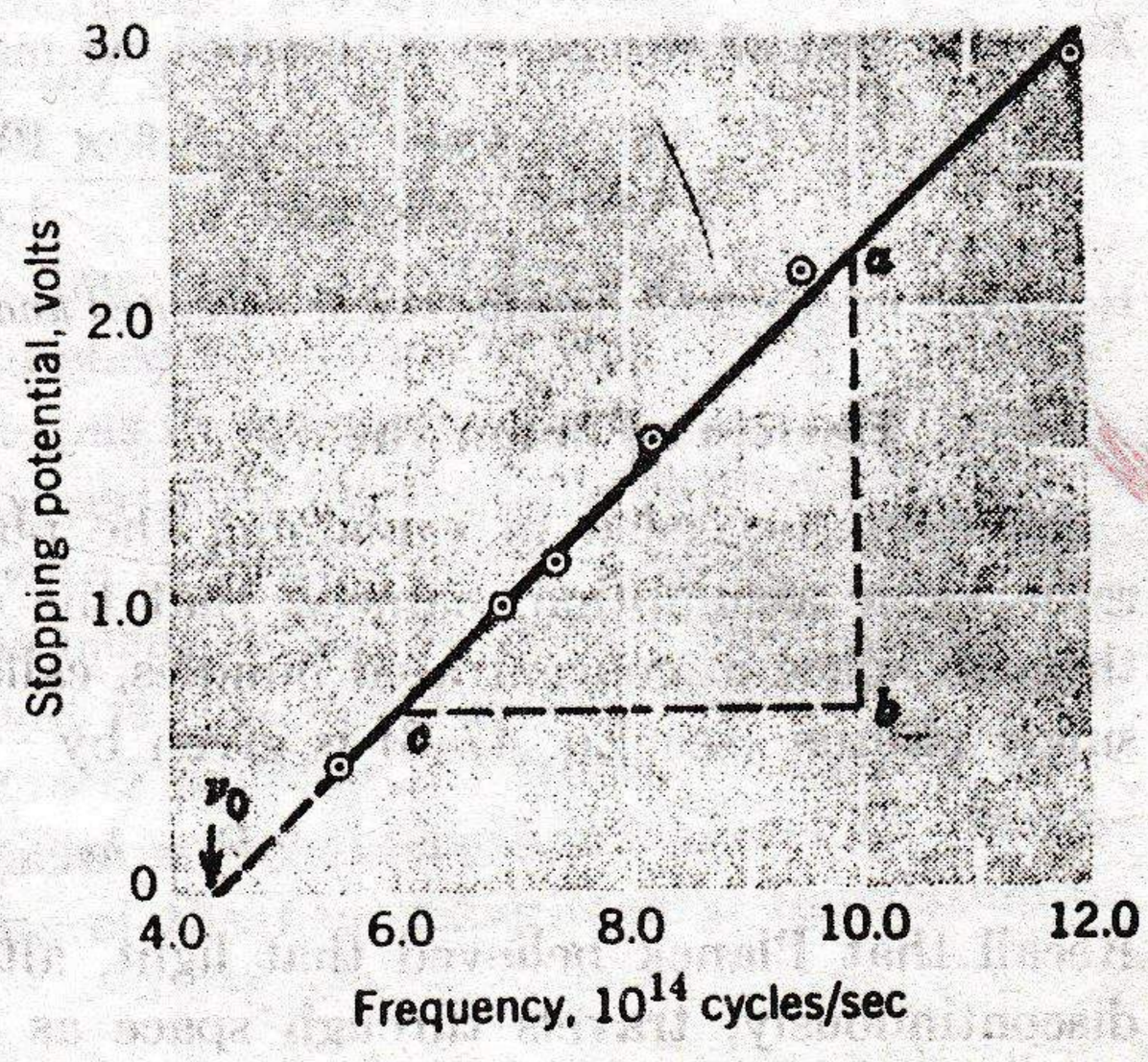


Fig. 47–8 A plot of Millikan's measurements of the stopping potential at various frequencies for sodium. The cutoff frequency ν_0 is 4.39×10^{14} cycles/sec.

1. Wave theory suggests that the kinetic energy of the photoelectrons should increase as the light beam is made more intense. However, Fig. 47–7 shows that K_{max} $(= eV_0)$ *is independent of the light intensity;* this has been tested over a range of intensities of 10^7.

2. According to the wave theory, the photoelectric effect should occur for *any* frequency of the light, provided only that the light is intense enough. However, Fig. 47–8 shows that there exists, for each surface, *a characteristic cutoff frequency* ν_0. For frequencies less than this, the photoelectric effect disappears, no matter how intense the illumination.

3. If the energy of the photoelectrons is "soaked up" from the incident wave by the metal plate, it is not likely that the "effective target area" for an electron in the metal is much more than a few atomic diameters. Thus, if the light is feeble enough, there should be a measurable time lag (see Example 2) between the impinging of the light on the surface and the ejection of the photoelectron. During this interval the electron should be "soaking up" energy from the beam until it had accumulated enough energy to escape. However, *no detectable time lag has ever been measured.* This disagreement is particularly striking when the photoelectric substance is a gas; under these circumstances the energy of the emitted photoelectron must certainly be "soaked out of the beam" by a single atom.

▶ **Example 2.** A metal plate is placed 5 meters from a monochromatic light source whose power output is 10^{-3} watt. Consider that a given ejected photoelectron may collect its energy from a circular area of the plate as large as ten atomic diameters (10^{-9} meter) in radius. The energy required to remove an electron through the metal surface is about 5.0 ev. *Assuming light to be a wave,* how long would it take for such a "target" to soak up this much energy from such a light source?

The target area is $\pi\ (10^{-9}\ \text{meter})^2$ or $3 \times 10^{-18}\ \text{meter}^2$; the area of a 5-meter sphere centered on the light source is $4\pi\ (5\ \text{meters})^2 \cong 300\ \text{meters}^2$. Thus, if the light source radiates uniformly in all directions, the rate P at which energy falls on the target is given by

$$P = (10^{-3}\ \text{watt})\left(\frac{3 \times 10^{-18}\ \text{meter}^2}{300\ \text{meters}^2}\right) = 10^{-23}\ \text{joule/sec}.$$

Assuming that all this power is absorbed, we may calculate the time required from

$$t = \left(\frac{5\ \text{ev}}{10^{-23}\ \text{joule/sec}}\right)\left(\frac{1.6 \times 10^{-19}\ \text{joule}}{1\ \text{ev}}\right) \cong 20\ \text{hr}.$$

However, *no detectable time lag can be measured under any circumstances.* ◀

47–5 Einstein's Photon Theory

Einstein succeeded in explaining the photoelectric effect by making a remarkable assumption, namely, that the energy in a light beam travels through space in concentrated bundles, called *photons.* The energy E of a single photon (see Eq. 47–8) is given by

$$E = h\nu. \tag{47–10}$$

Recall that Planck believed that light, although emitted from its source discontinuously, travels through space as an electromagnetic wave. Ein-

stein's hypothesis suggests that light traveling through space behaves not like a wave at all but like a particle. Millikan, whose experiments verified Einstein's ideas in every detail, spoke of Einstein's "bold, not to say reckless, hypothesis."

Applying the photon concept to the photoelectric effect, Einstein wrote

$$h\nu = E_0 + K_{\max} \tag{47–11}$$

where $h\nu$ is the energy of the photon. Equation 47–11 says that a photon carries an energy $h\nu$ into the surface. Part of this energy (E_0) is used in causing the electron to pass through the metal surface. The excess energy ($h\nu - E_0$) is given to the electron in the form of kinetic energy; if the electron does not lose energy by internal collisions as it escapes from the metal, it will exhibit it all as kinetic energy after it emerges. Thus $K_{\max}$ represents the *maximum* kinetic energy that the photoelectron can have outside the surface; in nearly all cases it will have less energy than this because of internal losses.

Consider how Einstein's photon hypothesis meets the three objections raised against the wave-theory interpretation of the photoelectric effect. As for objection 1 (the lack of dependence of $K_{\max}$ on the intensity of illumination), there is complete agreement of the photon theory with experiment. Doubling the light intensity merely doubles the number of photons and thus doubles the photoelectric current; it does not change the energy ($= h\nu$) of the individual photons or the nature of the individual photoelectric processes described by Eq. 47–11.

Objection 2 (the existence of a cutoff frequency) follows from Eq. 47–11. If $K_{\max}$ equals zero, we have

$$h\nu_0 = E_0,$$

which asserts that the photon has just enough energy to eject the photoelectrons and none extra to appear as kinetic energy. This quantity E_0 is called the work function of the substance. If ν is reduced below ν_0, the individual photons, no matter how many of them there are (that is, no matter how intense the illumination), will not have enough energy to eject photoelectrons.

Objection 3 (the absence of a time lag) follows from the photon theory because the required energy is supplied in a concentrated bundle. It is *not* spread uniformly over a large area, as in the wave theory.

Although the photon hypothesis certainly fits the facts of photoelectricity, it seems to be in direct conflict with the wave theory of light which, as we have seen in earlier chapters, has been verified in many experiments. Our modern view of the nature of light is that *it has a dual character, behaving like a wave under some circumstances and like a particle, or photon, under others.* We discuss the wave-particle duality at length in Chapter 48. Meanwhile, let us continue our studies of the firm experimental foundation on which the photon concept rests.

Let us rewrite Einstein's photoelectric equation (Eq. 47–11) by substituting eV_0 for K_{max} (see Eq. 47–9). This yields, after rearrangement,

$$V_0 = \frac{h}{e}\nu - \frac{E_0}{e}. \qquad (47\text{–}12)$$

Thus Einstein's theory predicts a linear relationship between V_0 and ν, in complete agreement with experiment; see Fig. 47–8. The slope of the experimental curve in this figure should be h/e, or

$$\frac{h}{e} = \frac{ab}{bc} = \frac{2.20 \text{ volt} - 0.65 \text{ volt}}{(10 \times 10^{14} - 6 \times 10^{14}) \text{ cycles/sec}} = 3.9 \times 10^{-15} \text{ volt-sec.}$$

We can find h by multiplying this ratio by the electron charge e,

$$h = (3.9 \times 10^{-15} \text{ volt-sec})(1.6 \times 10^{-19} \text{ coul}) = 6.2 \times 10^{-34} \text{ joule-sec.}$$

From a more careful analysis of this and other data, including data taken with lithium surfaces, Millikan found the value $h = 6.57 \times 10^{-34}$ joule-sec, with an accuracy of about 0.5%. This agreement with the value of h derived from Planck's radiation formula is a striking confirmation of Einstein's photon concept.

▶ **Example 3.** Deduce the work function for sodium from Fig. 47–8.

The intersection of the straight line in Fig. 47–8 with the horizontal axis is the cut-off frequency ν_0. Substituting these values yields

$$E_0 = h\nu_0 = (6.63 \times 10^{-34} \text{ joule-sec})(4.39 \times 10^{14} \text{ cycles/sec})$$
$$= 2.92 \times 10^{-19} \text{ joule} = 1.82 \text{ ev.} \quad ◀$$

47–6 The Compton Effect

Compelling confirmation of the concept of the photon as a concentrated bundle of energy was provided in 1923 by A. H. Compton (1892–1962) who earned a Nobel prize for this work in 1927.* Compton allowed a beam of X-rays of sharply defined wavelength λ to fall on a graphite block, as in Fig. 47–9, and he measured, for various angles of scattering, the intensity of the scattered X-rays as a function of their wavelength. Figure 47–10 shows his experimental results. We see that although the incident beam consists essentially of a single wavelength λ the scattered X-rays have intensity peaks at *two* wavelengths; one of them is the same as the incident wavelength, the other, λ', being larger by an amount $\Delta\lambda$. This so-called *Compton shift* $\Delta\lambda$ varies with the angle at which the scattered X-rays are observed.

The presence of a scattered wave of wavelength λ' cannot be understood if the incident X-rays are regarded as an electromagnetic wave like that of Fig. 39–11. On this picture the incident wave of frequency ν causes electrons in the scattering block to oscillate at that same frequency. These oscillating electrons, like charges surging back and forth in a small radio transmitting

* For an historical account of Compton's researches read "The Scattering of X Rays as Particles," A. H. Compton, p. 817, *Am. J. Phys.*, December 1961.

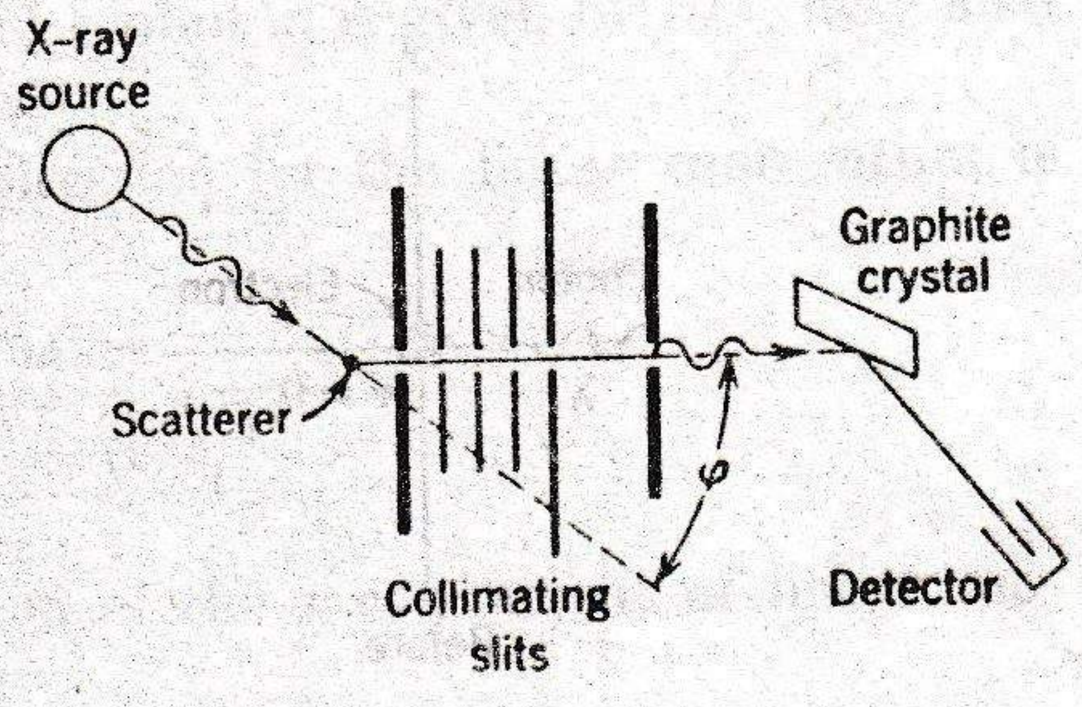


Fig. 47–9 Compton's experimental arrangement. Monochromatic X-rays of wavelength λ fall on a graphite scatterer. The distribution of intensity with wavelength is measured for X-rays scattered at any selected angle φ. The scattered wavelengths are measured by observing Bragg reflections from a crystal; see Eq. 45–8. Their intensities are measured by a detector, such as an ionization chamber.

antenna, radiate electromagnetic waves that again have this same frequency ν. Thus, on the wave picture the scattered wave should have the same frequency ν and the same wavelength λ as the incident wave.

Compton * was able to explain his experimental results by postulating that the incoming X-ray beam was not a wave but an assembly of photons of energy E $(= h\nu)$ and that these photons experienced billiard-ball-like collisions with the free electrons in the scattering block. The "recoil" photons emerging from the block constitute, on this view, the scattered radiation. Since the incident photon transfers some of its energy to the electron with

* P. W. Debye simultaneously and independently offered the same interpretation.

