Conservation of Linear Momentum

CHAPTER 9

9–1 Center of Mass

So far we have treated objects as though they were particles, having mass but no size. In translational motion each point on a body experiences the same displacement as any other point as time goes on, so that the motion of one particle represents the motion of the whole body. But even when a body rotates or vibrates as it moves, there is one point on the body, called the *center of mass*, that moves in the same way that a single particle subject to the same external forces would move. Figure 9–1 shows the simple parabolic motion of the center of mass of an Indian club thrown from one performer to another; no other point in the club moves in such a simple way. Note that, if the club were moving in pure translation (see Fig. 3–1), then *every* point in it would experience the same displacements as does the center of mass in Fig. 9–1. For this reason the motion of the center of mass of a body is called the translational motion of the body.

When the system with which we deal is not a rigid body, a center of mass, whose motion can also be described in a relatively simple way, can be assigned, even though the particles that make up the system may be changing their positions with respect to each other in a relatively complicated way as the motion proceeds. In this section we define the center of mass and show how to calculate its position. In the next section we discuss the properties that make it useful in describing the motion of extended objects or systems of particles.

Consider first the simple case of a system of two particles m_1 and m_2 at distances x_1 and x_2 respectively, from some origin O. We define a point C

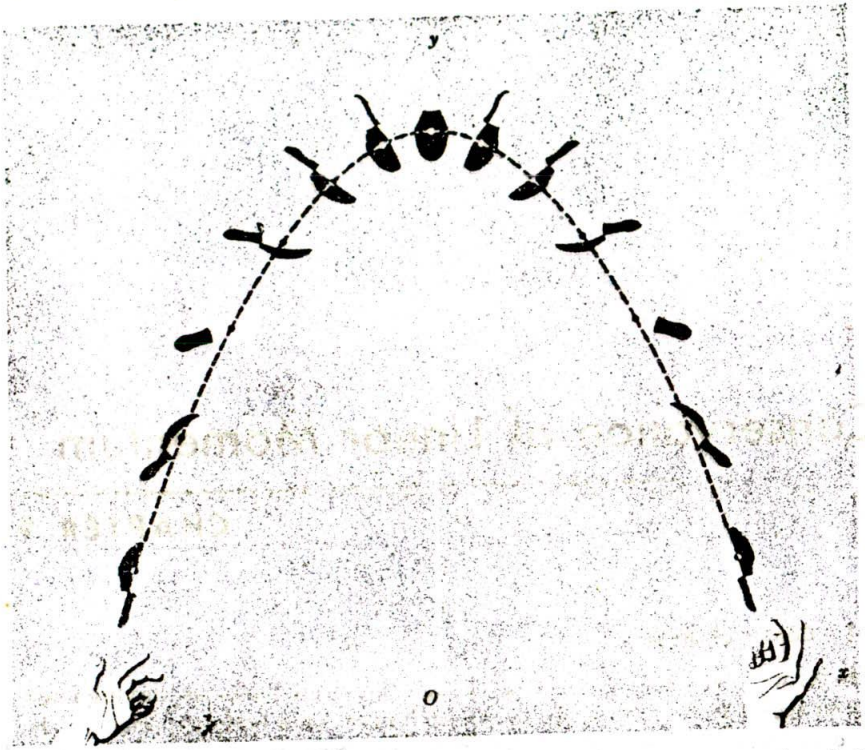


Fig. 9–1 An Indian club is thrown from one performer to another. Even though it rotates and spins around its axis, as shown, there is one point on its axis, the *center of mass*, that follows a simple parabolic path.

the center of mass of the system, as a distance x_{cm} from the origin O, where x_{cm} is defined by

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \tag{9–1}$$

This point (Fig. 9–2) has the property that the product of the total mass of the system $M(= m_1 + m_2)$ times the distance of this point from the origin is equal to the sum of the products of the mass of each particle by its distance from the origin; that is,

$$(m_1 + m_2)x_{cm} = Mx_{cm} = m_1 x_1 + m_2 x_2.$$

In Eq. 9–1, x_{cm} can be regarded as the *mass-weighted mean* of x_1 and x_2.

An analogy might help to fix this idea. Suppose, for example, that we are given two boxes of nails. In one box we have n_1 nails all having the same length l_1; in the other box we have n_2 nails all having the same length l_2. We are asked to get the mean length of the nails. If $n_1 = n_2$, the mean length is simply $(l_1 + l_2)/2$. But if $n_1 \neq n_2$, we must allow for the

fact that there are more nails of one length than another by a "weighting" factor for each length. For l_1 this factor is $n_1/(n_1 + n_2)$ and for l_2 this factor is $n_2/(n_1 + n_2)$, the fraction of the total number of nails in each box. Then the weighted-mean length is

$$\bar{l} = \left(\frac{n_1}{n_1 + n_2}\right) l_1 + \left(\frac{n_2}{n_1 + n_2}\right) l_2$$

or

$$\bar{l} = \frac{n_1 l_1 + n_2 l_2}{n_1 + n_2}.$$

The center of mass, defined in Eq. 9–1, is then a weighted-mean displacement where the "weighting" factor for each particle is the fraction of the total mass that each particle has.

If we have n particles, $m_1, m_2, \cdots, m_n$, *along a straight line*, by definition the center of mass of these particles relative to some origin is

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\Sigma\, m_i x_i}{\Sigma\, m_i}, \qquad (9\text{–}2)$$

where $x_1, x_2, \cdots, x_n$ are the distances of the masses from the origin from which x_{cm} is measured. The symbol Σ represents a summation operation, in this case over all n particles. The sum

$$\Sigma\, m_i = M$$

is the total mass of the system. We can then rewrite Eq. 9–2 in the form

$$M x_{cm} = \Sigma\, m_i x_i. \qquad (9\text{–}2a)$$

Suppose now that we have three particles *not in a straight line;* they will lie in a plane, as in Fig. 9–3. The center of mass C is defined and located by

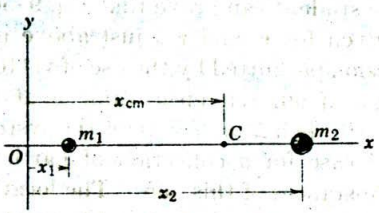


Fig. 9–2 The center of mass of the two masses m_1 and m_2 lies on the line joining m_1 and m_2 at C, a distance x_{cm} from the origin.

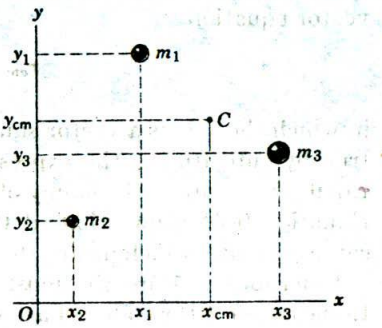


Fig. 9–3 The center of mass of the three masses m_1, m_2, and m_3 lies at point C, with coordinates x_{cm}, y_{cm}. C lies in the same plane as that of the triangle formed by the three masses.

the coordinates x_{cm} and y_{cm}, where

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}, \qquad (9\text{–}3)$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3},$$

in which x_1, y_1 are the coordinates of the particle of mass m_1; x_2, y_2 are those of m_2; and x_3, y_3 are those of m_3. The coordinates x_{cm}, y_{cm} of the center of mass are measured from the same arbitrary origin.

For a large number of particles lying in a plane, the center of mass is at x_{cm}, y_{cm}, where

$$x_{cm} = \frac{\Sigma\, m_ix_i}{\Sigma\, m_i} = \frac{1}{M}\sum m_ix_i \quad \text{and} \quad y_{cm} = \frac{\Sigma\, m_iy_i}{\Sigma\, m_i} = \frac{1}{M}\sum m_iy_i \qquad (9\text{–}4)$$

in which M ($= \Sigma\, m_i$) is the total mass of the system.