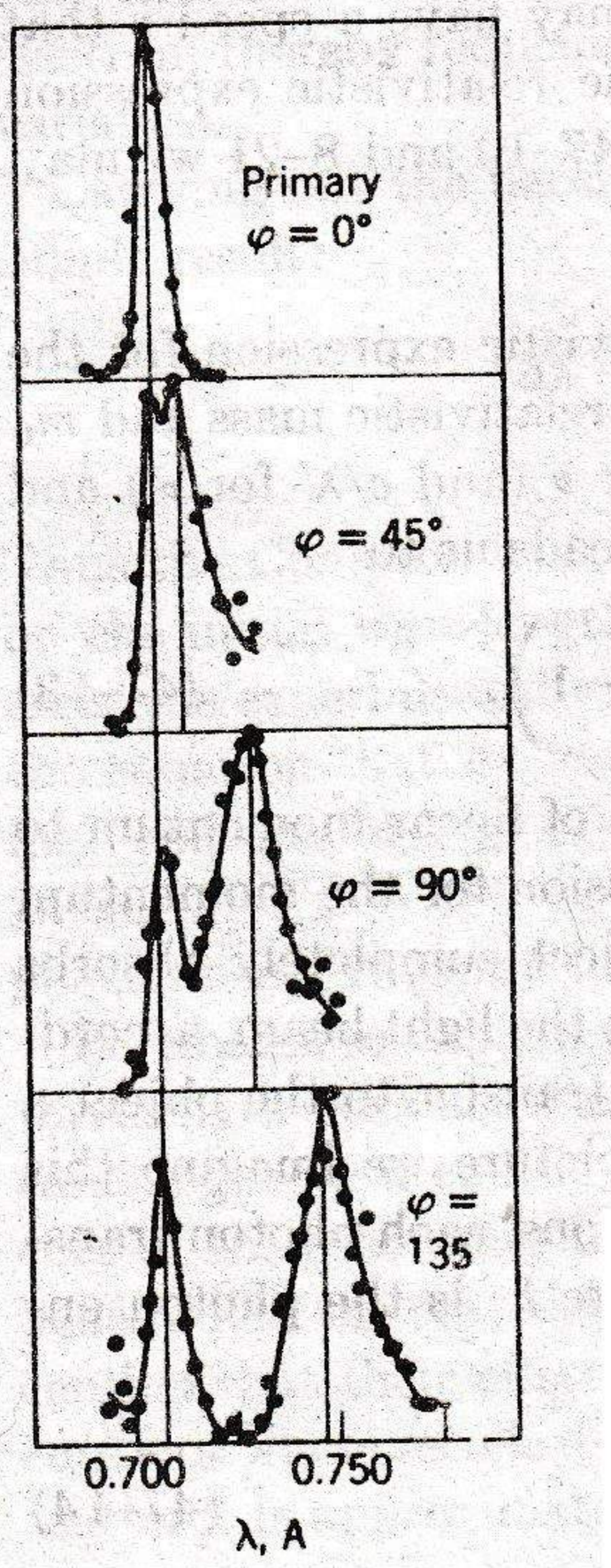


Fig. 47–10 Compton's experimental results. The solid vertical line on the left corresponds to the wavelength λ, that on the right to λ'. Results are shown for four different angles of scattering φ. Note that the Compton shift $\Delta\lambda$ for $\varphi = 90°$ is $h/m_0c = 0.242$ A.

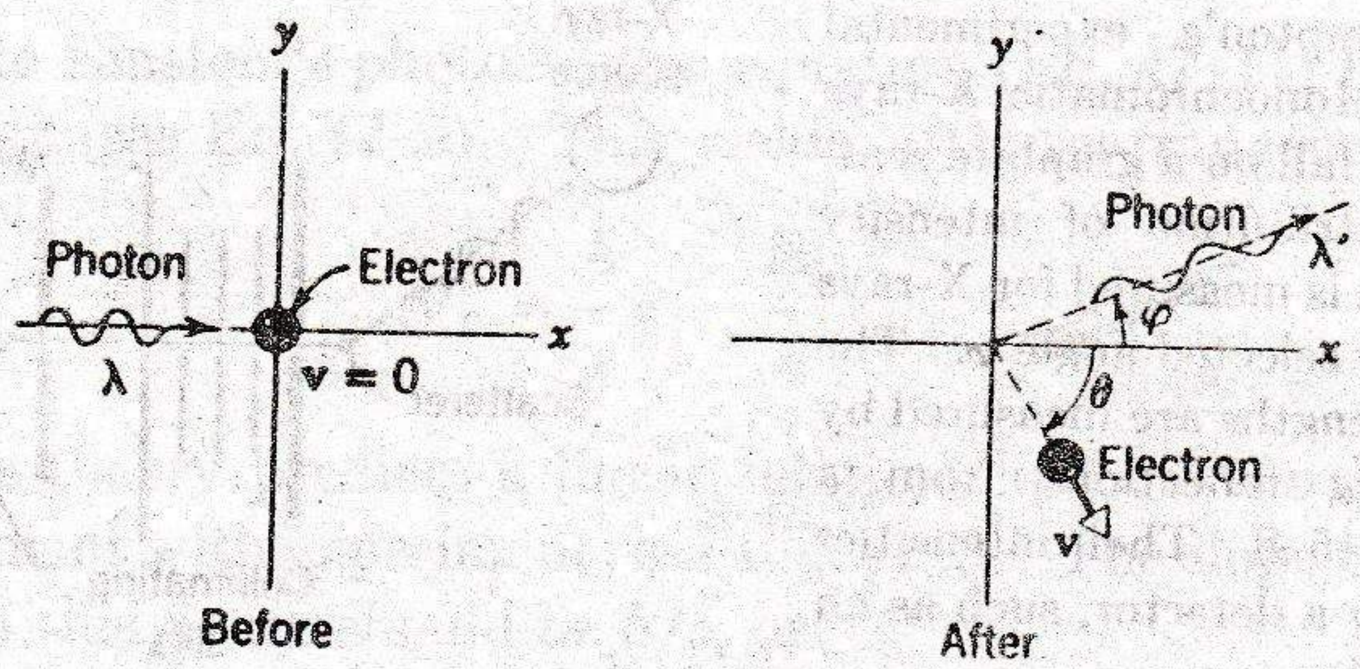


Fig. 47–11 A photon of wavelength λ is incident on an electron at rest. On collision, the photon is scattered at an angle φ with increased wavelength λ′, while the electron moves off with speed v in direction θ.

which it collides, the scattered photon must have a lower energy E'; it must therefore have a lower frequency $\nu'(= E'/h)$, which implies a larger wavelength $\lambda'(= c/\nu')$. This point of view accounts, at least qualitatively, for the wavelength shift $\Delta\lambda$. Notice how different this particle model of X-ray scattering is from that based on the wave picture. Now let us analyze a single photon-electron collision quantitatively.

Figure 47–11 shows a collision between a photon and an electron, the electron assumed to be initially at rest and essentially free, that is, not bound to the atoms of the scatterer. Let us apply the law of conservation of energy to this collision. Since the recoil electrons may have a speed v that is comparable with that of light, we must use the relativistic expression for the kinetic energy of the electron. From Eqs. 47–10 and 8–21 we may write

$$h\nu = h\nu' + (m - m_0)c^2,$$

in which the second term on the right is the relativistic expression for the kinetic energy of the recoiling electron, m being the relativistic mass and m_0 the rest mass of that particle. Substituting c/λ for ν (and c/λ' for ν') and using Eq. 8–20 to eliminate the relativistic mass m leads us to

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + m_0c^2\left(\frac{1}{\sqrt{1 - (v/c)^2}} - 1\right). \qquad (47\text{–}13)$$

Now let us apply the (vector) law of conservation of linear momentum to the collision of Fig. 47–11. We first need an expression for the momentum of a photon. In Section 40–2 we saw that if an object completely absorbs an energy U from a parallel light beam that falls on it the light beam, according to the wave theory of light, will simultaneously transfer to the object a linear momentum given by U/c. On the photon picture we imagine this momentum to be carried along by the individual photons, each photon transporting linear momentum in amount $p = h\nu/c$, where $h\nu$ is the photon energy. Thus, if we substitute λ for c/ν, we can write

$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}. \qquad (47\text{–}14)$$

This conclusion, that the momentum of a photon is given by h/λ, may also be deduced from the theory of relativity.

For the electron, the relativistic expression for the linear momentum is given by Eq. 9–13, or

$$p_e = \frac{m_0 v}{\sqrt{1 - (v/c)^2}}.$$

We can then write for the conservation of the x component of linear momentum

$$\frac{h}{\lambda} = \frac{h}{\lambda'} \cos \varphi + \frac{m_0 v}{\sqrt{1 - (v/c)^2}} \cos \theta \qquad (47\text{–}15)$$

and for the y component

$$0 = \frac{h}{\lambda'} \sin \varphi - \frac{m_0 v}{\sqrt{1 - (v/c)^2}} \sin \theta. \qquad (47\text{–}16)$$

Our immediate aim is to find $\Delta\lambda$ $(= \lambda' - \lambda)$, the wavelength shift of the scattered photons, so that we may compare it with the experimental results of Fig. 47–10. Compton's experiment did not involve observations of the recoil electron in the scattering block. Of the five collision variables (λ, λ', v, φ, and θ) that appear in the three equations (47–13, 47–15, and 47–16) we may eliminate two. We chose to eliminate v and θ, which deal only with the electron, thereby reducing the three equations to a single relation among the variables.

Carrying out the necessary algebraic steps (see Problem 15) leads to this simple result:

$$\Delta\lambda \; (= \lambda' - \lambda) = \frac{h}{m_0 c} (1 - \cos \varphi). \qquad (47\text{–}17)$$

Thus the Compton shift $\Delta\lambda$ depends only on the scattering angle φ and *not* on the initial wavelength λ. Equation 47–17 predicts within experimental error the experimentally observed Compton shifts of Fig. 47–10. Note from the equation that $\Delta\lambda$ varies from zero (for $\varphi = 0$, corresponding to a "grazing" collision in Fig. 47–11, the incident photon being scarely deflected) to $2h/m_0c$ (for $\varphi = 180°$, corresponding to a "head-on" collision, the incident photon being reversed in direction).

It remains to explain the presence of the peak in Fig. 47–10 for which the wavelength does *not* change on scattering. This peak can be understood as resulting from a collision between a photon and electrons bound in an ionic core in the scattering block. During photon collisions the bound electrons behave like the free electrons that we considered in Fig. 47–11, with the exception that their effective mass is much greater. This is because the ionic core as a whole recoils during the collision. The effective mass M for a carbon scatterer is approximately the mass of a carbon nucleus. Since this nucleus

contains 6 protons and 6 neutrons, we have approximately that $M = 12 \times 1840m_0 = 22{,}000m_0$. If we replace m_0 by M in Eq. 47–17, we see that the Compton shift for collisions with tightly bound electrons is immeasurably small.

As in the cavity radiation problem (see Eq. 47–7) and the photoelectric effect (see Eq. 47–11), Planck's constant h is centrally involved in the Compton effect. The quantity h is the central constant of quantum physics. In a universe in which $h = 0$ there would be no quantum physics and classical physics would be valid in the subatomic domain. In particular, as Eq. 47–17 shows, there would be no Compton effect (that is, $\Delta\lambda = 0$) in such a universe.

▶ **Example 4.** X-rays with $\lambda = 1.00$ A are scattered from a carbon block. The scattered radiation is viewed at 90° to the incident beam. (*a*) What is the Compton shift $\Delta\lambda$? (*b*) What kinetic energy is imparted to the recoiling electron?

(*a*) Putting $\varphi = 90°$ in Eq. 47–17, we have, for the Compton shift,

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\varphi)$$

$$= \frac{6.63 \times 10^{-34} \text{ joule-sec}}{(9.11 \times 10^{-31} \text{ kg})(3.00 \times 10^{8} \text{ meters/sec})}(1 - \cos 90°)$$

$$= 2.43 \times 10^{-12} \text{ meter} = 0.0243 \text{ A}.$$

(*b*) If we put K for the kinetic energy of the electron, we can write Eq. 47–13 as

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + K.$$

Since $\lambda' = \lambda + \Delta\lambda$, we obtain

$$\frac{hc}{\lambda} = \frac{hc}{\lambda + \Delta\lambda} + K,$$

which reduces to

$$K = \frac{hc\,\Delta\lambda}{\lambda(\lambda + \Delta\lambda)}$$

$$= \frac{(6.63 \times 10^{-34} \text{ joule-sec})(3.00 \times 10^{8} \text{ meters/sec})(2.43 \times 10^{-12} \text{ meter})}{(1.00 \times 10^{-10} \text{ meter})(1.00 + 0.024) \times 10^{-10} \text{ meter}}$$

$$= 4.73 \times 10^{-17} \text{ joule} = 295 \text{ ev}.$$

The student may show that the initial photon energy E in this case ($= h\nu = hc/\lambda$) is 12,400 ev so that the photon lost about 2.3% of its energy in this collision. A photon whose energy was ten times as large ($= 124{,}000$ ev) can be shown to lose 23% of its energy in a similar collision. This follows from the fact that $\Delta\lambda$ does not depend on the initial wavelength. Hence more energetic X-rays, which have smaller wavelengths, will experience a larger *per cent* increase in wavelength and thus a larger per cent loss in energy. ◀

47–7 Line Spectra

We have seen how Planck successfully explained the nature of the radiation from heated solid objects of which the cavity radiator formed the prototype. Such radiations form *continuous spectra* and are contrasted with *line spectra* such as that of Fig. 47–12, which shows the radiation emitted from iron ions

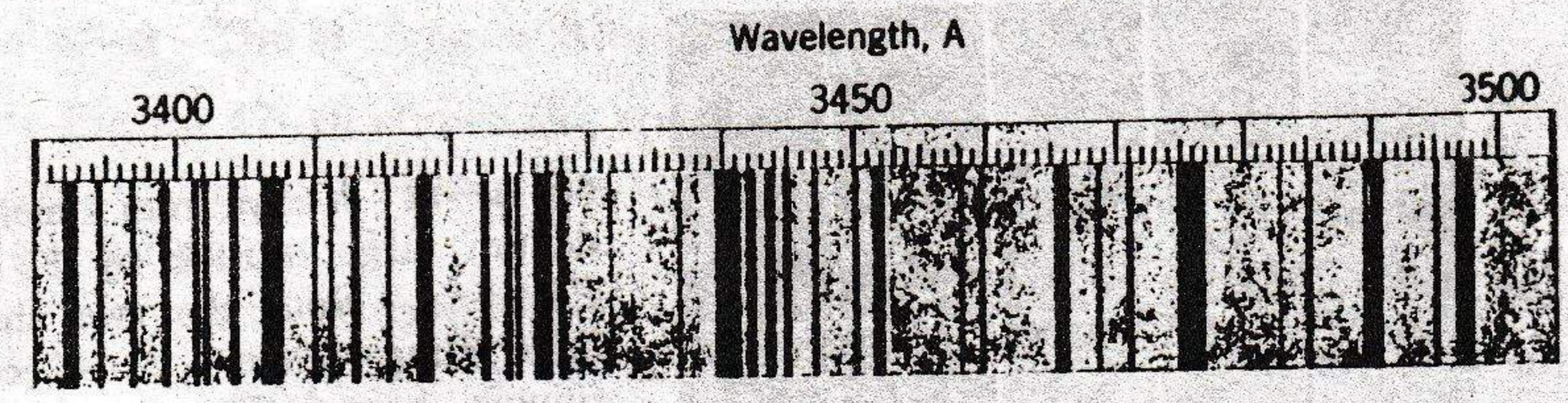


Fig. 47–12 A small portion of the spectrum of iron, in the region 3400 to 3500 A.

and atoms in an electric arc struck between iron electrodes. We shall see that Planck's quantization ideas, suitably extended, lead to an understanding of line spectra also. The prototype for the study of line spectra is that of atomic hydrogen; being the simplest atom it has the simplest spectrum.

Line spectra are common in all parts of the electromagnetic spectrum. Figure 47–13 shows a spectrum of the γ-rays ($\lambda \cong 10^{-12}$ meter) emitted from a particular radioactive nucleus, an isotope of mercury. Figure 47–14 shows a spectrum of X-rays ($\lambda \cong 10^{-10}$ meter) emitted from a molybdenum target when struck by a 35-kev electron beam. The sharp emission lines are superimposed on a continuous background.

Figure 47–15 shows a spectrum associated with the molecule HCl. It occurs in the infrared, with $\lambda \cong 10^{-6}$ meter. This is an *absorption* spectrum rather than an emission spectrum, as in Fig. 47–12. Experiment shows that isolated atoms and molecules absorb radiation, as well as emit it, at discrete wavelengths.

Figure 47–16 shows a portion of the absorption spectrum of ammonia (NH_3) in the microwave region ($\lambda \cong 10^{-2}$ meter). Finally, Fig. 47–17 shows how radiation in the radio-frequency region ($\lambda \cong 43$ meters) is absorbed by hydrogen molecules placed in a magnetic field.

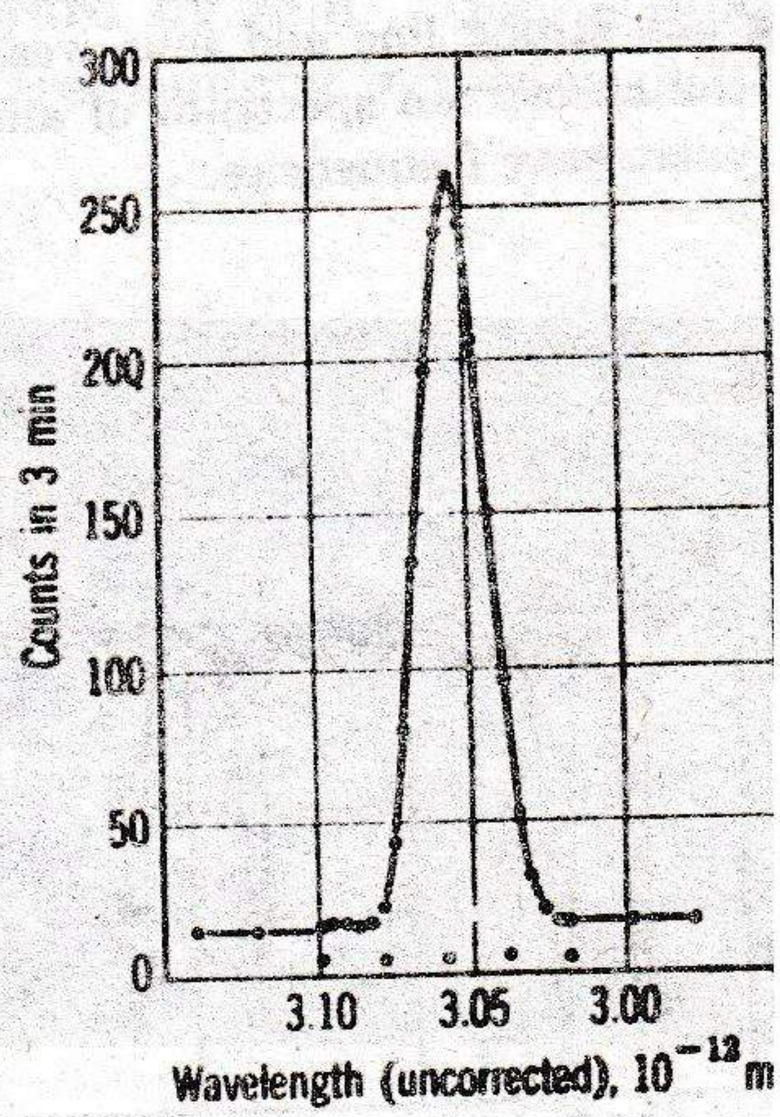


Fig. 47–13 A wavelength plot for a gamma ray emitted by the nucleus Hg^{198}. (From data by DuMond and co-workers.)