For a large number of particles not necessarily confined to a plane but *distributed in space*, the center of mass is at x_{cm}, y_{cm}, z_{cm}, where

$$x_{cm} = \frac{1}{M}\sum m_ix_i, \quad y_{cm} = \frac{1}{M}\sum m_iy_i, \quad z_{cm} = \frac{1}{M}\sum m_iz_i \qquad (9\text{–}5a)$$

In vector notation each particle in the system can be described by a position vector $\mathbf{r}_i$ in a particular reference frame and the center of mass can be located by a position vector $\mathbf{r}_{cm}$. These vectors are related to x_i, y_i, z_i and x_{cm}, y_{cm}, z_{cm} in Eq. 9–5*a* by

$$\mathbf{r}_i = \mathbf{i}x_i + \mathbf{j}y_i + \mathbf{k}z_i$$

and

$$\mathbf{r}_{cm} = \mathbf{i}x_{cm} + \mathbf{j}y_{cm} + \mathbf{k}z_{cm}.$$

Thus the three scalar equations of Eq. 9–5*a* can be replaced by a single vector equation

$$\mathbf{r}_{cm} = \frac{1}{M}\sum m_i\mathbf{r}_i \qquad (9\text{–}5b)$$

in which the sum is a vector sum. The student can prove that Eq. 9–5*b* is true by substituting the expressions given for $\mathbf{r}_i$ and $\mathbf{r}_{cm}$ just above into Eq. 9–5*b*. Note the economy of expression permitted by the use of vectors. Equation 9–5*b* shows that, if the origin of our reference frame is at the center of mass (which means that $\mathbf{r}_{cm} = 0$), then $\Sigma\, m_i\mathbf{r}_i = 0$ for the system.

Equations 9–5 are the most general case for a collection of particles. Equations 9–1 through 9–4 are special instances of this one. The location of the center of mass is independent of the reference frame used to locate it (see Problem 1). *The center of mass of a system of particles depends only on the masses of the particles and the positions of the particles relative to one another.*

A rigid body, such as a meter stick, can be thought of as a system of closely packed particles. Hence it also has a center of mass. The number of particles (atoms, for example) in the body is so large and their spacing so small, however, that we can treat the body as though it has a continuous distribution of mass. To obtain the expression for the center of mass of a continuous body, let us begin by subdividing the body into n small elements of mass Δm_i located approximately at the points x_i, y_i, z_i. The coordinates of the center of mass are then given approximately, by

$$x_{cm} = \frac{\Sigma\, \Delta m_i x_i}{\Sigma\, \Delta m_i} \qquad y_{cm} = \frac{\Sigma\, \Delta m_i y_i}{\Sigma\, \Delta m_i} \qquad z_{cm} = \frac{\Sigma\, \Delta m_i z_i}{\Sigma\, \Delta m_i}.$$

Now let the elements of mass be further subdivided so that the number of elements n tends to infinity. The points x_i, y_i, z_i will locate the mass elements more precisely as n is increased and will locate them exactly as n becomes infinite. The continuous body is then subdivided into an infinite number of infinitesimal mass elements. The coordinates of the center of mass can now be given precisely as

$$\begin{aligned} x_{cm} &= \lim_{\Delta m_i \to 0} \frac{\Sigma\, \Delta m_i x_i}{\Sigma\, \Delta m_i} = \frac{\int x\, dm}{\int dm} = \frac{1}{M} \int x\, dm, \\ y_{cm} &= \lim_{\Delta m_i \to 0} \frac{\Sigma\, \Delta m_i y_i}{\Sigma\, \Delta m_i} = \frac{\int y\, dm}{\int dm} = \frac{1}{M} \int y\, dm, \\ z_{cm} &= \lim_{\Delta m_i \to 0} \frac{\Sigma\, \Delta m_i z_i}{\Sigma\, \Delta m_i} = \frac{\int z\, dm}{\int dm} = \frac{1}{M} \int z\, dm. \end{aligned} \tag{9–6a}$$

In these expressions dm is the differential element of mass at the point x, y, z, and $\int dm$ equals M, where M is the total mass of the object. For a continuous body the summation of Eq. 9–5*a* is replaced by the integration of Eq. 9–6*a*.

The vector expression that is equivalent to the three scalar expressions of Eq. 9–6*a* is

$$\mathbf{r}_{cm} = \frac{1}{M} \int \mathbf{r}\, dm \tag{9–6b}$$

As before, the summation of Eq. 9–5*b* has been replaced by an integration. Once again we see that if the origin of our reference frame is at the center of mass (that is, if $\mathbf{r}_{cm} = 0$), then $\int \mathbf{r}\, dm = 0$ for the body. This integral, and the corresponding sum $\Sigma\, m_i \mathbf{r}_i$ of Eq. 9–5*b*, is called the *first moment of mass for the system*.

Often we deal with homogeneous objects having a point, a line, or a plane of symmetry. Then the center of mass will lie at the point, on the line, or in the plane of symmetry. For example, the center of mass of a homogeneous sphere (which has a point of symmetry) will be at the center of the sphere, the center of mass of a cone (which has a line of symmetry) will be on the axis of the cone, etc. We can understand that this is so

because, from symmetry, the first moment of mass ($\int \mathbf{r}\, dm$) is zero at the center of a sphere, somewhere along the axis of a cone, etc. It follows from Eq. 9–6*b* that $\mathbf{r}_{cm} = 0$ for such points which means that the center of mass is located at these points.

▶ **Example 1.** Locate the center of mass of three particles of mass $m_1 = 1.0$ kg, $m_2 = 2.0$ kg, and $m_3 = 3.0$ kg at the corners of an equilateral triangle 1.0 meter on a side.

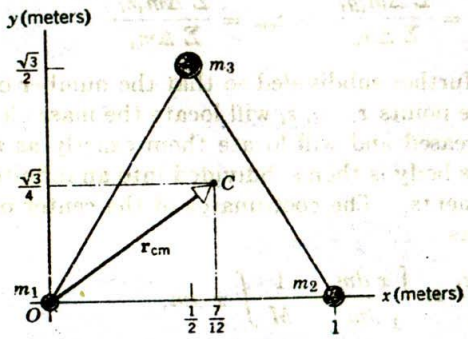


Fig. 9–4 Example 1. Finding the center of mass C of three unequal masses forming an equilateral triangle.

Choose the x-axis along one side of the triangle as shown in Fig. 9–4. Then,

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{(1.0\text{ kg})(0) + (2.0\text{ kg})(1.0\text{ meter}) + (3.0\text{ kg})(\frac{1}{2}\text{ meter})}{(1.0 + 2.0 + 3.0)\text{ kg}} = \tfrac{7}{12}\text{ meter},$$

$$y_{cm} = \frac{\sum m_i y_i}{\sum m_i} = \frac{(1.0\text{ kg})(0) + (2.0\text{ kg})(0) + (3.0\text{ kg})(\sqrt{3}/2\text{ meter})}{(1.0 + 2.0 + 3.0)\text{ kg}} = \tfrac{\sqrt{3}}{4}\text{ meter}.$$

The center of mass C is shown in the figure. Why is it not at the geometric center of the triangle?

Example 2. Find the center of mass of the triangular plate of Fig. 9–5.

If a body can be divided into parts such that the center of mass of each part is known, the center of mass of the body can usually be found simply. The triangular plate may be divided into narrow strips parallel to one side. The center of mass of each strip lies on the line which joins the middle of that side to the opposite vertex. But we can divide up the triangle in three different ways, using this process for each of three sides. Hence the center of mass lies at the intersection of the



Fig. 9–5 Example 2. Finding the center of mass C of a triangular plate.

three lines which join the middle of each side with the opposite vertices. This is the only point that is common to the three lines. ◀

9–2 Motion of the Center of Mass

Now we can discuss the physical importance of the center-of-mass concept. Consider the motion of a group of particles whose masses are $m_1, m_2, \ldots, m_n$ and whose total mass is M. For the time being we will assume that mass neither enters nor leaves the system so that the total mass M of the system remains constant with time. In Section 9–7 we shall consider systems in which M is not constant; a familiar example is a rocket, which expels hot gases as its fuel burns, thus reducing its mass.

From Eq. 9–5*b* we have, for our fixed system of particles,

$$M\mathbf{r}_{cm} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n,$$

where $\mathbf{r}_{cm}$ is the position vector identifying the center of mass in a particular reference frame. Differentiating this equation with respect to time, we obtain

$$M\frac{d\mathbf{r}_{cm}}{dt} = m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} + \cdots + m_n\frac{d\mathbf{r}_n}{dt} \tag{9–7}$$

or

$$M\mathbf{v}_{cm} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots + m_n\mathbf{v}_n,$$

where $\mathbf{v}_1$ is the velocity of the first particle, etc., and $d\mathbf{r}_{cm}/dt$ $(= \mathbf{v}_{cm})$ is the velocity of the center of mass.

Differentiating Eq. 9–7 with respect to time, we obtain

$$M\frac{d\mathbf{v}_{cm}}{dt} = m_1\frac{d\mathbf{v}_1}{dt} + m_2\frac{d\mathbf{v}_2}{dt} + \cdots + m_n\frac{d\mathbf{v}_n}{dt} \tag{9–8}$$

$$= m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \cdots + m_n\mathbf{a}_n,$$

where $\mathbf{a}_1$ is the acceleration of the first particle, etc., and $d\mathbf{v}_{cm}/dt$ $(= \mathbf{a}_{cm})$ is the acceleration of the center of mass of the system. Now, from Newton's second law, the force $\mathbf{F}_1$ acting on the first particle is given by $\mathbf{F}_1 = m_1\mathbf{a}_1$. Likewise, $\mathbf{F}_2 = m_2\mathbf{a}_2$, etc. We can then write Eq. 9–8 as

$$M\mathbf{a}_{cm} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n. \tag{9–9}$$

Hence *the total mass of the group of particles times the acceleration of its center of mass is equal to the vector sum of all the forces acting on the group of particles.*

Among all these forces will be *internal* forces exerted by the particles on each other. However, from Newton's third law, these internal forces will occur in equal and opposite pairs, so that they contribute nothing to the sum. Hence the internal forces can be removed from the problem. The right-hand sum in Eq. 9–9 represents the sum of only the *external* forces acting on all the particles. We can then rewrite Eq. 9–9 as simply

$$M\mathbf{a}_{cm} = \mathbf{F}_{ext}. \tag{9–10}$$

This states that *the center of mass of a system of particles moves as though all the mass of the system were concentrated at the center of mass and all the external forces were applied at that point.*

Notice that we obtain this simple result without specifying the nature of the system of particles. The system can be a rigid body in which the particles are in fixed positions with respect to one another, or it can be a collection of particles in which there may be all kinds of internal motion. Whatever the system is, and however its individual parts may be moving, its center of mass moves according to Eq. 9–10.

Hence, instead of treating bodies as single particles as we have done in previous chapters, we can treat them as collections of particles. Then we can obtain the translational motion of the body, that is, the motion of its center of mass, by assuming that all the mass of the body is concentrated at its center of mass and all the external forces are applied at that point.* This, in fact, is the procedure that we followed implicitly in all our force-diagrams and problem solving.

Aside from justifying and making more concrete our previous procedure, we have now found how to describe the translational motion of a *system* of particles and how to describe the *translational* motion of a body which may be rotating as well. In this chapter and the next we apply this result to the linear motion of a system of particles. In later chapters we shall see how it simplifies the analysis of rotational motion.

▶ **Example 3.** Consider three particles of different masses acted on by external forces, as shown in Fig. 9–6. Find the acceleration of the center of mass of the system.

First we find the coordinates of the center of mass. From Eq. 9–3,

$$x_{cm} = \frac{(8.0 \times 4) + (4.0 \times -2) + (4.0 \times 1)}{16} \text{ meters} = 1.8 \text{ meters},$$

$$y_{cm} = \frac{(8.0 \times 1) + (4.0 \times 2) + (4.0 \times -3)}{16} \text{ meters} = 0.25 \text{ meter}.$$

These are shown as C in Fig. 9–6.

To obtain the acceleration of the center of mass, we first determine the resultant external force acting on the system consisting of the three particles. The x-component of this force is

$$F_x = 14 \text{ nt} - 6.0 \text{ nt} = 8.0 \text{ nt},$$

and the y-component is

$$F_y = 16 \text{ nt}.$$

Hence the resultant external force has a magnitude

$$F = \sqrt{(8.0)^2 + (16)^2} \text{ nt} = 18 \text{ nt},$$

* When the external force is gravity, it acts through the *center of gravity* of the body. In every case we have considered, the center of gravity coincides with the center of mass, which is a more general concept. The conditions under which these points are different for a body will be discussed in Chapter 14.

and makes an angle θ with the x-axis given by

$$\tan\theta = \frac{16\text{ nt}}{8.0\text{ nt}} = 2.0 \quad \text{or} \quad \theta = 63°.$$

Then, from Eq. 9–10, the acceleration of the center of mass is

$$a_{cm} = \frac{F}{M} = \frac{18\text{ nt}}{16\text{ kg}} = 1.1\text{ meters/sec}^2,$$

making an angle of 63° with the x-axis.

Although the three particles will change their relative positions as time goes on, the center of mass will move, as shown, with this constant acceleration. ◀

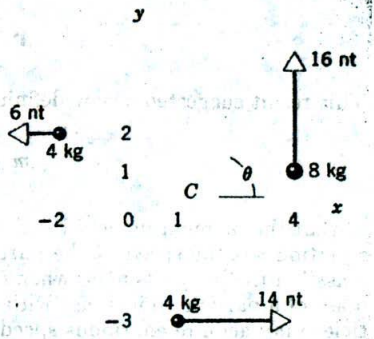


Fig. 9–6 Example 3. Finding the motion of the center of mass of three masses, each subjected to a different force. The forces all lie in the plane defined by the particles. The distances indicated along the axes are in meters.

9–3 Linear Momentum of a Particle

The *momentum* of a single particle is a vector $\mathbf{p}$ defined as the product of its mass m and its velocity $\mathbf{v}$. That is,

$$\mathbf{p} = m\mathbf{v}. \tag{9–11}$$

Momentum, being the product of a scalar by a vector, is itself a vector. Because it is proportional to $\mathbf{v}$, the momentum $\mathbf{p}$ of a particular particle depends on the reference frame of the observer; we must always specify this frame.

Newton, in his famous *Principia*, expressed the second law of motion in terms of momentum (which he called "quantity of motion"). Expressed in modern terminology Newton's second law reads: *The rate of change of momentum of a body is proportional to the resultant force acting on the body and is in the direction of that force.* In symbolic form this becomes

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{9–12}$$

If our system is a single particle of (constant) mass m, this formulation of the second law is equivalent to the form $\mathbf{F} = m\mathbf{a}$, which we have used up to now. That is, if m is a constant, then

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}.$$

The relations $\mathbf{F} = m\mathbf{a}$ and $\mathbf{F} = d\mathbf{p}/dt$ for single particles are completely equivalent in classical mechanics.