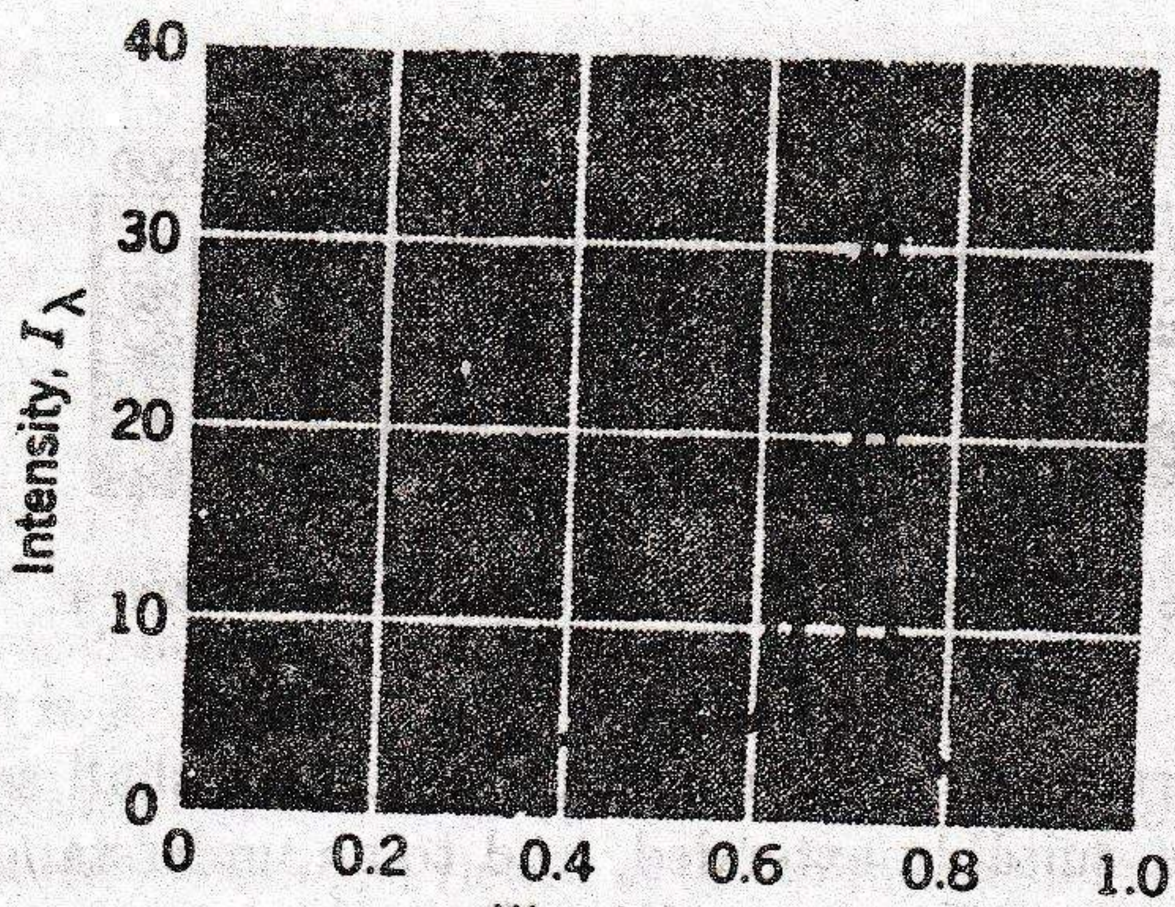


Fig. 47–14 X-rays from a molybdenum target struck by 35-kev electrons. Note the two sharp lines rising above a broad continuous base. The wavelength of the most intense line is 7.1×10^{-11} meter or 0.71 A. (From data by Ulrey.)

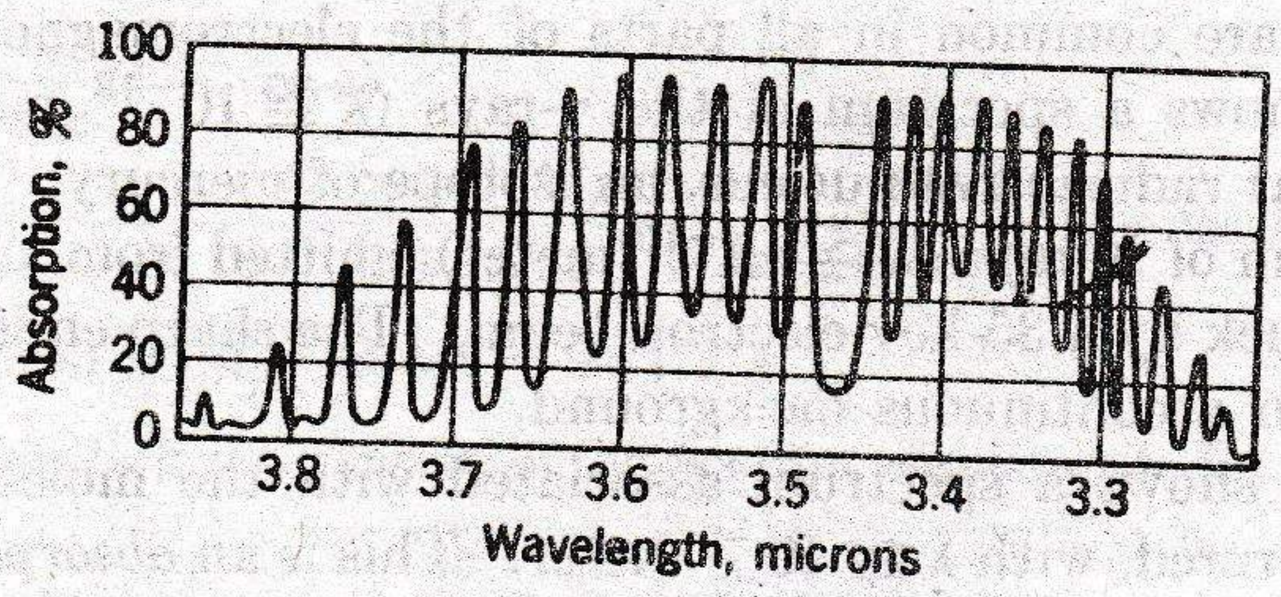
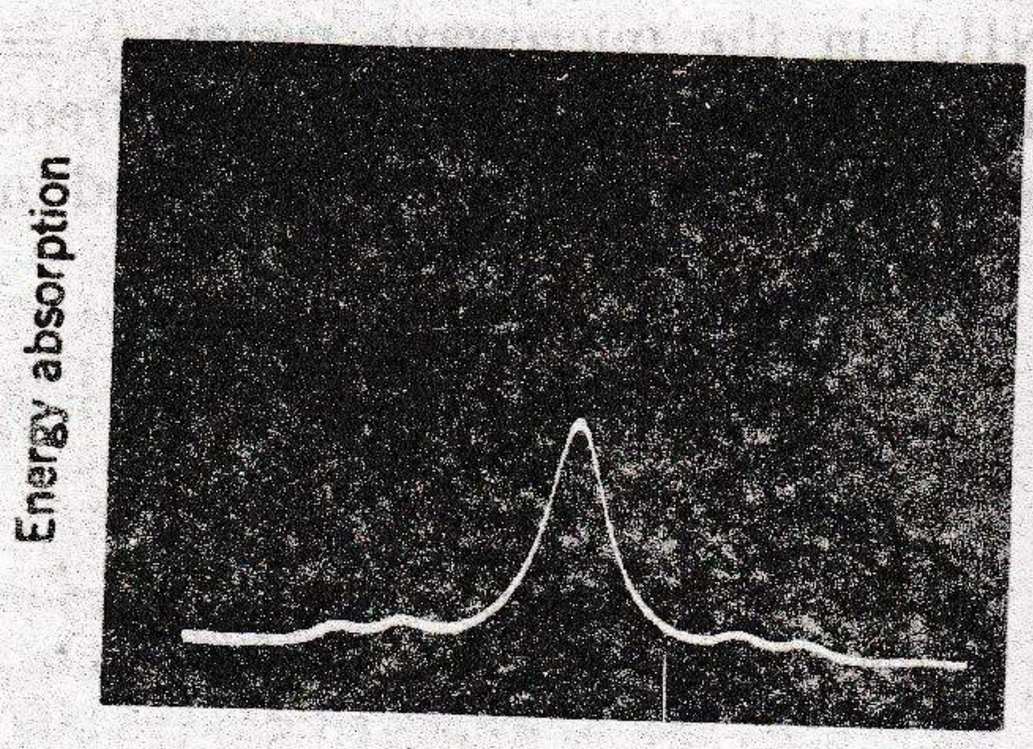


Fig. 47–15 An absorption spectrum of the HCl molecule near $\lambda = 3.5 \times 10^{-6}$ meter = 3.5 μ. (From data by E. S. Imes.)

Fig. 47–16 An oscilloscope trace showing one strong line and four weak lines in the absorption spectrum of ammonia at microwave frequencies.



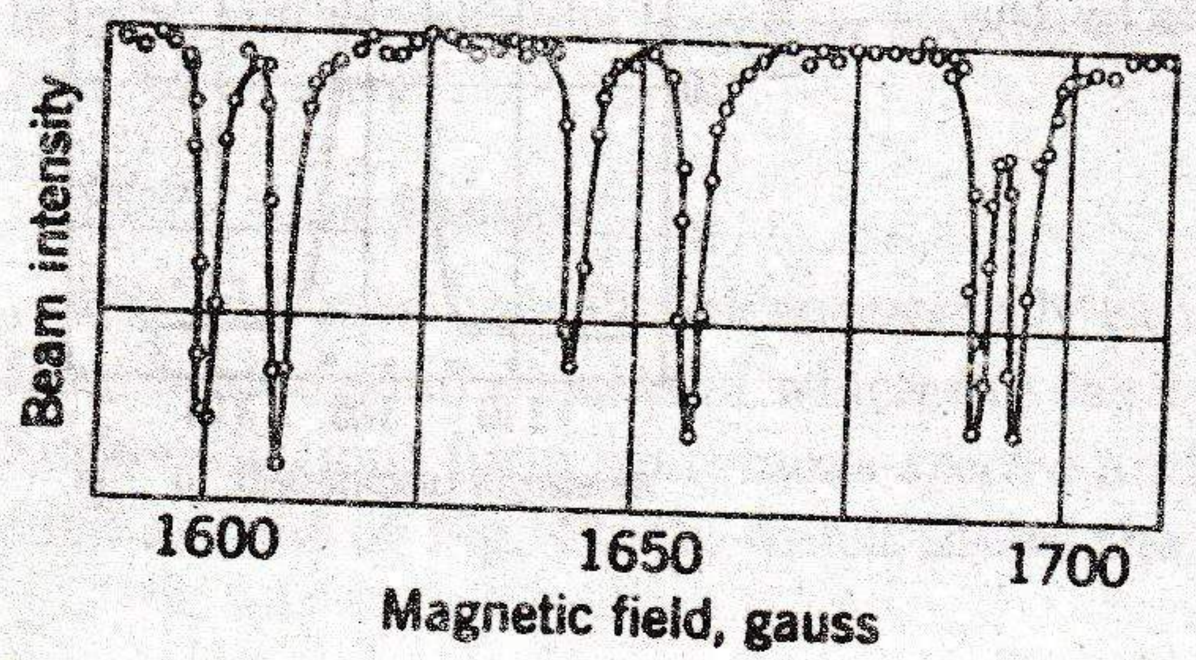


Fig. 47–17 A portion of the absorption spectrum of the protons in molecular hydrogen at $\lambda \cong 43$ meters. In this technique the frequency is left fixed and the sample is placed in a magnetic field, which is varied to scan the spectrum. (From data by Kellogg, Rabi, and Zacharias.)

47–8 The Hydrogen Atom

Figure 47–18 shows the spectrum of hydrogen. A theoretical interpretation of the hydrogen spectrum, based on quantum ideas, was first given by Niels Bohr in 1913.

We might wish to associate the frequency of an emitted spectrum line with the frequency of an electron revolving in an orbit inside the atom. Classical electromagnetism predicts that charges will radiate energy when they are accelerated. In this way electromagnetic waves are emitted from a radio transmitting antenna in which electrons are caused to surge back and forth. This radiation represents a loss of energy for the moving electrons which, in a radio antenna, is compensated for by supplying energy from an oscillator. In an isolated atom, however, no energy is supplied from external sources. We would expect the frequency of the electron and thus that of the emitted radiation to change continuously as the energy drains away. This prediction of classical theory cannot be reconciled with the existence of sharp spectrum lines. Thus classical physics cannot explain the hydrogen, or any other, spectrum.

Bohr circumvented this difficulty by assuming that, like Planck's oscillators, the hydrogen atom exists in certain *stationary states* in which it does not radiate. Radiation occurs *only* when the atom makes a transition from one state, with energy E_k, to a state with lower energy E_j. In equation form

$$h\nu = E_k - E_j, \tag{47–18}$$

where $h\nu$ is the quantum of energy carried away by the photon that is emitted from the atom during the transition.

To learn the allowed frequencies predicted by Eq. 47–18, it is necessary to know the energies of the various stationary states in which a hydrogen atom can exist. This calculation was first carried out by Bohr on the basis of a specific model for the hydrogen atom put forward by him. Bohr's model was highly successful for hydrogen and had a tremendous influence on the

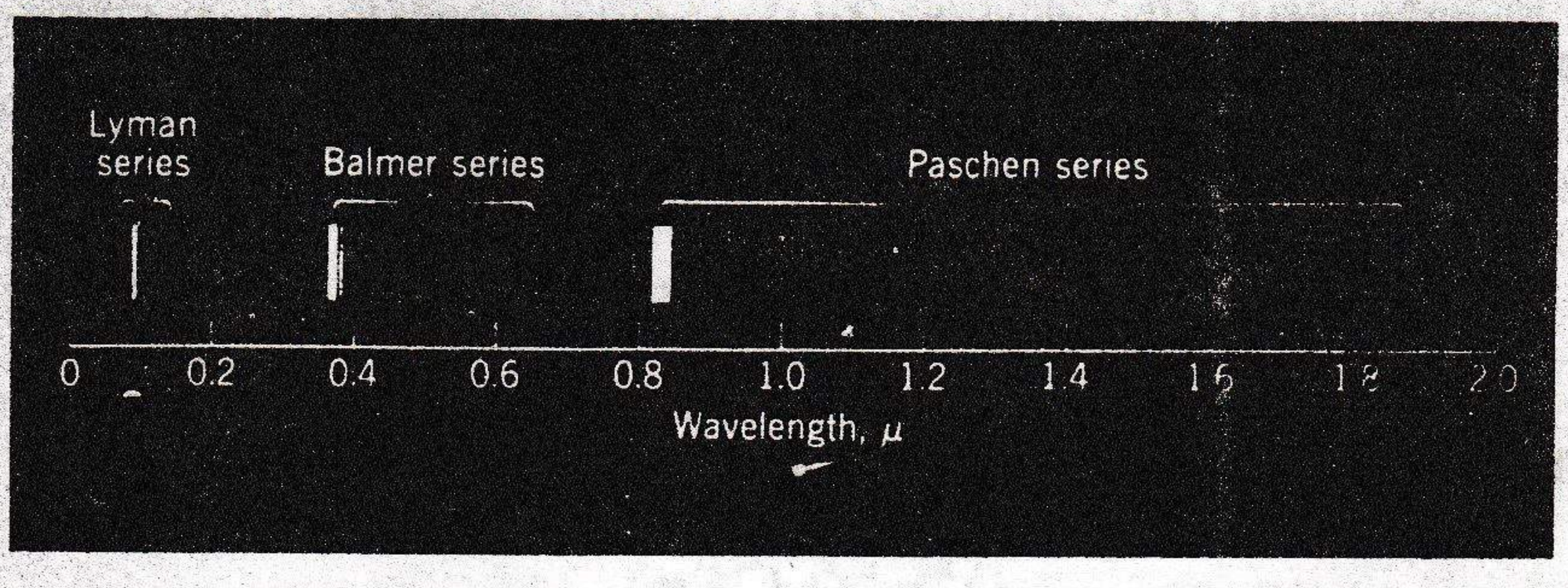


Fig. 47–18 The spectrum of hydrogen. It consists of a number of series of lines, three of which are shown. Within each series the spectrum lines follow a regular pattern, approaching a so-called *series limit* at the short-wave end of the series.

further development of the subject; it is now regarded as an important preliminary stage in the development of a more complete theory of quantum physics.