In relativity theory the second law for a single particle in the form $\mathbf{F} = m\mathbf{a}$ is not valid. However, it turns out that Newton's second law in the form $\mathbf{F} = d\mathbf{p}/dt$ *is*

still a valid law *if* the momentum **p** for a single particle is defined not as $m_0\mathbf{v}$ but as

$$\mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1 - v^2/c^2}}. \tag{9–13}$$

This result suggested a new definition of mass (compare Eqs. 9–11 and 9–13)

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

so that the momentum could still be written as $\mathbf{p} = m\mathbf{v}$; see Section 8–9. In this equation v is the speed of the particle, c is the speed of light, and m_0 is the "rest mass" of the body (its mass when $v = 0$). From this definition we must expect the mass of a particle to increase with its speed. In atomic and nuclear systems particles may acquire enormous speeds, comparable to the speed of light. This concept can be put to a direct test in such systems because the increase in mass over the rest mass for such particles is large enough to measure accurately. Results of all such experiments indicate that this effect is real and given exactly by the equation above. (See for example, Fig. 8–8.)

9–4 Linear Momentum of a System of Particles

Suppose that instead of a single particle we have a system of n particles, with masses m_1, m_2, etc. *We shall continue to assume*, as we did in Section 9–2, *that no mass enters or leaves the system*, so that the mass M ($= \Sigma m_i$) of the system remains constant with time. The particles may interact with each other and external forces may act on them as well. Each particle will have a velocity and a momentum. Particle 1 of mass m_1 and velocity $\mathbf{v}_1$ will have a momentum $\mathbf{p}_1 = m_1\mathbf{v}_1$, for example. The system as a whole will have a *total momentum* **P** in a particular reference frame, which is defined to be simply the vector sum of the momenta of the individual particles in that same frame, or

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n \tag{9–14}$$
$$= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots + m_n\mathbf{v}_n.$$

If we compare this relation with Eq. 9–7 we see at once that

$$\mathbf{P} = M\mathbf{v}_{cm}, \tag{9–15}$$

which is an equivalent definition for the momentum of a system of particles. In words, Eq. 9–15 states: *The total momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its center of mass.*

We have seen (Eq. 9–10) that Newton's second law for a system of particles can be written as

$$\mathbf{F}_{ext} = M\mathbf{a}_{cm} \tag{9–10}$$

in which $\mathbf{F}_{ext}$ is the vector sum of all the *external* forces acting on the system; we recall that the internal forces acting between particles cancel in pairs because of Newton's third law (see Fig. 9–7). If we differentiate

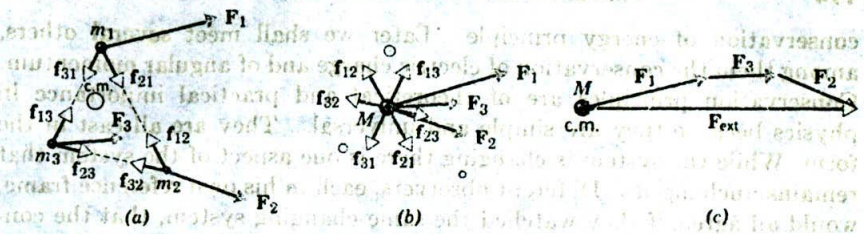


Fig. 9–7 Relationship between the forces acting on a system of three masses m_1, m_2, and m_3. (*a*) All the forces acting on each mass are shown here, as well as the location of the center of mass. On m_1 act forces $\mathbf{f}_{21}$ and $\mathbf{f}_{31}$ exerted by m_2 and m_3 respectively, as well as $\mathbf{F}_1$, a force from some external agent. Similar sets of forces act on m_2 and m_3. However, according to Newton's third law, internal forces $\mathbf{f}_{31}$ and $\mathbf{f}_{13}$ must be equal and opposite and must both lie along the line of centers of m_1 and m_3. Similar statements hold for the other two pairs of action-reaction forces. (*b*) If we are interested only in the motion of the system as a whole, we may consider all the forces to act on a mass $M = m_1 + m_2 + m_3$, located at the center of mass. Owing to the equality of the action-reaction pairs of internal forces as just stated, they cancel each other identically, leaving only the three external forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$. These are added graphically in (*c*) to yield a net force $\mathbf{F}_{ext}$ acting on the center of mass of the system.

Eq. 9–15 with respect to time we obtain, for an assumed constant mass M,

$$\frac{d\mathbf{P}}{dt} = M\frac{d\mathbf{v}_{cm}}{dt} = M\mathbf{a}_{cm}. \tag{9–16}$$

Comparison of Eqs. 9–10 and 9–16 allows us to write Newton's second law for a system of particles in the form

$$\mathbf{F}_{ext} = \frac{d\mathbf{P}}{dt}. \tag{9–17}$$

This equation is the generalization of the single-particle equation $\mathbf{F} = d\mathbf{p}/dt$ (Eq. 9–12) to a system of many particles, no mass entering or leaving the system. Equation 9–17 reduces to Eq. 9–12 for the special case of a single particle, there being only external forces on a one-particle system.

9–5 Conservation of Linear Momentum

Suppose that the sum of the external forces acting on a system is zero. Then, from Eq. 9–17,

$$\frac{d\mathbf{P}}{dt} = 0 \quad \text{or} \quad \mathbf{P} = \text{constant}.$$

When the resultant external force acting on a system is zero, the total vector momentum of the system remains constant. This simple but quite general result is called *the principle of the conservation of linear momentum*. We shall see that it is applicable to many important physical situations.

The conservation of linear momentum principle is the second of the great conservation principles that we have met so far, the first being the

conservation of energy principle. Later we shall meet several others, among them the conservation of electric charge and of angular momentum. Conservation principles are of theoretical and practical importance in physics because they are simple and universal. They are all cast in the form: While the system is changing there is one aspect of the system that remains unchanged. Different observers, each in his own reference frame, would all agree, if they watched the same changing system, that the conservation laws applied to the system. For the conservation of linear momentum, for example, observers in different reference frames would assign different values of **P** to the linear momentum of the system, but each would agree (assuming $\mathbf{F}_{ext} = 0$) that his own value of **P** remained unchanged as the particles that make up the system move about.

The total momentum of a system can only be changed by external forces acting on the system. The internal forces, being equal and opposite, produce equal and opposite changes in momentum which annul one another. For a system of particles

$$\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n = \mathbf{P},$$

so that when the total momentum **P** is constant we have

$$\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n = \text{constant} = \mathbf{P}_0. \qquad (9\text{–}18)$$

The momenta of the individual particles may change, but their sum remains constant if there is no net external force.

Momentum is a vector quantity. Equation 9–18 is therefore equivalent to three scalar equations, one for each coordinate direction. Hence the conservation of linear momentum supplies us with three conditions on the motion of a system to which it applies. The conservation of energy on the other hand supplies us with only one condition on the motion of a system to which it applies, because energy is a scalar.