Let us assume that the electron in the hydrogen atom moves in a circular orbit of radius r centered on its nucleus. We assume that the nucleus, which is a single proton, is so massive that the center of mass of the system is essentially at the position of the proton. Let us calculate the energy E of such an atom.

Writing Newton's second law for the motion of the electron, we have (using Coulomb's law)

$$F = ma,$$

or

$$\frac{e^2}{4\pi\epsilon_0 r^2} = m\frac{v^2}{r}.$$

This allows us to calculate the kinetic energy of the electron, which is

$$K = \tfrac{1}{2}mv^2 = \frac{e^2}{8\pi\epsilon_0 r}. \tag{47–19}$$

The potential energy U of the proton-electron system is given by

$$U = V(-e) = -\frac{e^2}{4\pi\epsilon_0 r}, \tag{47–20}$$

where $V\ (= e/4\pi\epsilon_0 r)$ is the potential of the proton at the radius of the electron.

The total energy E of the system is

$$E = K + U = -\frac{e^2}{8\pi\epsilon_0 r}. \tag{47–21}$$

Since the orbit radius can apparently take on any value, so can the energy E. The problem of quantizing E reduces to that of quantizing r.

Every property of the orbit is fixed if the radius is given. Equations 47–19, 47–20, and 47–21 show this specifically for the energies K, U, and E. From Eq. 47–19 we can show that the *linear speed* v for the electron is also given in terms of r by

$$v = \sqrt{\frac{e^2}{4\pi\epsilon_0 mr}}. \tag{47–22}$$

The *rotational frequency* ν_0 follows at once from

$$\nu_0 = \frac{v}{2\pi r} = \sqrt{\frac{e^2}{16\pi^3\epsilon_0 mr^3}}. \tag{47–23}$$

The *linear momentum* p follows from Eq. 47–22:

$$p = mv = \sqrt{\frac{me^2}{4\pi\epsilon_0 r}}. \tag{47–24}$$

The *angular momentum* L is given by

$$L = pr = \sqrt{\frac{me^2 r}{4\pi\epsilon_0}}. \tag{47–25}$$

Thus if r is known, the orbit parameters K, U, E, v, ν_o, p, and L are also known. If any one of these quantities is quantized, all of them must be.

At this stage Bohr had no rules to guide him and so made (after some indirect reasoning which we do not reproduce) a bold hypothesis, namely, that the necessary quantization of the orbit parameters shows up most simply when applied to the angular momentum and that, specifically, L can take on only values given by

$$L = n\frac{h}{2\pi} \qquad n = 1, 2, 3, \ldots. \tag{47–26}$$

Planck's constant appears again in a fundamental way; the integer n is a *quantum number*.

Combining Eqs. 47–25 and 47–26 leads to

$$r = n^2 \frac{h^2\epsilon_0}{\pi me^2} \qquad n = 1, 2, 3, \ldots, \tag{47–27}$$

which tells how r is quantized. Substituting Eq. 47–27 into Eq. 47–21 produces

$$E = -\frac{me^4}{8\epsilon_0{}^2h^2}\frac{1}{n^2} \qquad n = 1, 2, 3, \ldots, \tag{47–28}$$

which gives directly the energy values of the allowed stationary states.

Figure 47–19 shows the energies of the stationary states and their associated quantum numbers. Equation 47–27 shows that the orbit radius increases as n^2. The upper level in Fig. 47–19, marked $n = \infty$, corresponds to a state in which the electron is completely removed from the atom (that is, $E = 0$ and $r = \infty$). Figure 47–19 also shows some of the quantum jumps that take place between the different stationary states.

Combining Eqs. 47–18 and 47–28 allows us to write a completely theoretical formula for the frequencies of the lines in the hydrogen spectrum. It is

$$\nu = \frac{me^4}{8\epsilon_0{}^2h^3}\left(\frac{1}{j^2} - \frac{1}{k^2}\right) \tag{47–29}$$

in which j and k are integers describing, respectively, the lower and the upper stationary states. The corresponding wavelengths can easily be

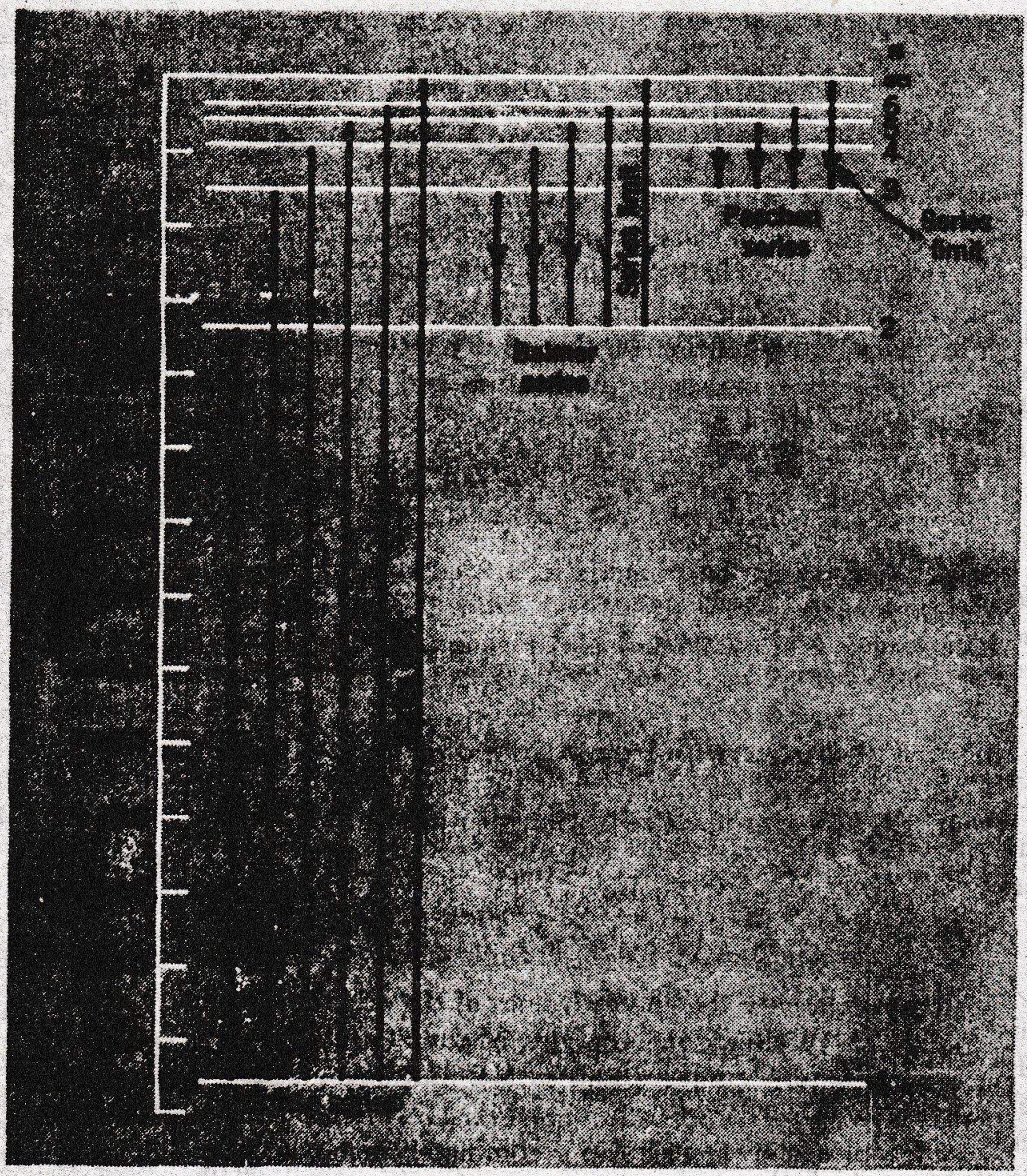


Fig. 47–19 An energy level diagram for hydrogen showing the quantum number n for each level and some of the transitions that appear in the spectrum. An infinite number of levels is crowded in between the levels marked $n = 6$ and $n = \infty$. Compare this figure carefully with Fig. 47–18.

found from $\lambda = c/\nu$. Table 47–1 shows some wavelengths so calculated; it should be compared carefully with Figs. 47–18 and 47–19.

Table 47–1

THE HYDROGEN SPECTRUM

(Some selected lines)

Name of series	Quantum Number		Wavelength, A
	j (lower state)	k (upper state)	
Lyman	1	2	1216
	1	3	1026
	1	4	970
	1	5	949
	1	6	940
	1	∞	912
Balmer	2	3	6563
	2	4	4861
	2	5	4341
	2	6	4102
	2	7	3970
	2	∞	3650
Paschen	3	4	18751
	3	5	12818
	3	6	10938
	3	7	10050
	3	8	9546
	3	∞	8220

▶ **Example 5.** Calculate the binding energy of the hydrogen atom (the energy binding the electron to the nucleus) from Eq. 47–28.

The binding energy is numerically equal to the energy of the lowest state in Fig. 47–19. The largest negative value of E in Eq. 47–28 is found for $n = 1$. This yields

$$E = -\frac{me^4}{8\epsilon_0^2 h^2}$$

$$= -\frac{(9.11 \times 10^{-31}\ \text{kg})(1.60 \times 10^{-19}\ \text{coul})^4}{(8)(8.85 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)^2(6.63 \times 10^{-34}\ \text{joule-sec})^2}$$

$$= -2.17 \times 10^{-18}\ \text{joule} = -13.6\ \text{ev},$$

which agrees with the experimentally observed binding energy for hydrogen. ◀

47–9 The Correspondence Principle

Although all theories in physics have limitations, they usually do not break down abruptly but in a continuous way, yielding results that agree less and less well with experiment. Thus the predictions of Newtonian mechanics become less and less accurate as the speed is made to approach that of light. A similar relationship must exist between quantum physics and classical physics; it remains to find the circumstances under which the latter theory is revealed as a special case of the former.

The radius of the lowest energy state in hydrogen (the so-called *ground state*) is found by putting $n = 1$ in Eq. 47–27; it turns out to be 5.3×10^{-11} meter. If $n = 10{,}000$, however, the radius is $(10{,}000)^2$ times as large or 5.3 mm. This "atom" is so large that we suspect that its behavior should be accurately described by classical physics. Let us test this by computing the frequency of the emitted light on the basis of both classical and quantum assumptions. These calculations should differ at small quantum numbers but should agree at large quantum numbers. The fact that *quantum physics reduces to classical physics at large quantum numbers* is called the *correspondence principle*. This principle, credited to Niels Bohr, was very useful during the years in which quantum physics was being developed. Bohr, in fact, based his theory of the hydrogen atom on correspondence principle arguments.

Classically, the frequency of the light emitted from an atom is equal to ν_0, its frequency of revolution * in its orbit. This can be expressed in terms of a quantum number n by combining Eqs. 47–23 and 47–27 to obtain

$$\nu_0 = \frac{me^4}{8\epsilon_0{}^2h^3}\frac{2}{n^3}. \tag{47–30}$$

Quantum physics predicts that the frequency ν of the emitted light is given by Eq. 47–29. Considering a transition between an orbit with quantum number $k = n$ and one with $j = n - 1$ leads to

$$\nu = \frac{me^4}{8\epsilon_0{}^2h^3}\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right]$$

$$= \frac{me^4}{8\epsilon_0{}^2h^3}\left[\frac{2n-1}{(n-1)^2n^2}\right]. \tag{47–31}$$

As $n \to \infty$ the expression in the square brackets above approaches $2/n^3$ so that $\nu \to \nu_0$ as $n \to \infty$. Table 47–2 illustrates this example of the correspondence principle.

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* Integral multiples of the frequency also exist but may be ignored without affecting the present argument.

Table 47–2

THE CORRESPONDENCE PRINCIPLE AS APPLIED TO THE HYDROGEN ATOM

Quantum Number, n	Frequency of Revolution in Orbit (Eq. 47–30) cycles/sec	Frequency of Transition to Next Lowest State (Eq. 47–31) cycles/sec	Difference, %
2	8.20×10^{14}	24.6×10^{14}	67
5	5.26×10^{13}	7.38×10^{13}	29
10	6.57×10^{12}	7.72×10^{12}	14
50	5.25×10^{10}	5.42×10^{10}	3
100	6.578×10^{9}	6.677×10^{9}	1.5
1,000	6.5779×10^{6}	6.5878×10^{6}	0.15
10,000	6.5779×10^{3}	6.5789×10^{3}	0.015

QUESTIONS

1. "Pockets" formed by the coals in a coal fire seem brighter than the coals themselves. Is the temperature in such pockets appreciably higher than the surface temperature of an exposed glowing coal?

2. The relation $R = \sigma T^4$ (Eq. 47–2) is exact for true cavities and holds for all temperatures. Why don't we use this relation as the basis of a definition of temperature at, say, 100°C?

3. Do all incandescent solids obey the fourth-power law of temperature, as Eq. 47–3 seems to suggest?

4. A hole in the wall of a cavity radiator is sometimes called a *black body*. Why?

5. It is stated that if we look into a cavity whose walls are maintained at a constant temperature no details of the interior are visible. Does this seem reasonable?

6. How can a *photon* energy be given by $E = h\nu$ (Eq. 47–10) when the very presence of the frequency ν in the formula implies that light is a *wave*?

7. In the photoelectric effect, why does the existence of a cutoff frequency speak in favor of the photon theory and against the wave theory?

8. Why are photoelectric measurements so sensitive to the nature of the photoelectric surface?

9. Does Einstein's theory of photoelectricity, in which light is postulated to be a photon, invalidate Young's interference experiment?

10. List and discuss carefully the assumptions made by Planck in connection with the cavity radiation problem, by Einstein in connection with the photoelectric effect, and by Bohr in connection with the hydrogen atom problem.

11. In Bohr's theory for the hydrogen atom orbits, what is the implication of the fact that the potential energy is negative and is greater in magnitude than the kinetic energy?

12. Can a hydrogen atom absorb a photon whose energy exceeds its binding energy (13.6 ev)?

13. Discuss Example 1 in terms of the correspondence principle.

14. According to classical mechanics, an electron moving in an orbit should be able to do so with any angular momentum whatever. According to Bohr's theory of the hydrogen atom, however, the angular momentum is quantized according to $L = nh/2\pi$. Reconcile these two statements, using the correspondence principle.

PROBLEMS

1. At what wavelength does a cavity radiator at 6000°K radiate most per unit wavelength? Solve either analytically or graphically.

2. Using Fig. 47–1, verify that the emissivity of tungsten at 2000°K is 0.259.

3. A cavity radiator at 6000°K has a hole 0.10 mm in diameter drilled in its wall. At what rate do photons in the range 5500 to 5510 A escape from this hole?

4. A cavity whose walls are held at 4000°K has a circular aperture 5.0 mm in diameter. (*a*) At what rate does energy in the visible range (defined to extend from 0.40 to 0.70 μ) escape from this hole? (*b*) What fraction of the total radiation escaping from the cavity does this represent? Solve either analytically or graphically.

5. Show that Wien's law (p. 1085) is a special case of Planck's law (Eq. 47–6) for short wavelengths or low temperatures.

6. Solar radiation falls on the earth at a rate of 2.0 cal/cm^2-min. How many photons/cm^2-min is this, assuming an average wavelength of 5500 A?

7. A spectral emission line, important in radioastronomy, has a wavelength of 21 cm. To what photon energy does this correspond?

8. The energy required to remove an electron from sodium is 2.3 ev. Does sodium show a photoelectric effect for orange light, with $\lambda = 6800$ A?

9. Light of a wavelength 2000 A falls on an aluminum surface. In aluminum 4.2 ev are required to remove an electron. What is the kinetic energy of (*a*) the fastest and (*b*) the slowest emitted photoelectrons? (*c*) What is the stopping potential? (*d*) What is the cutoff wavelength for aluminum?

10. The work function for a clean lithium surface is 2.3 ev. Make a rough plot of the stopping potential V_0 versus the frequency of the incident light for such a surface.

11. In Example 2 suppose that the "target" is a single gas atom of 1.0 A radius and that the intensity of the light source is reduced to 10^{-5} watt. If the binding energy of the most loosely bound electron in the atom is 20 ev, what time lag for the photoelectric effect is expected on the basis of the wave theory of light?

12. A 100-watt sodium vapor lamp radiates uniformly in all directions. (*a*) At what distance from the lamp will the average density of photons be 10/cm^3? (*b*) What is the average density of photons 2.0 meters from the lamp? Assume the light to be monochromatic, with $\lambda = 5890$ A.