The law of the conservation of linear momentum holds true even in atomic and nuclear physics, although Newtonian mechanics does not. Hence this conservation law must be more fundamental than the Newtonian principles. In our derivation of this principle we must have made more rigid assumptions than we needed to. This is true even in the framework of classical mechanics. The student should recall the key role played by Newton's third law in this deduction of momentum conservation. This law was used to justify the assumption that the sum of the internal forces acting on all the particles is zero. However, it is somewhat artificial to regard the internal forces in a piece of matter as resulting from pairs of equal and opposite forces between the various pairs of atoms. These internal forces are actually many-body forces, depending on not only the relative separation and orientation of two atoms but also on the positions and orientations of neighboring atoms. If it were possible to prove our assumption without using Newton's third law, the law of conservation of linear momentum would not depend on the validity of the third law of motion. Actually we can prove this assumption on the basis of a much less stringent requirement than that the third law should hold. The proof lies outside the scope of this text.*

* See "On Newton's Third Law and the Conservation of Momentum" E. Gerjuoy, *American Journal of Physics*, November 1949.

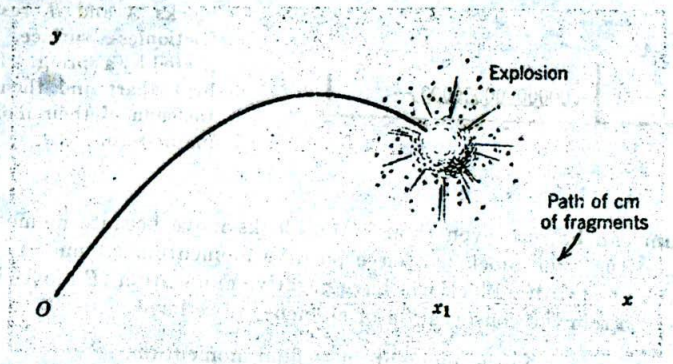


Fig. 9–8 Example 4. A projectile, following the usual parabolic trajectory, bursts at x_1. The center of mass of the fragments continues along the same parabolic path.

9–6 Some Applications of the Momentum Principle

▶ **Example 4.** Consider first a problem in which an external force acts on a system of particles. Recall our previous discussion of projectile motion (Chapter 4). Now let us imagine that our projectile is a shell that explodes while in flight. The path of the shell is shown in Fig. 9–8. We assume that the air resistance is negligible. The system is the shell, the earth is our reference frame, and the external force is that of gravity. At the point x_1 the shell explodes and shell fragments are blown in all directions. What can we say about the motion of this system thereafter?

The forces of the explosion are all *internal forces;* they are forces exerted by part of the system on other parts of the system. These forces may change the momenta of all the *individual fragments* from the values they had when they made up the shell, but they cannot change the *total* vector momentum of the system. Only an external force can change the total momentum. The external force, however, is simply that due to gravity. Since a system of particles as a whole moves as though all its mass were concentrated at the center of mass with the external force applied there, the center of mass of the fragments will continue to move in the parabolic trajectory that the unexploded shell would have followed. The change in the total momentum of the system attributable to gravity is the same whether the shell explodes or not.

What can you say about the *mechanical energy* of the system before and after the explosion?

Example 5. Consider now two blocks A and B, of masses m_A and m_B, coupled by a spring and resting on a horizontal frictionless table. Let us pull the blocks apart and stretch the spring, as in Fig. 9–9, and then release the blocks. Describe the subsequent motion.

If the system consists of the two blocks and spring, then after we have released the blocks there is no net *external* force acting on the system. We can therefore apply the conservation of linear momentum to the motion. The momentum of the system before the blocks were released was zero in the reference frame shown attached to the table, so the momentum must remain zero thereafter. The total

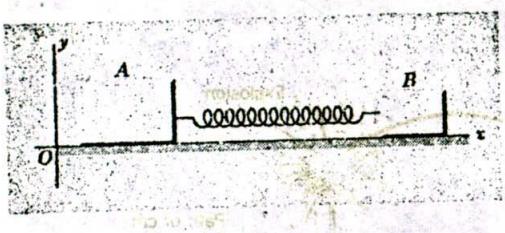


Fig. 9–9 Example 5. Two blocks A and B, resting on a frictionless surface, are connected by a spring. If they are held apart and then released, the sum of their momenta remains zero.

momentum can be zero even though the blocks move because momentum is a *vector* quantity. One block will have positive momentum (A moves in the $+x$ direction) and the other block will have negative momentum (B moves in the $-x$ direction). From the conservation of momentum we have

$$\text{initial momentum} = \text{final momentum},$$

$$0 = m_B v_B + m_A v_A.$$

Therefore

$$m_B v_B = -m_A v_A$$

or

$$v_A = -\frac{m_B}{m_A} v_B.$$

For example, if m_A is 2 slugs and m_B is 1 slug, then v_A will always be one-half v_B in magnitude and oppositely directed as the blocks move.

The kinetic energy of block A is $\frac{1}{2}m_A v_A^2$ and can be written as $(m_A v_A)^2/2m_A$ and that of block B is $\frac{1}{2}m_B v_B^2$ and can be written as $(m_B v_B)^2/2m_B$. But

$$\frac{K_A}{K_B} = \frac{2m_B(m_A v_A)^2}{2m_A(m_B v_B)^2} = \frac{m_B}{m_A},$$

in which $m_A v_A$ equals $m_B v_B$ because of momentum conservation. The kinetic energies of the blocks at any instant are inversely proportional to their respective masses. Because mechanical energy is conserved also, the blocks will continue to oscillate back and forth, the energy being partly kinetic and partly potential. What is the motion of the center of mass of this system?

If mechanical energy is not conserved, as would be true if friction were present, the motion will die out as the energy is dissipated. Can we apply the conservation of linear momentum in this case? Explain.

Example 6. As an example of recoil, consider radioactive decay. An α-particle (the nucleus of a helium atom) is emitted from a uranium-238 nucleus, originally at rest, with a speed of 1.4×10^7 meters/sec and a kinetic energy of 4.1 Mev (million electron volts). Find the recoil speed of the residual nucleus (thorium-234).

We think of the system (thorium + α-particle) as initially bound and forming the uranium nucleus. The system then fragments into two separate parts. The momentum of the system before fragmentation is zero. In the absence of external forces, the momentum after fragmentation is also zero. Hence,

$$\text{initial momentum} = \text{final momentum},$$

$$0 = M_\alpha v_\alpha + M_{Th} v_{Th},$$

$$v_{Th} = -\frac{M_\alpha}{M_{Th}} v_\alpha.$$

The ratio of the α-particle mass to the thorium nucleus mass, M_α/M_{Th}, is 4/234 and $v_\alpha = 1.4 \times 10^7$ meters/sec. Hence,

$$v_{Th} = -(4/234)(1.4 \times 10^7 \text{ meters/sec}) = -2.4 \times 10^5 \text{ meters/sec}.$$

The minus sign indicates that the residual thorium nucleus recoils in a direction exactly opposite to the motion of the α-particle, so as to give a resultant vector momentum of zero.

How can we compute the kinetic energy of the recoiling nucleus (see previous example)? Where does the energy of the fragments come from?

Example 7. Consider now the apparently simple example of a ball thrown up from the earth by a person and then caught by him on its return. To simplify matters we can consider the person to be part of the earth since he does not lose contact with it. We also assume that air resistance is negligible.

The system being considered consists of the earth and the ball. The gravitational forces between the parts of the system are now internal forces. Let us choose a reference frame in which the system (earth + ball) is at rest. When the ball is thrown up, the earth must recoil as seen by an observer in this reference frame. The momentum of the system (earth + ball) is zero initially and no external forces act. Therefore, momentum is conserved and the total momentum remains zero throughout the motion. The upward momentum acquired by the ball is balanced by an equal and opposite downward momentum of the earth. We have

$$\text{initial momentum} = \text{final momentum},$$

$$0 = m_B\mathbf{v}_B + m_E\mathbf{v}_E,$$

$$m_B\mathbf{v}_B = -m_E\mathbf{v}_E.$$

Here m_B and m_E are the masses of ball and earth respectively and $\mathbf{v}_B$ and $\mathbf{v}_E$ are the velocities of the ball and the earth in our selected reference frame. Owing to the enormous mass of the earth in comparison with the ball, the recoil speed of the earth is negligibly small.