13. Show, by analyzing a collision between a photon and a free electron (using relativistic mechanics), that it is impossible for a photon to give *all* of its energy to the free electron. In other terms, the photoelectric effect cannot occur for completely free electrons; the electrons must be bound in a solid or in an atom.

14. Calculate the per cent change in photon energy for a Compton collision with φ in Fig. 47–11 equal to 90° for radiation in (*a*) the microwave range, with $\lambda = 3.0$ cm, (*b*) the visible range, with $\lambda = 5000$ A, (*c*) the X-ray range, with $\lambda = 1.00$ A, and (*d*) the gamma ray range, the energy of the gamma-ray photons being 1.0 Mev. What are your conclusions about the importance of the Compton effect in these various regions of the electromagnetic spectrum, judged solely by the criterion of energy loss in a single Compton encounter?

15. Carry out the necessary algebra to eliminate v and θ from Eqs. 47–13, 47–15, and 47–16 to obtain the Compton shift relation (Eq. 47–17).

16. (*a*) Using Bohr's formula, calculate the three longest wavelengths in the Balmer series. (*b*) Between what wavelength limits does the Balmer series lie?

17. In the ground state of the hydrogen atom, according to Bohr's theory, what are (*a*) the quantum number, (*b*) the orbit radius, (*c*) the angular momentum, (*d*) the linear momentum, (*e*) the angular velocity, (*f*) the linear speed, (*g*) the force on the electron, (*h*) the acceleration of the electron, (*i*) the kinetic energy, (*j*) the potential energy, and (*k*) the total energy?

18. How do the quantities (*b*) to (*k*) in Problem 17 vary with the quantum number?

19. How much energy is required to remove an electron from a hydrogen atom in a state with $n = 8$?

20. A hydrogen atom is excited from a state with $n = 1$ to one with $n = 4$. (*a*) Calculate the energy that must be absorbed by the atom. (*b*) Calculate and display on an energy-level diagram the different photon energies that may be emitted if the atom returns to its $n = 1$ state. (*c*) Calculate the recoil speed of the hydrogen atom, assumed initially at rest, if it makes the transition from $n = 4$ to $n = 1$ in a single quantum jump.

21. A hydrogen atom in a state having a *binding energy* (this is the energy required to remove an electron) of 0.85 ev makes a transition to a state with an *excitation energy* (this is the difference in energy between the state and the ground state) of 10.2 ev. (*a*) Find the energy of the emitted photon. (*b*) Show this transition on an energy-level diagram for hydrogen, labeling the appropriate quantum numbers.

22. Show on an energy-level diagram for hydrogen the quantum numbers corresponding to a transition in which the wavelength of the emitted photon is 1216 A.

23. A neutron, with kinetic energy of 6.0 ev, collides with a resting hydrogen atom in its ground state. Apply the laws of conservation of momentum and of energy to this collision and show that it must be elastic (that is, *kinetic* energy must be conserved).

24. Apply Bohr's theory to singly ionized helium, that is, to a helium atom with one electron removed. What relationship exists between this spectrum and the hydrogen spectrum?

25. Using Bohr's theory, calculate the energy required to remove the electron from singly ionized helium.

26. *Positronium.* Apply Bohr's theory to the positronium atom. This consists of a positive and a negative electron revolving around their center of mass, which lies halfway between them. (*a*) What relationship exists between this spectrum and the hydrogen spectrum? (*b*) What is the radius of the ground state orbit? (Hint: It will be necessary to analyze this problem from first principles because this "atom" has no nucleus; both particles revolve about a point halfway between them.)

27. *Muonic atoms.* Apply Bohr's theory to a muonic atom, which consists of a nucleus of charge Ze with a negative muon (an elementary particle with a charge of $-e$ and a mass m that is 207 times as large as the electron mass; see Appendix *F*) circulating about it. Calculate (*a*) the radius of the first Bohr orbit, (*b*) the ionization energy, and (*c*) the wavelength of the most energetic photon that can be emitted. Assume that the muon is circulating about a hydrogen nucleus ($Z = 1$).

28. If an electron is rotating in an orbit at frequency ν_0, classical electromagnetism predicts that it will radiate energy not only at this frequency but also at $2\nu_0$, $3\nu_0$, $4\nu_0$, etc.; see footnote on p. 1104. Show that this is also predicted by Bohr's theory of the hydrogen atom in the limiting case of large quantum numbers.

29. In Table 47–2 show that the quantity in the last column is given by

$$\frac{100(\nu - \nu_0)}{\nu} \cong \frac{150}{n}$$

for large quantum numbers.

30. If the angular momentum of the earth due to its motion around the sun were quantized according to Bohr's relation $L = nh/2\pi$, what would the quantum number be? Could such quantization be detected if it existed?

Waves and Particles

CHAPTER 48

48–1 Matter Waves

In 1924 Louis de Broglie of France reasoned that (*a*) nature is strikingly symmetrical in many ways; (*b*) our observable universe is composed entirely of light and matter; (*c*) if light has a dual, wave-particle nature, perhaps matter has also. Since matter was then regarded as being composed of particles, de Broglie's reasoning suggested that one should search for a wave-like behavior for matter.

De Broglie's suggestion might not have received serious attention had he not predicted what the expected wavelength of the so-called matter waves would be. We recall that about 1680 Huygens put forward a wave theory of light that did not receive general acceptance, in part because Huygens was not able to state what the wavelength of the light was. When Thomas Young rectified this defect in 1800, the wave theory of light started on its way to acceptance.

De Broglie assumed that the wavelength of the predicted matter waves was given by the same relationship that held for light namely, Eq. 47–14, or

$$\lambda = \frac{h}{p}, \tag{48-1}$$

which connects the wavelength of a light wave with the momentum of the associated photons. The dual nature of light shows up strikingly in this equation and also in Eq. 47–10 ($E = h\nu$). Each equation contains within its structure both a wave concept (ν and λ) and a particle concept (E and p) *De Broglie predicted that the wavelength of matter waves would also be given by Eq.* 48–1, *where p would now be the momentum of the particle of matter.*

▶ **Example 1.** What wavelength is predicted by Eq. 48–1 for a beam of electrons whose kinetic energy is 100 ev?

The velocity of the electrons is found from $K = \frac{1}{2}mv^2$, or

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{(2)(100 \text{ ev})(1.6 \times 10^{-19} \text{ joule/ev})}{9.1 \times 10^{-31} \text{ kg}}}$$

$$= 5.9 \times 10^6 \text{ meters/sec.}$$

The momentum follows from

$$p = mv = (9.1 \times 10^{-31} \text{ kg})(5.9 \times 10^6 \text{ meters/sec}) = 5.4 \times 10^{-24} \text{ kg-m/sec.}$$

The wavelength (called the *de Broglie wavelength*) is found from Eq. 48–1 or

$$\lambda = \frac{h}{p} = \frac{6.6 \times 10^{-34} \text{ joule-sec}}{5.4 \times 10^{-24} \text{ kg-m/sec}} = 1.2 \text{ A.}$$

This is the same order of magnitude as the size of an atom or the spacing between adjacent planes of atoms in a solid. ◀

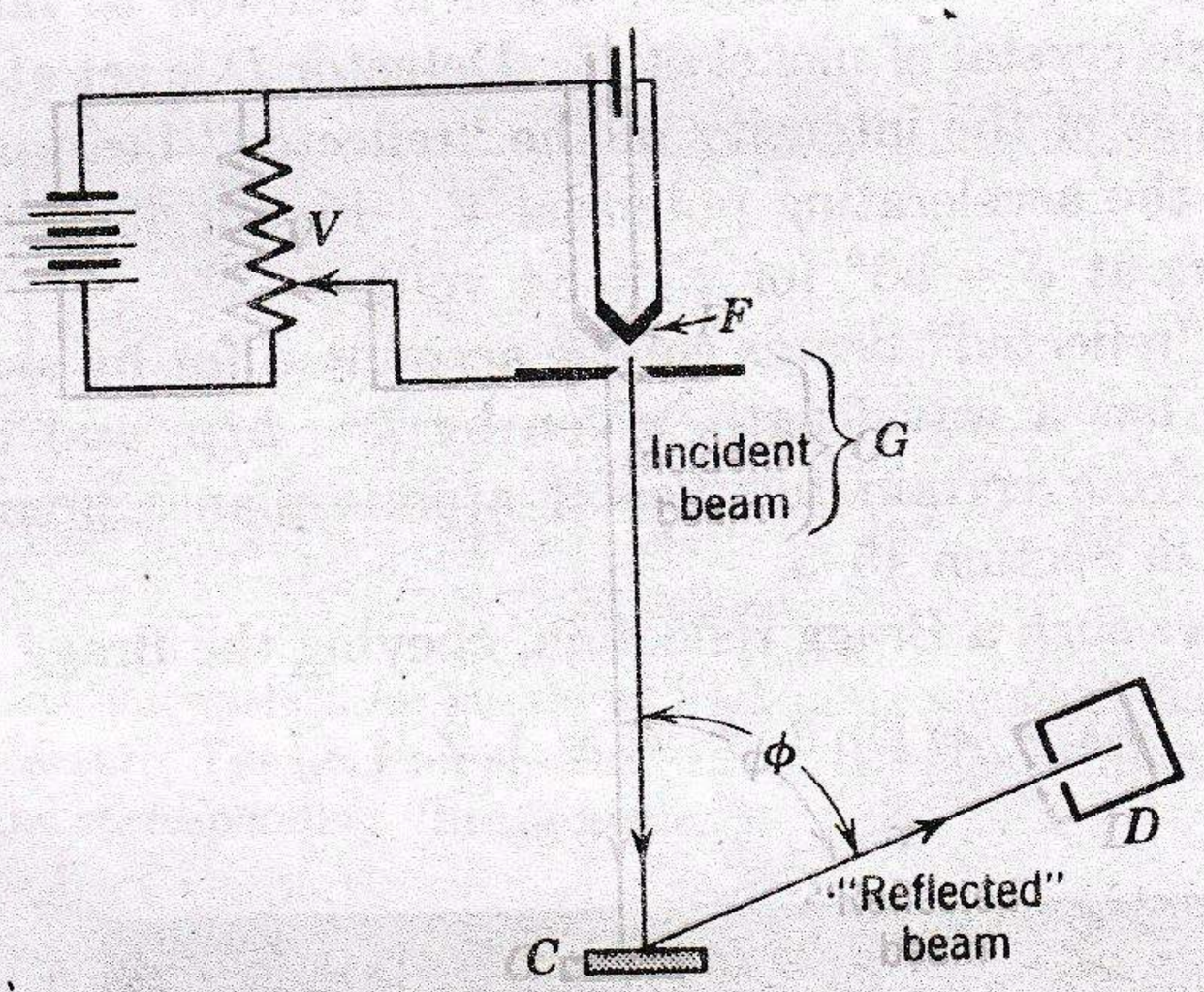


Fig. 48–1 The apparatus of Davisson and Germer. Electrons from filament F are accelerated by a variable potential difference V. After "reflection" from crystal C they are collected by detector D.

In 1926 Elsasser pointed out that the wave nature of matter might be tested in the same way that the wave nature of X-rays was first tested, namely, by allowing a beam of electrons of the appropriate energy to fall on a crystalline solid. The atoms of the crystal serve as a three-dimensional array of diffracting centers for the electron "wave"; we should look for strong diffracted peaks in certain characteristic directions, just as for X-ray diffraction.

This idea was tested by C. J. Davisson and L. H. Germer in this country and by G. P. Thomson in Scotland.* Figure 48–1 shows the apparatus of

* For an historical account of Thomson's researches see "Early Work in Electron Diffraction," Sir George Thomson, p. 821, *Am. J Phys.*, December, 1961.

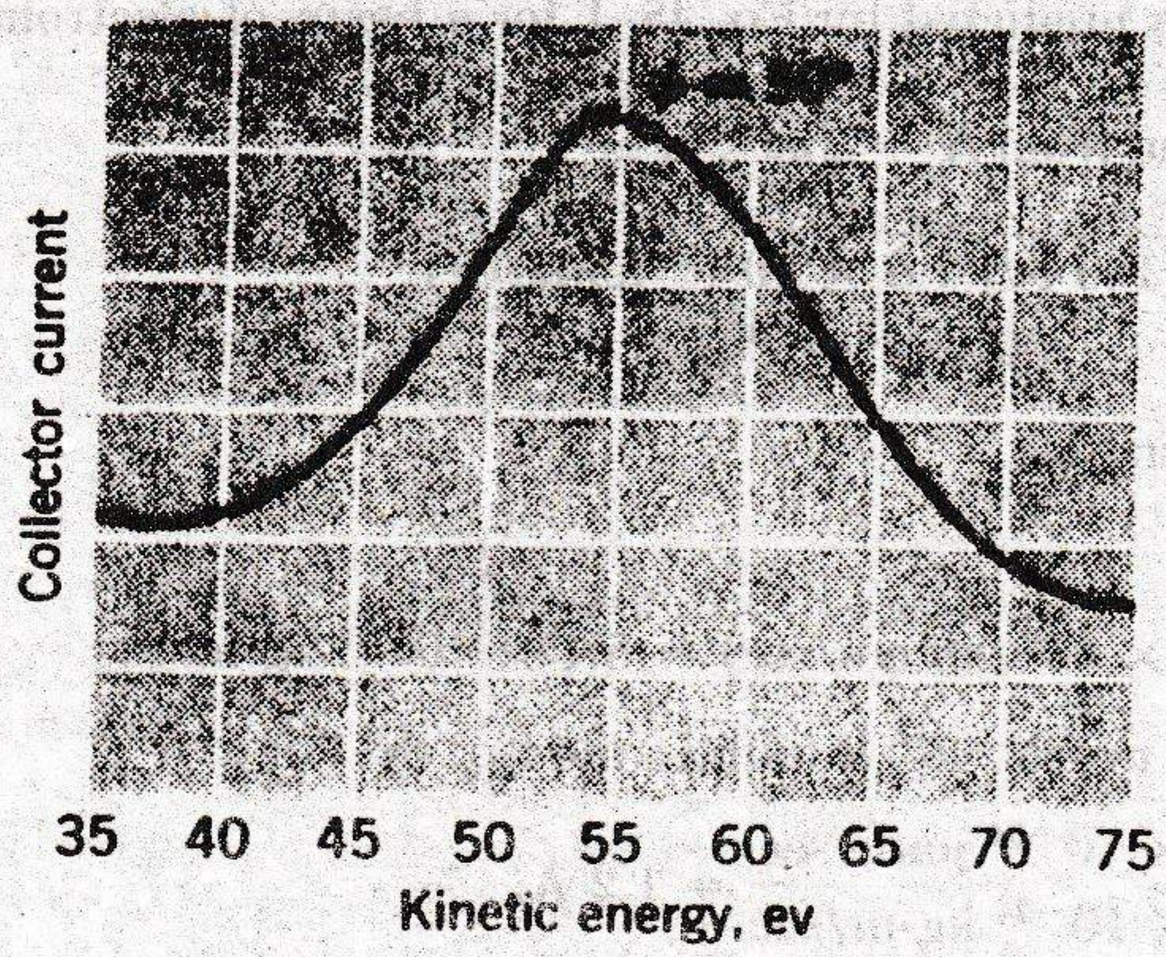


Fig. 48–2 The collector current in detector D in Fig. 48–1 as a function of the kinetic energy of the incident electrons, showing a diffraction maximum. The angle ϕ in Fig. 48–1 is adjusted to 50°. If an appreciably smaller or larger value is used, the diffraction maximum disappears.

Davisson and Germer. Electrons from a heated filament are accelerated by a variable potential difference V and emerge from the "electron gun" G with kinetic energy eV. This electron beam is allowed to fall at normal incidence on a single crystal of nickel at C. Detector D is set at a particular angle ϕ and readings of the intensity of the "reflected" beam are taken at various values of the accelerating potential V. Figure 48–2 shows that a strong beam occurs at $\phi = 50°$ for $V = 54$ volts.

All such strong "reflected" beams can be accounted for by assuming that the electron beam has a wavelength, given by $\lambda = h/p$, and that "Bragg reflections" occur from certain families of atomic planes precisely as described for X-rays in Section 45–5.

Figure 48–3 shows such a Bragg reflection, obeying the Bragg relationship

$$m\lambda = 2d \sin \theta \qquad m = 1, 2, 3, \ldots \tag{48–2}$$

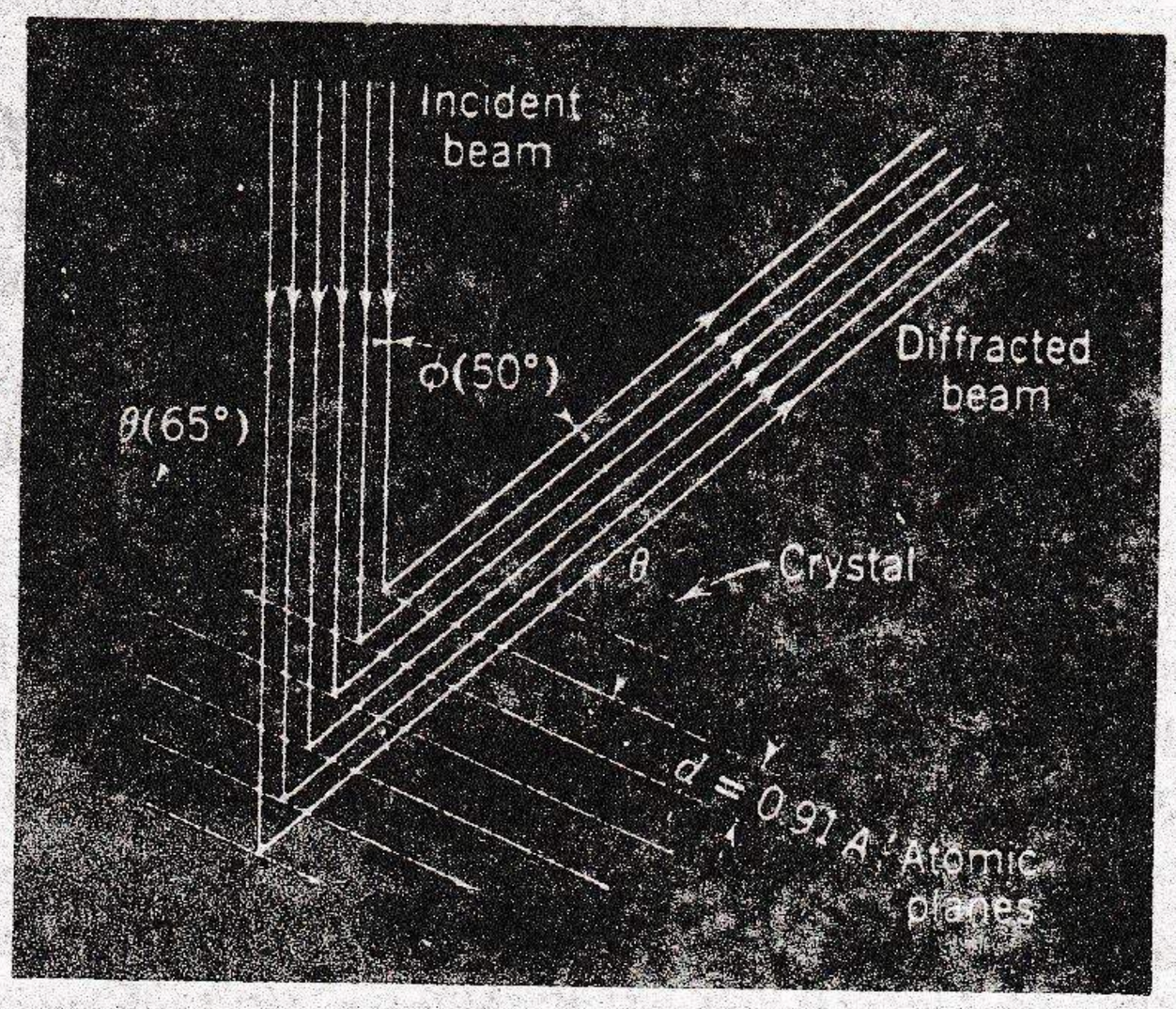


Fig. 48–3 The strong diffracted beam at $\phi = 50°$ and $V = 54$ volts arises from wave-like "reflection" from the family of atomic planes shown, for $d = 0.91$ A. The Bragg angle θ is 65°. For simplicity, refraction of the diffracted wave as it leaves the surface is ignored.

For the conditions of Fig. 48–3 the effective interplanar spacing d can be shown by X-ray analysis to be 0.91 A. Since ϕ equals 50°, it follows that θ equals $90° - \frac{1}{2} \times 50°$ or 65°. The wavelength to be calculated from Eq. 48–2, if we assume $m = 1$, is

$$\lambda = 2d \sin \theta = 2(0.91 \text{ A})(\sin 65°) = 1.65 \text{ A}.$$

The wavelength calculated from the de Broglie relationship $\lambda = h/p$ is, for 54-ev electrons (see Example 1), 1.64 A. This excellent agreement, combined with much similar evidence, is a convincing argument for believing that electrons are wave-like in some circumstances.

Not only electrons but all other particles, charged or uncharged, show wave-like characteristics. Beams of slow neutrons from nuclear reactors are routinely used to investigate the atomic structure of solids. Figure 48–4 shows a "neutron diffraction pattern" for finely powdered lead.

The evidence for the existence of matter waves with wavelengths given by Eq. 48–1 is strong indeed. Nevertheless, the evidence that matter is

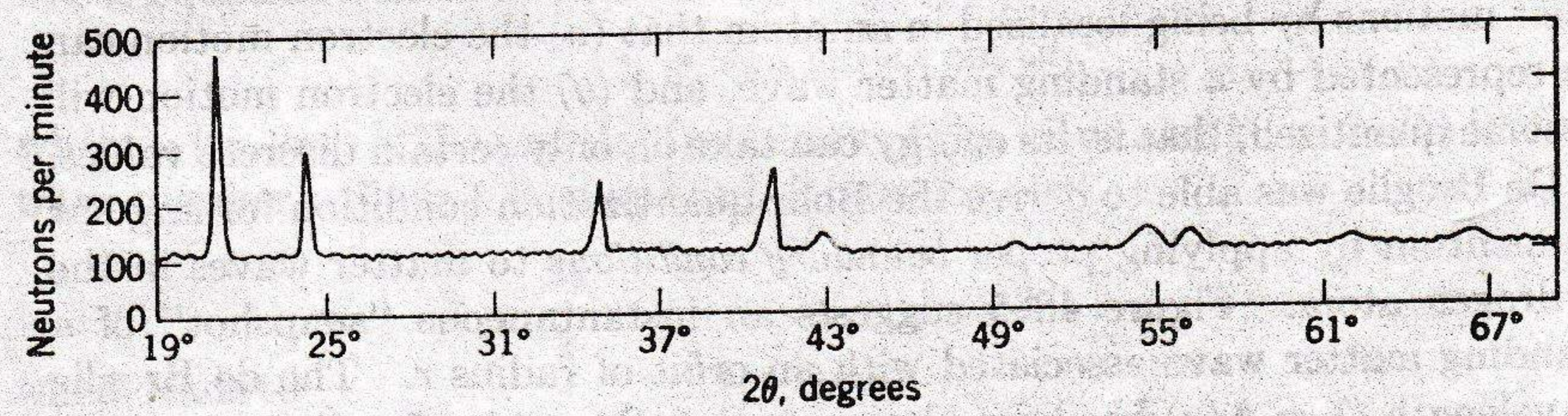


Fig. 48–4 A diffraction pattern for powdered lead, using a monochromatic neutron beam from a nuclear reactor. The peaks represent "Bragg reflections" from the various atomic planes, θ being the corresponding "Bragg angle."

composed of particles remains equally strong; see Fig. 10–10. Thus, for matter as for light, we must face up to the existence of a dual character; *matter behaves in some circumstances like a particle and in others like a wave.*

48–2 Atomic Structure and Standing Waves

The motion of electrons in beams is not bounded or limited in the beam direction. We can make an analogy to a sound wave in a long gas-filled tube, a wave traveling down a long string, or an electromagnetic wave in a long waveguide. All four cases can be described by appropriate traveling waves and, significantly, waves of *any* wavelength (within a certain range) can be propagated.

Let these last three waves be *bounded* by imposing physical restrictions. For the sound wave this corresponds to inserting end walls on a section of the long gas-filled pipe, thus forming an acoustic resonant cavity (Section 38–5). For the waves in the string it corresponds to removing a finite section of string and clamping it at each end, as a violin string (Section 20–5).

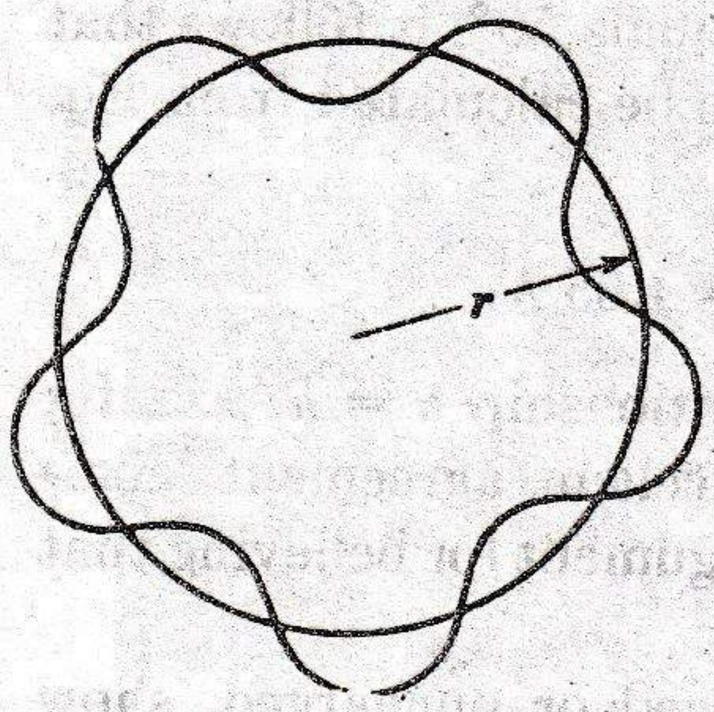


Fig. 48–5 Showing how an electron wave can be adjusted in wavelength to fit an integral number of times around the circumference of a Bohr orbit of radius r. This concept, like the Bohr orbit concept, is now regarded as oversimplified.

For the electromagnetic wave it corresponds to inserting end caps on a finite length of waveguide, thus forming an electromagnetic resonant cavity (Section 38–6).

Two important changes occur: (*a*) the motions are now represented by *standing* rather than traveling waves and (*b*) *only certain wavelengths* (*or frequencies*) *can now exist.* This *quantization* of the wavelength is a direct result of bounding or limiting the wave. We expect that if electrons are limited in their motions by being localized in an atom that (*a*) the electron motion can be represented by a standing matter wave, and (*b*) the electron motion will become quantized, that is, its energy can take on only certain discrete values.

De Broglie was able to derive the Bohr quantization condition for angular momentum by applying proper *boundary conditions* to matter waves in the hydrogen atom. Figure 48–5 suggests an instantaneous "snapshot" of a standing matter wave associated with an orbit of radius r. The de Broglie wavelength ($\lambda = h/p$) has been chosen so that the orbit of radius r contains an integral number n of the matter waves, or

$$\frac{2\pi r}{\lambda} = \frac{2\pi r}{(h/p)} = n \qquad n = 1, 2, 3, \ldots.$$

This leads at once to

$$L = pr = n\frac{h}{2\pi} \qquad n = 1, 2, 3, \ldots,$$

which is the Bohr quantization condition for L.

48–3 Wave Mechanics

The idea that the stationary states in atoms correspond to standing matter waves was taken up by Erwin Schrödinger in 1926 and used by him as the foundation of *wave mechanics*, one of several equivalent formulations of quantum physics.

An important quantity in wave mechanics is the *wave function* Ψ, which measures the "wave disturbance" of matter waves. For waves on strings the "wave disturbance" may be measured by a transverse displacement y; for sound waves it may be measured by a pressure variation p; for electromagnetic waves it may be measured by the electric field vector **E**.

We make the physical meaning of the wave disturbance Ψ clear in Section 48–4. Meanwhile, let us study the wave function $\Psi(x, t)$ for a simple, one-dimensional problem, that of the possible motions a particle of mass m confined between rigid walls of separation l as in Fig. 48–6*b*. The wave function can be obtained by analogy with a known mechanical problem, that of the natural modes of vibration of a string of length l, clamped at each end as in Fig. 48–6*a*.

In the vibrating string the boundary conditions require that nodes exist at each end. This means that the wavelength λ must be chosen so that

$$l = n\frac{\lambda}{2} \qquad n = 1, 2, 3, \ldots,$$

or that the wavelength λ is "quantized" by the requirement that

$$\lambda = \frac{2l}{n} \qquad n = 1, 2, 3, \ldots. \tag{48–3}$$

The wave disturbance for the string is represented by a standing wave whose equation was shown in Section 19–9 to be

$$y = 2y_m \sin kx \cos \omega t,$$

where $\omega\ (= 2\pi\nu)$ is the angular frequency of the wave and $k\ (= 2\pi/\lambda)$ is the wave number. Since λ is quantized, k must be also, or

$$k = \frac{2\pi}{\lambda} = \frac{n\pi}{l} \qquad n = 1, 2, 3, \ldots.$$

which leads to

$$y = \left[2y_m \sin \frac{n\pi x}{l}\right] \cos \omega t \qquad n = 1, 2, 3, \ldots. \tag{48–4}$$

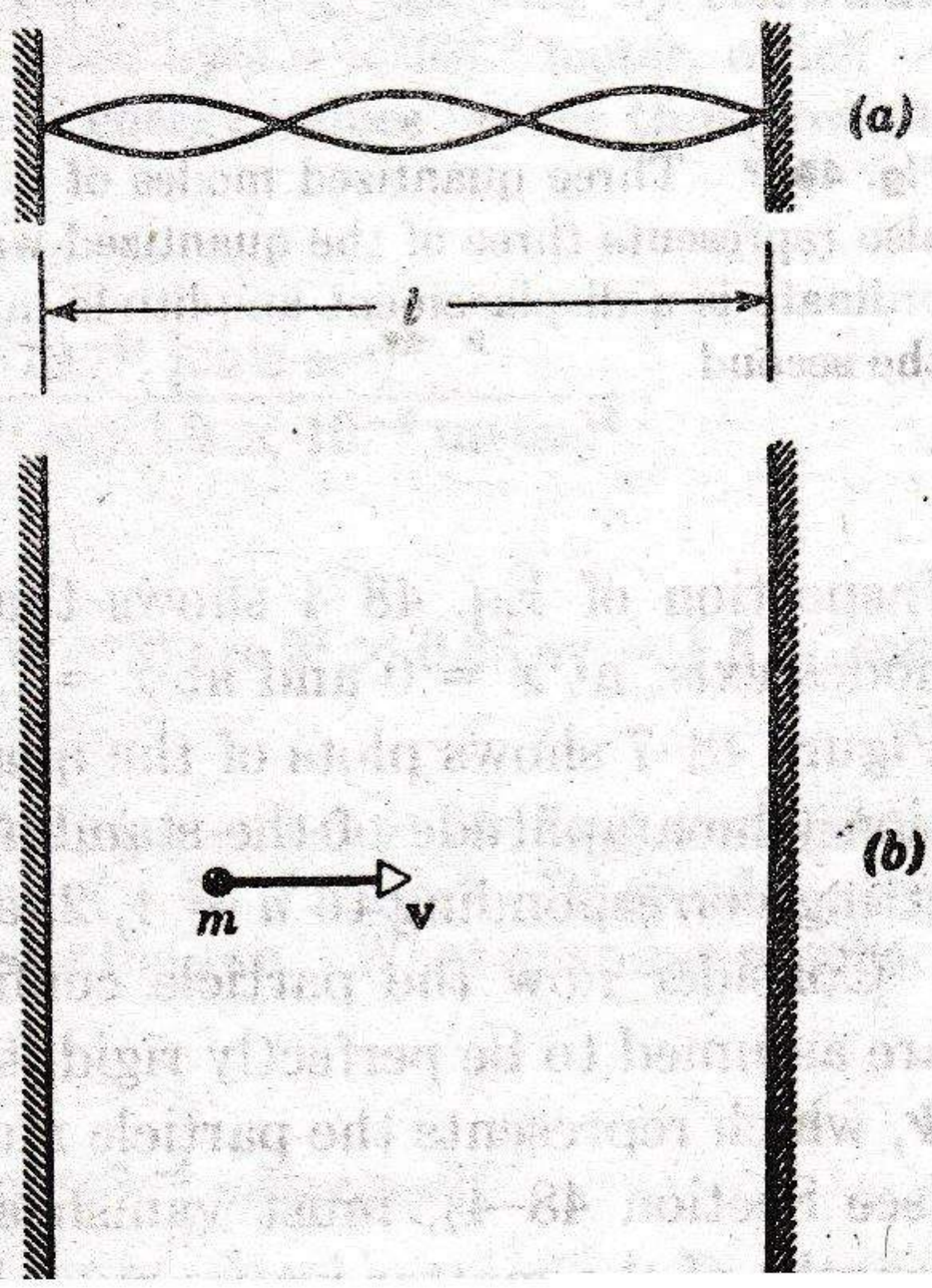


Fig. 48–6 (*a*) A stretched string of length l clamped between rigid supports. (*b*) A particle of mass m and velocity v confined to move between rigid walls a distance l apart.