As the ball and earth separate, the internal force of gravitational attraction pulls them together until they cease separating and begin to approach one another. As the ball falls toward the earth, the earth falls toward the ball with an equal but oppositely directed momentum. As the ball is caught, its momentum is neutralized by (and it neutralizes) the momentum of the earth. Both objects lose their relative motion, the total momentum is still zero, and the original situation before throwing is restored.

You will recall that when we discussed the conservation of energy in the presence of gravitational potential, we neglected to consider the motion of the earth itself. We took the surface of the earth as our zero level of gravitational potential energy. The reference position did not matter, since we were concerned only with *changes* in potential energy. However, in computing changes in kinetic energy, we assumed that the earth remained stationary, as in the case of the ball thrown up from the earth.

In principle, we cannot ignore the change in the kinetic energy of the earth itself. For example, when the ball falls toward the earth, the earth is slightly accelerated toward the ball. We neglected this fact before because we assumed that the change in kinetic energy of the earth is negligible. This result is not obvious, because although the earth's speed will certainly be small, its mass is enormous and the kinetic energy acquired may be significant. To settle the point we compute the

ratio of the kinetic energy of the earth to that of the ball. Using $m_E v_E = m_B v_B$ from momentum conservation, we have

$$\frac{K_E}{K_B} = \frac{\frac{1}{2}m_E v_E^2}{\frac{1}{2}m_B v_B^2} = \frac{\frac{1}{2}(m_E v_E)^2}{\frac{1}{2}(m_B v_B)^2} \cdot \frac{m_B}{m_E} = \frac{m_B}{m_E}.$$

Since the mass of the ball m_B is negligibly small compared to the mass of the earth m_E, the kinetic energy acquired by the earth, K_E, is negligibly small compared to that of the ball, K_B. For example, if $m_B = 6$ kg (a rather massive ball), then, since $m_E = 6 \times 10^{24}$ kg, $K_E/K_B = 10^{-24}$!

Notice that this problem is identical in principle to Example 5. The differences are only those of detail; in one the potential energy is elastic and in the other the potential energy is gravitational; in one the masses are pictured as of the same order of magnitude, and in the other they are of very different orders of magnitude. ◀

9–7 Systems of Variable Mass

So far we have dealt only with systems in which the total system mass M remained constant with time. Now we consider systems in which mass enters or leaves the system while we are observing it, dM/dt being positive in the former case and negative in the latter.

Figure 9–10*a* shows a system of mass M whose center of mass is moving with velocity $\mathbf{v}$ as seen from a particular reference frame. An external force $\mathbf{F}_{ext}$ acts on the system. At a time Δt later the configuration has changed to that shown in Fig. 9–10*b*. A mass ΔM has been ejected from the system, its center of mass moving with velocity $\mathbf{u}$ as seen by our observer. The system mass is reduced to $M - \Delta M$ and the velocity $\mathbf{v}$ of the center of mass of the system is changed to $\mathbf{v} + \Delta\mathbf{v}$.

The student may imagine the system of Fig. 9–10 to represent a rocket. It ejects hot gas from its orifice at a fairly high speed, decreasing its own mass and increasing its own speed. In a rocket the loss of mass is continuous during the burning process. The external force $\mathbf{F}_{ext}$ is *not* the thrust of the rocket but is the force of gravity on the rocket and the resisting force of the atmosphere.

To analyze the situation let us, for the time being, define the system to be one of constant mass. This means that in Fig. 9–10*b*, we shall include in our system not only the mass $M - \Delta M$ of the body but also the ejected mass ΔM, the total mass of the system being the M of Fig. 9–10*a*. Doing so permits us to apply the results that we have derived so far for constant mass systems. We shall see that this approach leads us to the form of Newton's second law for systems in which the mass is not constant.

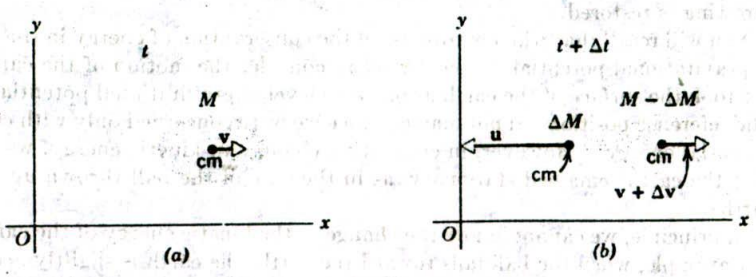


Fig. 9–10 A mass M moving with velocity $\mathbf{v}$ ejects a mass ΔM during a time interval Δt. An external force $\mathbf{F}_{ext}$ (not shown) acts on the system.

From Eq. 9–17

$$\mathbf{F}_{ext} = \frac{d\mathbf{P}}{dt} \tag{9–17}$$

we can write, as an approximate result for the finite time interval Δt,

$$\mathbf{F}_{ext} \cong \frac{\Delta \mathbf{P}}{\Delta t} = \frac{\mathbf{P}_f - \mathbf{P}_i}{\Delta t}$$

in which $\mathbf{P}_f$ is the (final) system momentum in Fig. 9–10*b* and $\mathbf{P}_i$ is the (initial) system momentum for Fig. 9–10*a*. But $\mathbf{P}_f = (M - \Delta M)(\mathbf{v} + \Delta \mathbf{v}) + \Delta M \mathbf{u}$ and $\mathbf{P}_i = M\mathbf{v}$. This leads to

$$\mathbf{F}_{ext} \cong \frac{[(M - \Delta M)(\mathbf{v} + \Delta \mathbf{v}) + \Delta M \mathbf{u}] - [M\mathbf{v}]}{\Delta t}$$

$$= M \frac{\Delta \mathbf{v}}{\Delta t} + [\mathbf{u} - (\mathbf{v} + \Delta \mathbf{v})] \frac{\Delta M}{\Delta t}. \tag{9–19}$$

Now, if we let Δt approach zero, the configuration of Fig. 9–10*b* approaches that of Fig. 9–10*a*; that is, $\Delta \mathbf{v}/\Delta t$ approaches $d\mathbf{v}/dt$, the acceleration of the body in Fig. 9–10*a*. The quantity ΔM is the mass ejected in Δt: this leads to a decrease in the mass M of the original body. Since dM/dt, the change in mass of the body with time, is intrinsically negative in this case, the positive quantity $\Delta M/\Delta t$ is replaced by $-dM/dt$ as Δt approaches zero. Finally, $\Delta \mathbf{v}$ goes to zero as Δt approaches zero. Making these changes in Eq. 9–19 leads to

$$\mathbf{F}_{ext} = M \frac{d\mathbf{v}}{dt} + \mathbf{v} \frac{dM}{dt} - \mathbf{u} \frac{dM}{dt} \tag{9–20a}$$

or

$$\mathbf{F}_{ext} = \frac{d}{dt}(M\mathbf{v}) - \mathbf{u} \frac{dM}{dt}, \tag{9–20b}$$

which is Newton's second law, defining the external forces on a body (like that of Fig. 9–10*a*) whose mass is changing.

Note that these equations reduce to the familiar forms $\mathbf{F}_{ext} = M\mathbf{a}$ and $\mathbf{F}_{ext} = (d/dt)(M\mathbf{v})$ respectively for the special case of a body of constant mass ($dM/dt = 0$). It is important to note that we *cannot* derive a general expression for Newton's second law for variable mass systems by treating the mass in $\mathbf{F} = d\mathbf{P}/dt = d(M\mathbf{v})/dt$ as a *variable*. For this leads to

$$\mathbf{F} = d(M\mathbf{v})/dt = M\, d\mathbf{v}/dt + \mathbf{v}\, dM/dt,$$

which is only a special case of the more general Eq. 9–20, namely, the case in which either (*a*) $dM/dt = 0$, a system of constant mass, or (*b*) $\mathbf{u} = 0$, a special choice of reference frame. We *can* use $\mathbf{F} = d\mathbf{P}/dt$ to analyze variable mass systems *only* if we apply it to an *entire system of constant mass* having parts among which there is an interchange of mass. This indeed is what we have done in deriving Eqs. 9–20. The importance of the momentum formulation $\mathbf{F} = d\mathbf{P}/dt$ in classical physics lies in the fact that it highlights momentum conservation and gives us a simple, physical way to treat complicated systems. Since the choice of what we will take as the system is ours to make, we can always choose a system of constant mass by defining our system broadly enough.

However, it is often convenient, as in rocket problems, to choose a system whose mass varies with time. In such cases we apply Newton's second law of Eqs. 9–20 in a form that is sometimes more convenient and interpretable more physically. The quantity $\mathbf{u} - (\mathbf{v} + \Delta \mathbf{v})$ in Eq. 9–19 is just $\mathbf{v}_{rel}$, the relative velocity of the

ejected mass with respect to the main body. Therefore Eqs. 9–20 may be written as

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_{ext} + (\mathbf{u} - \mathbf{v})\frac{dM}{dt} \tag{9–21a}$$

or

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_{ext} + \mathbf{v}_{rel}\frac{dM}{dt}. \tag{9–21b}$$

The last term in Eq. 9–21*b*, $[\mathbf{v}_{rel}(dM/dt)]$, is the rate at which momentum is being transferred into (or out of) the system by the mass that the system has ejected (or collected). It can be interpreted as the force exerted *on* the system by the mass that leaves it (or joins it). For a rocket, this term is called the *thrust* and it is the rocket designer's aim to make it as large as possible. Inspection of Eq. 9–21 shows that this requires that the rocket eject as much mass per unit time as possible and that the speed of the ejected mass relative to the rocket be as high as possible. We can rewrite Eq. 9–21 as

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_{ext} + \mathbf{F}_{reaction}$$

in which $\mathbf{F}_{reaction}$ $(= \mathbf{v}_{rel}\,dM/dt)$ is the reaction force exerted on the system by the mass that leaves it.

▶ **Example 8.** A machine gun is mounted on a car on a horizontal frictionless surface as in Fig. 9–11*a*. The mass of the system (car + gun) at a particular instant is M. At that same instant the gun is firing bullets of mass m whose velocity, in the reference frame shown, is $\mathbf{u}$. The velocity of the car in this same frame is $\mathbf{v}$ and the velocity of the bullets *with respect to the car* is $\mathbf{u} - \mathbf{v}$. The number of bullets fired per unit time is n. What is the acceleration of the car?

We select the car and gun as our system. Since its mass M is variable, we apply Newton's second law in the form given in Eq. 9–21. Since no net external force acts on the system, we have $\mathbf{F}_{ext} = 0$ in that equation, yielding

$$M\frac{d\mathbf{v}}{dt} = \mathbf{v}_{rel}\frac{dM}{dt}.$$

Now $d\mathbf{v}/dt$ is $\mathbf{a}$, the acceleration of the system; $\mathbf{v}_{rel}$ is $\mathbf{u} - \mathbf{v}$, pointing to the left in Fig. 9–11*a*, and dM/dt is $-mn$. Inserting these in the equation above yields

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\frac{\mathbf{v}_{rel}(mn)}{M}.$$

This shows that $\mathbf{a}$ points in the direction opposite to $\mathbf{v}_{rel}$, that is, $\mathbf{a}$ points to the right in Fig. 9–11*a*. If $v_{rel} = 500$ meters/sec, $m = 10$ gm, $n = 10$/sec, and $M = 200$ kg at some instant, then at that instant

$$a = \frac{(500\text{ meters/sec})(10^{-2}\text{ kg})(10/\text{sec})}{200\text{ kg}} = 0.25\text{ meter/sec}^2.$$

The magnitude of the average "thrust" of the ejected bullets on the system (car + gun) at this instant is given by

$$F = v_{rel}nm = (500\text{ meters/sec})(10/\text{sec})(10^{-2}\text{ kg})$$
$$= 50\text{ nt}.$$

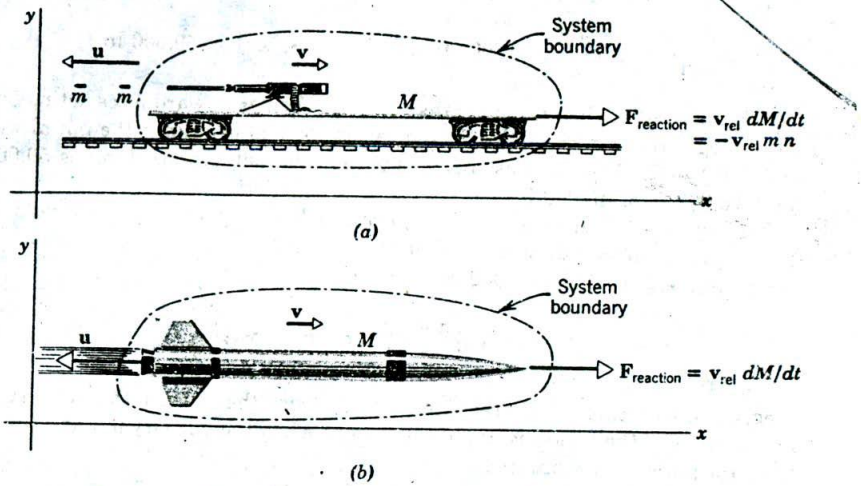


Fig. 9–11 (*a*) Example 8. A machine gun is fixed to a car that rolls with negligible friction. The gun fires bullets of mass m at a rate (number per unit time) n, the velocity of the bullets with respect to the gun being $\mathbf{u} - \mathbf{v}$. At the instant shown some bullets have already left the system. The velocities indicated for the car and the bullets are those that would be measured by an observer in a reference frame fixed to the rails as shown. The reaction force on the system is $\mathbf{F} = -mn\mathbf{v}_{rel} = (dM/dt)\mathbf{v}_{rel}$. (*b*) A rocket moves through space with negligible external forces. Gas particles are ejected from the exhaust, the particles having a velocity $\mathbf{u} - \mathbf{v}$ with respect to the rocket. The rate at which mass is expelled at the exhaust is $-dM/dt$. The reaction force on the rocket is $\mathbf{F} = (dM/dt)\mathbf{v}_{rel}$. The velocities indicated for the rocket and exhaust gases are relative to the ground.

In Figure 9–11*b* we show the analagous situation for a rocket. It is instructive to view this problem from the point of view of Newton's third law and the momentum principle. Choose a fixed-mass system (rocket + gas) and attach a reference frame to its center of mass. The rocket forces a jet of hot gases from its exhaust; this is the action force. The jet of hot gases exerts a force on the rocket, propelling it forward. This is the reaction force. These forces are internal forces in the system (rocket + gas). In the absence of external forces the total momentum of the system is constant (the center of mass, initially at rest, remains at rest). The individual parts of the system (rocket and gases) may change their momentum, however; with respect to the center of mass frame, the hot gases acquire momentum in the backward direction and the rocket acquires an equal amount of momentum in the forward direction.