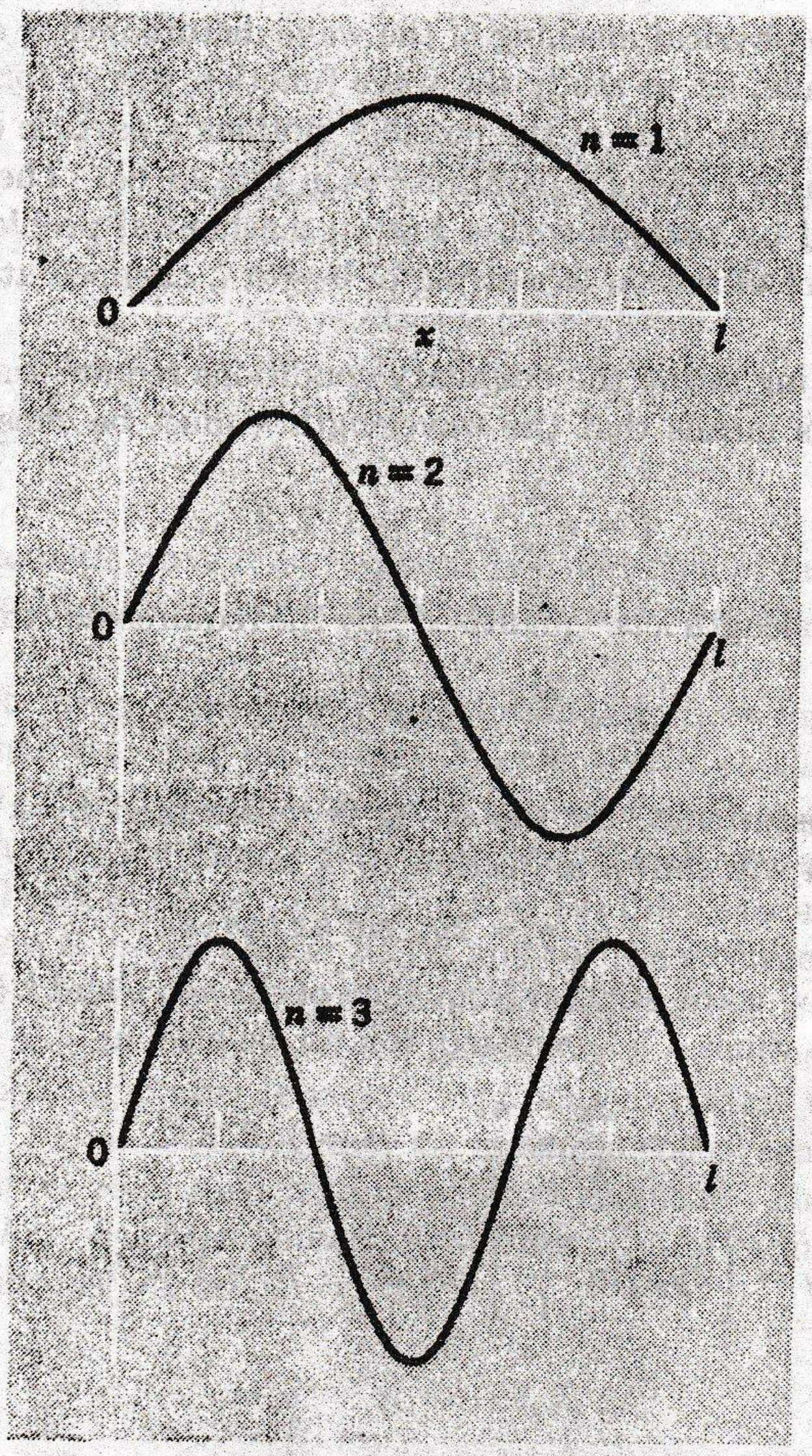


Fig. 48–7 Three quantized modes of vibration for the string of Fig. 48–6*a*. The figure also represents three of the quantized wave functions for the particle of Fig. 48–6*b*. The ordinate is a displacement amplitude in the first case and a wave-function amplitude in the second.

Inspection of Eq. 48-4 shows that no matter what value of n is selected nodes exist at $x = 0$ and at $x = l$, as required by the boundary conditions. Figure 48–7 shows plots of the quantity in the square brackets in this equation (the amplitude of the standing wave) for the modes of vibration of the string corresponding to $n = 1$, 2, and 3.

Consider now the particle confined between rigid walls. Since the walls are assumed to be perfectly rigid, the particle cannot penetrate them so that Ψ, which represents the particle motion in some way not yet clearly specified (see Section 48–4), must vanish at $x = 0$ and $x = l$. The allowed wavelengths of the matter waves must be given by Eq. 48–3, or

$$\lambda = \frac{2l}{n}.$$

Replacing λ by h/p (see Eq. 48–1) leads to

$$p = \frac{nh}{2l}, \tag{48–5}$$

which shows that the *linear momentum* of the particle is quantized. The momentum p ($= mv$) is related to the energy E (which is entirely kinetic and is equal to $\frac{1}{2}mv^2$) by

$$p = \sqrt{2mE}. \tag{48–6}$$

Combining Eqs. 48–5 and 48–6 leads to the quantization condition for E, or

$$E = n^2 \frac{h^2}{8ml^2} \qquad n = 1, 2, 3, \ldots. \tag{48–7}$$

The particle cannot have any energy, as we would expect classically, but only energies given by Eq. 48–7.

The matter wave is described, in strict analogy with Eq. 48–4, by

$$\Psi = \left[\Psi_m \sin \frac{n\pi x}{l}\right] \cos \omega t \qquad n = 1, 2, 3, \ldots. \tag{48–8}$$

Figure 48–7 can serve equally well to show how the amplitude of the standing matter waves for the states of motion corresponding to $n = 1$, 2, and 3 varies throughout the box. We see clearly in this problem how the act of localizing or bounding a particle leads to energy quantization.

▶ **Example 2.** Consider an electron ($m = 9.1 \times 10^{-31}$ kg) confined by electrical forces to move between two rigid "walls" separated by 1.0×10^{-9} meter, which is about five atomic diameters. Find the quantized energy values for the three lowest stationary states.

From Eq. 48–7, for $n = 1$, we have

$$E = n^2 \frac{h^2}{8ml^2} = (1)^2 \frac{(6.6 \times 10^{-34}\ \text{joule-sec})^2}{(8)(9.1 \times 10^{-31}\ \text{kg})(1.0 \times 10^{-9}\ \text{meter})^2}$$

$$= 6.0 \times 10^{-20}\ \text{joule} = 0.38\ \text{ev}.$$

The energies for the next two states ($n = 2$ and $n = 3$) are $2^2 \times 0.38$ ev $= 1.5$ ev and $3^2 \times 0.38$ ev $= 3.4$ ev.

Example 3. Consider a grain of dust ($m = 1.0\ \mu\text{gm} = 1.0 \times 10^{-9}$ kg) confined to move between two rigid walls separated by 0.1 mm ($= 10^{-4}$ meter). Its speed is only 10^{-6} meter/sec, so that it requires 100 sec to cross the gap. What quantum number describes this motion?

The energy is

$$E\ (= K) = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(10^{-9}\ \text{kg})(10^{-6}\ \text{meter/sec})^2$$

$$= 5 \times 10^{-22}\ \text{joule}.$$

Solving Eq. 48–7 for n yields

$$n = \sqrt{8mE}\,\frac{l}{h} = \sqrt{(8)(10^{-9}\text{ kg})(5\times 10^{-22}\text{ joule})}\left(\frac{10^{-4}\text{ meter}}{6.6\times 10^{-34}\text{ joule-sec}}\right)$$
$$= 3\times 10^{14}.$$

Even in these extreme conditions the quantized nature of the motion would never be apparent; we cannot distinguish experimentally between $n = 3\times 10^{14}$ and $n = 3\times 10^{14} + 1$. Classical physics, which fails completely for the problem of Example 2, works extremely well for this problem. ◀

48–4 Meaning of Ψ

Max Born first suggested that the quantity Ψ^2 at any particular point is a measure of *the probability that the particle will be near that point.* More exactly, if a volume element dV is constructed at that point, the probability that the particle will be found in the volume element at a given instant is $\Psi^2\,dV$. This interpretation of Ψ provides a statistical connection between the wave and the associated particle; it tells us where the particle is *likely to be*, not where it *is*.

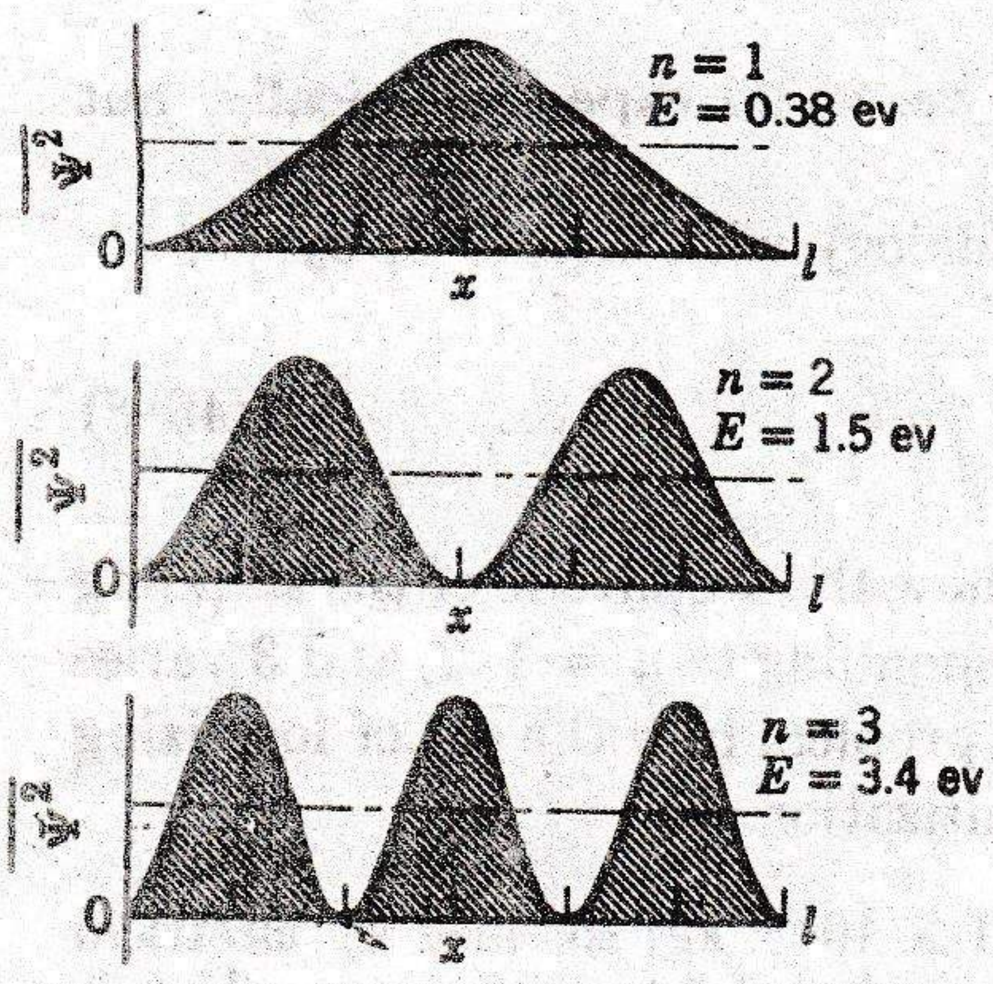


Fig. 48–8 The "probability functions" for three states of motion of the particle of Fig. 48–6*b*, along with the corresponding quantized energies for the conditions of Example 2. The horizontal lines show the predictions of classical mechanics, in which the probability function is constant for all positions of the particle.

For the particle confined between rigid walls the probability that the particle will lie between two planes that are distance x and $x + dx$ from one wall (see Fig. 48–8) is given by

$$\Psi^2\,dx = {\Psi_m}^2 \sin^2\frac{n\pi}{l}x\cos^2\omega t\,dx.$$

Since we are more interested in the particle's spatial location than in its time behavior, we average Ψ^2 over one cycle of the motion. This is equivalent to replacing $\cos^2\omega t$ by its average value for one cycle, namely one-half, or

$$\overline{\Psi^2} = \tfrac{1}{2}{\Psi_m}^2 \sin^2\frac{n\pi x}{l} \qquad n = 1, 2, 3, \ldots. \tag{48–9}$$

Figure 48–8 shows $\overline{\Psi^2}$ for the three stationary states corresponding to $n = 1$, 2, and 3. Note that for $n = 1$ the particle is more likely to be near

the center than the ends. This is in sharp contradiction to the results of classical physics, according to which the particle has the same probability of being located anywhere between the walls, as shown by the horizontal line in Fig. 48–8.

The problem of a particle confined between rigid walls has little real application in physics. We would prefer to illustrate the wave mechanics of Schrödinger by applying it to a more experimentally realizable situation, such as the hydrogen atom. Only mathematical complexity prevents us from doing this. We state without proof that when this problem is solved by wave mechanics the motion of the electron in the ground state of the atom, defined by putting $n = 1$ in Eq. 47–28, is described by the following wave function,

$$\Psi = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \cos \omega t, \tag{48–10}$$

where

$$a = \frac{h^2 \epsilon_0}{\pi m e^2}.$$

Putting $n = 1$ in Eq. 47–27 shows that a is the radius of the ground-state orbit in Bohr's theory. This special interpretation has little meaning in wave mechanics; a is taken here merely as a convenient unit of length when dealing with atomic problems, having the value 0.529 A.

▶ **Example 4.** Consider two hypothetical spherical shells centered on the nucleus of a hydrogen atom with radii r and $r + dr$. What is the probability $P(r)$ that the electron will lie between these shells, as a function of r?

This probability is $\Psi^2\, dV$, where dV is the volume between the shells, or $4\pi r^2\, dr$. Thus

$$\Psi^2\, dV = \left(\frac{1}{\sqrt{\pi a^3}} e^{-r/a} \cos \omega t\right)^2 (4\pi r^2\, dr) = P(r)\, dr.$$

Averaging over the time (that is, replacing $\cos^2 \omega t$ by $\frac{1}{2}$) yields for the probability $\overline{P(r)}$

$$\overline{P(r)} = \frac{2r^2}{a^3} e^{-2r/a}.$$

Figure 48–9 shows a plot of this function. Note that the most probable location for the electron corresponds to the first Bohr radius. Thus in wave mechanics we do not say that the electron in the $n = 1$ state in hydrogen goes around the nucleus in a circular orbit of 0.529 A radius but only that the electron is more likely to be found at this distance from the nucleus than at any other distance, either larger or smaller. ◀

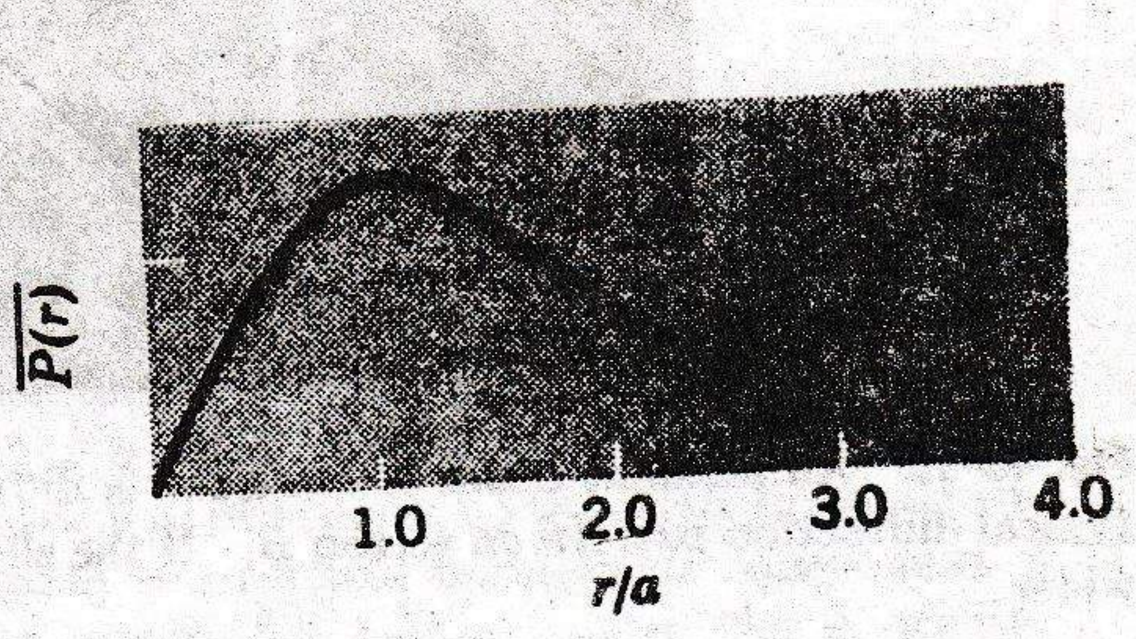


Fig. 48–9 The probability function for the ground state of the hydrogen atom, as calculated from wave mechanics. The separation between the nucleus and the electron is r; a is the radius of the first Bohr orbit (0.529×10^{-10} meter), used here merely as a convenient unit of distance.

48–5 The Uncertainty Principle

Only those quantities that can be measured have any real meaning in physics. If we could focus a "super" microscope on an electron in an atom and see it moving around in an orbit, we would declare that such orbits have meaning. However, we shall show that *it is fundamentally impossible to make such an observation*—even with the most ideal instruments that could conceivably be constructed. Therefore, we declare that such orbits have no physical meaning.

We observe the moon traveling around the earth by means of the sunlight that it reflects in our direction. Now light transfers linear momentum to an object from which it is reflected. In principle, this reflected light would disturb the course of the moon in its orbit, although a little thought shows that this disturbing effect is negligible.

For electrons the situation is quite different. Here, too, we can hope to "see" the electron only if we reflect light, or another particle, from it. In this case the recoil that the electron experiences when the light (photon) bounces from it completely alters the electron's motion in a way that cannot be avoided or even corrected for.

It is not surprising that the probability curve of Fig. 48–9 is the most detailed information that we can hope to obtain, by measurement, about the distribution of negative charge in the hydrogen atom. If orbits such as those envisaged by Bohr existed, they would be broken up completely in our attempts to verify their existence. Under these circumstances, we prefer to say that it is the probability function, and not the orbits, that represents physical reality.

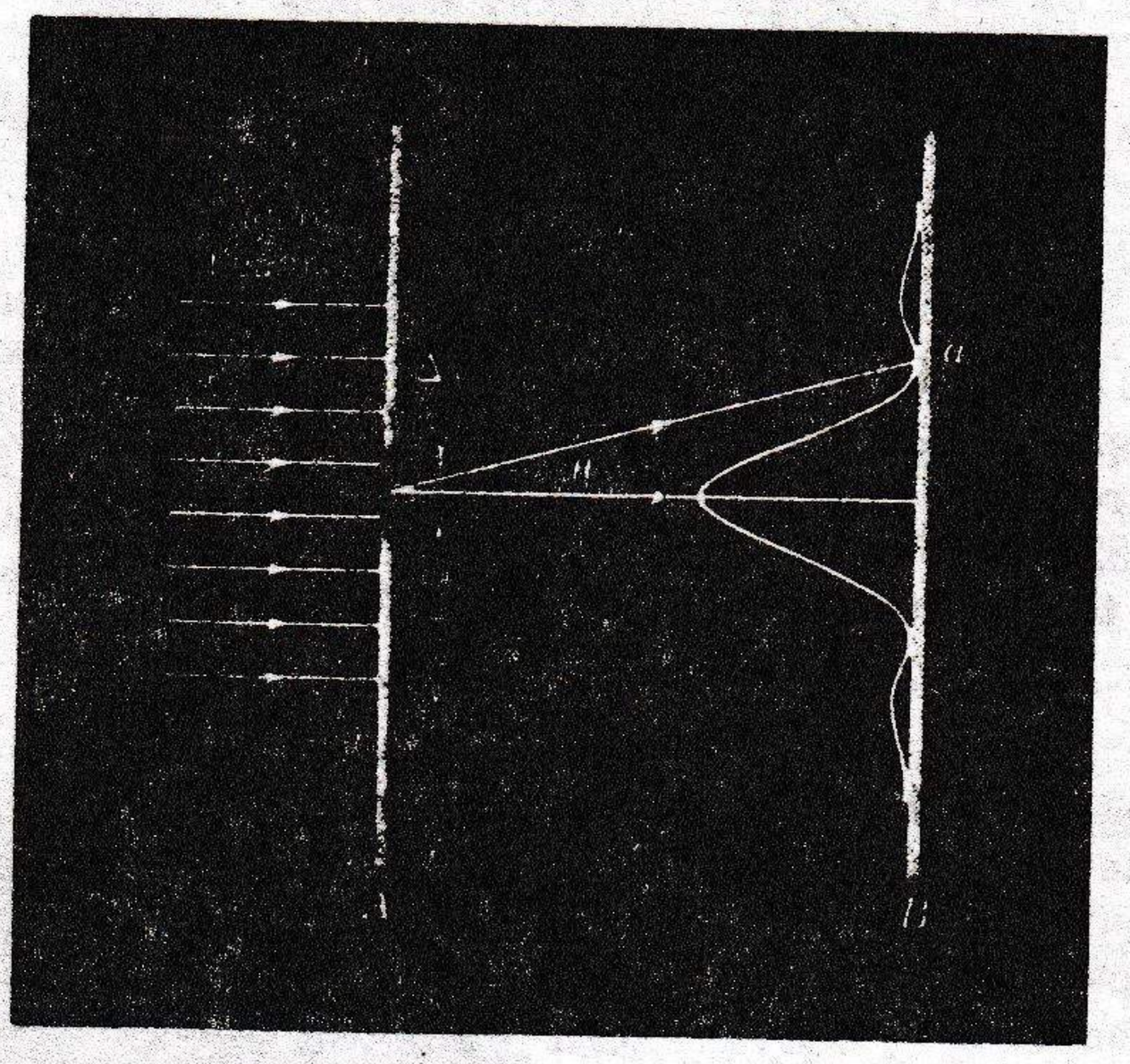


Fig. 48–10 An incident beam of electrons is diffracted at the slit in screen *A*, forming a typical diffraction pattern on screen *B*. If the slit is made narrower, the pattern becomes wider.

Our inherent inability to describe the motions of electrons in a classical way finds expression in the *uncertainty principle*, enunciated by Werner Heisenberg in 1927. To formulate this principle, consider a beam of monoenergetic electrons of speed v_0 moving from left to right in Fig. 48–10. Let us set ourselves the task of measuring the position of a particular electron in the vertical (y) direction and also its velocity component v_y in this direction. If we succeed in carrying out these measurements with unlimited accuracy, we can then claim to have established the position and motion of the electron (or one component of it at least) with precision. However, we shall see that *it is impossible to make these two measurements simultaneously with unlimited accuracy*.

To measure y we block the beam with an absorbing screen A in which we put a slit of width Δy. If an electron gets through the slit, its vertical position must be known to this accuracy. By making the slit narrower, we can improve the accuracy of this vertical position measurement as much as we wish.

Since the electron is a wave, it will undergo diffraction at the slit, and a photographic plate placed at B in Fig. 48–10 will reveal a typical *diffraction pattern*. The existence of this diffraction pattern means that there is an uncertainty in the values of v_y possessed by the electrons emerging from the slit. Let v_{ya} be the value of v_y that corresponds to an electron landing at the first minimum on the screen, marked by point a and described by a characteristic angle θ_a. We take v_{ya} as a *rough* measure of the uncertainty Δv_y in v_y for electrons emerging from the slit.

The first minimum in the diffraction pattern is given by Eq. 44–2, or

$$\sin \theta_a = \frac{\lambda}{\Delta y}.$$

If we assume that θ_a is small enough, we can write this equation as

$$\theta_a \cong \frac{\lambda}{\Delta y}. \tag{48–11}$$

To reach point a, v_{ya} ($= \Delta v_y$) must be such that

$$\theta_a \cong \frac{\Delta v_y}{v_0}. \tag{48–12}$$

Combining Eqs. 48–11 and 48–12 leads to

$$\frac{\Delta v_y}{v_0} = \frac{\lambda}{\Delta y},$$

which we rewrite as

$$\Delta v_y \, \Delta y \cong \lambda v_0. \tag{48–13}$$

Now λ, the wavelength of the electron beam, is given by h/p or h/mv_0; substituting this into Eq. 48–13 yields

$$\Delta v_y \, \Delta y \cong \frac{hv_0}{mv_0}.$$

We rewrite this as

$$\Delta p_y \, \Delta y \cong h. \tag{48–14}$$

In Eq. 48–14 Δp_y ($= m \, \Delta v_y$) is the uncertainty in our knowledge of the vertical momentum of the electrons; Δy is the uncertainty in our knowledge of their vertical position. The equation tells us that, since the product of these uncertainties is a constant, *we cannot measure p_y and y simultaneously with unlimited accuracy*.

If we want to improve our measurement of y (that is, if we want to reduce Δy), we use a finer slit. However (see Eq. 48–11), this will produce a wider diffraction pattern. A wider pattern means that our knowledge of the vertical momentum component of the electron has deteriorated, or, in other words, Δp_y has increased; this is exactly what Eq. 48–14 predicts.

The limits on measurement imposed by Eq. 48–14 have nothing to do with the crudity of our measuring instruments. We are permitted to postulate the existence of the finest conceivable measuring equipment. Equation 48–14 represents a *fundamental* limitation, imposed by nature.

Equation 48–14 is a derivation, for a special case, of a general principle known as the *uncertainty principle*. As applied to position and momentum measurements, it asserts that

$$\Delta p_x \, \Delta x \geq h$$

$$\Delta p_y \, \Delta y \geq h$$

$$\Delta p_z \, \Delta z \geq h. \qquad (48\text{–}15)$$

Thus no component of the motion of an electron, free or bound, can be described with unlimited precision.

Planck's constant h probably appears nowhere that has more deep-seated significance than in Eq. 48–15. If this product had been zero instead of h, the classical ideas about particles and orbits would be correct; it would then be possible to measure both momentum and position with unlimited precision. The fact that h appears means that the classical ideas are wrong; the magnitude of h tells us under what circumstances these classical ideas must be replaced by quantum ideas. Gamow * has speculated, in an interesting and readable fantasy, what our world would be like if the constant h were much larger than it is, so that nonclassical ideas would be apparent to our sense perceptions.

▶ **Example 5.** An electron has a speed of 300 meters/sec, accurate to 0.01%. With what fundamental accuracy can we locate the position of this electron?

The electron momentum is

$$p = mv = (9.1 \times 10^{-31} \text{ kg})(300 \text{ meters/sec}) = 2.7 \times 10^{-28} \text{ kg-m/sec}.$$

The uncertainty in momentum is given to be 0.01% of this, or

$$\Delta p = (0.0001)(2.7 \times 10^{-28} \text{ kg-m/sec}) = 2.7 \times 10^{-32} \text{ kg-m/sec}.$$

The minimum uncertainty in position, from Eq. 48–15, is

$$\Delta x = \frac{h}{\Delta p} = \frac{6.6 \times 10^{-34} \text{ joule-sec}}{2.7 \times 10^{-32} \text{ kg-m/sec}}$$

$$= 2.4 \text{ cm}.$$

If the electron momentum has really been determined by measurement to have the accuracy stated, there is no hope whatever that its position can be known to any better accuracy than that stated, namely about 1 in. The concept of the electron as a tiny dot is not very valid under these circumstances.

▶ **Example 6.** A bullet has a speed of 300 meters/sec, accurate to 0.01%. With what fundamental accuracy can we locate its position? Its mass is 50 gm (= 0.05 kg).

This example is the same as Example 5 in every respect save the mass of the particle involved. The momentum is

$$p = mv = (0.05 \text{ kg})(300 \text{ meters/sec}) = 15 \text{ kg-m/sec}$$

and

$$\Delta p = (0.0001)(15 \text{ kg-m/sec}) = 1.5 \times 10^{-3} \text{ kg-m/sec}.$$

Equation 48–15 yields

$$\Delta x = \frac{6.6 \times 10^{-34} \text{ joule-sec}}{1.5 \times 10^{-3} \text{ kg-m/sec}} = 4.4 \times 10^{-31} \text{ meter}.$$

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* *Mr. Tompkins in Wonderland*, Macmillan, 1940.

This is so far beyond the possibility of measurement (a nucleus is only about 10^{-15} meter in diameter) that we can assert that for heavy objects like bullets the uncertainty principle sets no limits whatever on our measuring procedures. Once again the correspondence principle shows us how quantum physics reduces to classical physics under the appropriate circumstances. ◀

The uncertainty relation shows us why it is possible for both light and matter to have a dual, wave-particle, nature. It is because these two views, so obviously opposite to each other, can never be brought face to face in the same experimental situation. If we devise an experiment that forces the electron to reveal its wave character strongly, its particle character will always be inherently fuzzy. If we modify the experiment to bring out the particle character more strongly, the wave character necessarily becomes fuzzy. Matter and light are like coins that can be made to display either face at will but not both simultaneously. Niels Bohr first pointed out in his *principle of complementarity* how the ideas of wave and of particle complement rather than contradict each other.

QUESTIONS

1. How can the *wavelength* of an electron be given by $\lambda = h/p$ when the very presence of the momentum p in this formula implies that the electron is a particle?

2. How could Davisson and Germer be sure that the "54-volt" peak of Fig. 48–2 was a first-order diffraction peak, that is, that $m = 1$ in Eq. 48–2?

3. In a repetition of Thomson's experiment for measuring e/m for the electron (see Section 33–8), a beam of 10^4-ev electrons is collimated by passage through a slit of width 0.50 mm. Why is the beam-like character of the emerging electrons not destroyed by diffraction of the electron wave at this slit?

4. Why is the wave nature of matter not more apparent to our daily observations?

5. Apply the correspondence principle to the problem of a particle confined between rigid walls, showing that those features which seem "strange" (that is, the quantization of energy and the nonuniformity of the probability functions of Fig. 48–8) become undetectable experimentally at large quantum numbers.

6. In the $n = 1$ mode, for a particle confined between rigid walls, what is the probability that the particle will be found in a small volume element at the surface of either wall?

7. A standing wave can be viewed as the superposition of two traveling waves. Can you apply this to the problem of a particle confined between rigid walls, giving an interpretation in terms of the motion of the electron?

8. What is the physical significance of the wave function Ψ?

9. Why does the concept of Bohr orbits violate the uncertainty principle?

10. How can the predictions of wave mechanics be so exact if the only information we have about the positions of the electrons is statistical?

11. Make up some numerical examples to show the difficulty of getting the uncertainty principle to reveal itself during experiments with an object whose mass is about 1 gm.

12. Figure 48–8 shows that for $n = 3$ the probability function for a particle confined between rigid walls is zero at two points between the walls. How can the particle ever move across these positions? (Hint: Consider the implications of the uncertainty principle.)

13. The uncertainty principle can be stated in terms of angular quantities (compare Eq. 48–15) as

$$\Delta L \, \Delta\phi \geq h$$

where ΔL is the uncertainty in the *angular* momentum and $\Delta\phi$ the uncertainty in the *angular* position. For electrons in atoms the angular momentum has definite quantized values, with no uncertainty whatever. What can we conclude about the uncertainty in the angular position and about the validity of the orbit concept?

PROBLEMS

1. A bullet of mass 40 gm travels at 1000 meters/sec. (*a*) What wavelength can we associate with it? (*b*) Why does the wave nature of the bullet not reveal itself through diffraction effects?

2. What wavelength do we associate with a beam of neutrons whose energy is 0.025 ev?

3. An electron and a photon each have a wavelength of 2.0 A. What are their (*a*) momenta and (*b*) energies?

4. The 50-Bev electron accelerator to be built at Stanford will provide an electron beam of small wavelength, suitable for probing the fine details of nuclear structure by scattering experiments. What will this wavelength be and how does it compare to the size of an average nucleus? (Hint: At these energies it is necessary to use the extreme relativistic relationship between momentum and energy, namely $p = E/c$. This is the same relationship used for light (Section 40–2) and is justified whenever the kinetic energy of a particle is very much greater than its rest energy m_0c^2, as in this case.)

5. In the experiment of Davisson and Germer (*a*) at what angles would the second- and third-order diffracted beams corresponding to the strong maximum of Fig. 48–2 occur, provided they are present, and (*b*) at what angle would the first-order diffracted beam occur if the accelerating potential were changed from 54 to 60 volts?

6. Make a plot of de Broglie wavelength against kinetic energy for (*a*) electrons and (*b*) protons. Restrict the range of energy values to those in which classical mechanics applies reasonably well. A convenient criterion is that the maximum kinetic energy on each plot be only about, say, 5% of the rest energy m_0c^2 for the particular particle.

7. What is the wavelength of a hydrogen atom moving with a velocity corresponding to the mean kinetic energy for thermal equilibrium at 20°C?

8. If an electron moves from a state represented by $n = 3$ in Fig. 48–8 to one represented by $n = 2$ in that figure, emitting electromagnetic radiation in the process, what are (*a*) the energy of the emitted single photon and (*b*) the corresponding wavelength?

9. A particle is confined between rigid walls separated by a distance l. What is the probability that it will be found within a distance $l/3$ of one wall (*a*) for $n = 1$, (*b*) for $n = 2$, (*c*) for $n = 3$, and (*d*) under the assumptions of classical physics?

10. In the ground state of the hydrogen atom show that the probability P_r that the electron lies *within* a sphere of radius r is given by

$$P_r = 1 - e^{-2r/a}\left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1\right).$$

Does this yield expected values for (*a*) $r = 0$ and (*b*) $r = \infty$? (*c*) State clearly the difference in meaning between this expression and that given on p. 1117.

11. In the ground state of the hydrogen atom, what is the probability that the electron will lie within a sphere whose radius is that of the first Bohr orbit?

12. The uncertainty in the position of an electron is given as about 0.5 A, which is the radius of the first Bohr orbit in hydrogen. What is the uncertainty in the linear momentum of the electron?

13. Show that if the uncertainty in the location of a particle is equal to its de Broglie wavelength the uncertainty in its velocity is equal to its velocity.

14. The highest achievable resolving power of a microscope is limited only by the wavelength used; that is, the smallest detail that can be separated is about equal to the wavelength. Suppose one wishes to "see" inside an atom. Assuming the atom to have a diameter of 1.0 A, this means that we wish to resolve detail of separation about 0.1 A. (*a*) If an electron microscope is used, what minimum energy of electrons is needed? (*b*) If a light microscope is used, what minimum energy of photons is needed? (*c*) Which microscope seems more practical for this purpose? Explain.

15. A microscope using photons is employed to locate an electron in an atom to within a distance of 0.1 A. What is the uncertainty in the momentum of the electron located in this way?