The student can analyze the system (bullets + car) in a similar way.

Example 9. A rocket weighs 30,000 lb when fueled up on the launching pad. It is fired vertically upward and, at burnout, weighs 10,000 lb. Gases are exhausted at the rate of 10 slugs/sec with a velocity of 5000 ft/sec, relative to the rocket (exhaust velocity), both quantities being assumed to be constant while the fuel is burning.

(*a*) What is the thrust? The thrust **F** is the last term in Eq. 9–21*b*, or

$$F = v_{\text{rel}} \frac{dM}{dt} = (5000 \text{ ft/sec})(10 \text{ slugs/sec}) = 50{,}000 \text{ lb}.$$

Note that initially, when the fuel tanks are full, the net upward force acting on the rocket (neglecting air resistance) is the thrust (50,000 lb) minus the initial weight (30,000 lb) or 20,000 lb. Just before burnout the net upward force is 50,000 lb minus 10,000 lb or 40,000 lb.

(*b*) If we could neglect *all external forces*, including gravity and air resistance, what would be the speed of the rocket at burnout?

If we put $\mathbf{F}_{\text{ext}} = 0$ in Eq. 9–21*b*, we have

$$M \frac{d\mathbf{v}}{dt} = \mathbf{v}_{\text{rel}} \frac{dM}{dt} \quad \text{or} \quad d\mathbf{v} = \mathbf{v}_{\text{rel}} \frac{dM}{M}.$$

Integrating this expression (see Appendix I) from the instant the velocity is $\mathbf{v}_0$ and the mass of the rocket is M_0 to the instant when the velocity is $\mathbf{v}$ and the mass of the rocket is M, we obtain

$$\int_{\mathbf{v}_0}^{\mathbf{v}} d\mathbf{v} = \mathbf{v}_{\text{rel}} \int_{M_0}^{M} \frac{dM}{M},$$

the exhaust velocity being assumed constant during this time. This yields

$$\mathbf{v} - \mathbf{v}_0 = -\mathbf{v}_{\text{rel}} \ln (M_0/M) = -\mathbf{v}_{\text{rel}} \ln \left(1 + \frac{M_0 - M}{M}\right).$$

Hence the change in speed of the rocket in any interval of time depends only on the exhaust velocity (being opposite in direction from it) and on the fraction of mass exhausted during that time interval.

In our example, $v_0 = 0$ and $M_0/M = (30{,}000/10{,}000) = 3.0$, so that the speed of the rocket at burnout is

$$v = v_{\text{rel}} \ln (M_0/M) = (5000 \text{ ft/sec}) \ln 3.0 = 3800 \text{ miles/hr}.$$

If the external forces of gravity and air resistance were taken into account, the final speed would be smaller.*

Assuming that the rocket starts from rest ($v_0 = 0$) with an initial mass M_0 and reaches a final velocity v_f at burnout when its mass is M_f, we can write the rocket equation above as

$$\frac{M_f}{M_0} = e^{-v_f/v_{\text{rel}}}$$

in which v_{rel} is the exhaust velocity.

The classical rocket (or variable mass) equations imply that the speed of the rocket can increase to any value provided only that the rocket expels enough propellant so that the final remaining mass is sufficiently small. However, we know from relativistic mechanics that a rocket cannot be accelerated to a speed equal to or greater than the speed of light. Once the rocket's speed approaches the relativistic range the classical equations are no longer applicable. One must take

* For an exact solution of the classical rocket problem see "Variable-Mass Dynamics" by J. L. Meriam, *Journal of Engineering Education*, December 1960.

into account the variation of inertial mass of a particle with speed and the relativistic velocity addition formula. The resulting equations apply to a relativistic rocket.*

Example 10. Sand drops from a stationary hopper at a rate dM/dt onto a conveyor belt moving with velocity $\mathbf{v}$ in the reference frame of the laboratory, as in Fig. 9–12. What force is required to keep the belt moving at a speed v?

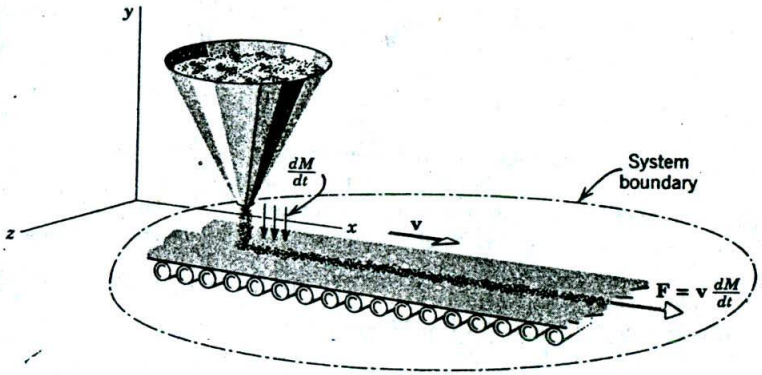


Fig. 9–12 Example 10. Sand drops from a hopper at a rate dM/dt onto a conveyer belt moving with velocity $\mathbf{v}$ in the reference frame of the laboratory. The force $\mathbf{F}$ required to keep the belt moving at constant velocity is $\mathbf{v}\,dM/dt$. The hopper is at rest in the reference frame shown.

This is a clear-cut example of a force associated with change of mass alone, the velocity being constant. We take as our system the belt of varying mass so that Eq. 9–21 applies. We must put $dv/dt = 0$ in that equation because the velocity of the belt is constant. Furthermore, to an observer at rest on the belt, the falling sand (and the hopper) would appear to have a horizontal motion with speed v in a direction opposite to that shown for the belt in the laboratory. Therefore $\mathbf{v}_{rel} = -\mathbf{v}$ in Eqs. 9–21. More formally, $\mathbf{v}_{rel} = \mathbf{u} - \mathbf{v}$; but $\mathbf{u} = 0$, so that $\mathbf{v}_{rel} = -\mathbf{v}$. Making these substitutions yields

$$0 = \mathbf{F}_{ext} - \mathbf{v}\frac{dM}{dt}$$

or

$$\mathbf{F}_{ext} = \mathbf{v}\frac{dM}{dt}.$$

In this example, dM/dt is positive because the system is gaining mass with time. Hence, as expected, the necessary external force must point in the direction in which the belt moves. Note that, in the absence of friction, the mass of the belt itself does not enter the problem.

* See "The Equation of Motion for Relativistic Particles and Systems with a Variable Rest Mass," by Kalman B. Pomeranz, in *American Journal of Physics*, December 1964.

The power supplied by the external force is

$$P = \mathbf{F} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{F} = \mathbf{v} \cdot (\mathbf{v}\, dM/dt) = v^2(dM/dt).$$

Since v = a constant, we can write this as

$$P = \frac{d(Mv^2)}{dt} = 2\frac{d}{dt}\left(\frac{1}{2}Mv^2\right) = 2\frac{dK}{dt}.$$

This tells us that the power required to keep the belt moving is *twice* the rate at which the kinetic energy of the system is increasing; note that we need not consider the kinetic energy of the belt itself because—its speed being constant—its kinetic energy does not change. It is clear that mechanical energy is not conserved in this case. Where is the other half of the power going? In which of the previous examples did we have conservation of momentum without conservation of mechanical energy?

The student should be able to solve Example 10 alternatively by choosing a fixed-mass system and applying the momentum principle. ◀

QUESTIONS

1. Must there necessarily be any mass at the center of mass of a system?

2. Does the center of mass of a solid body necessarily lie within the body? If not, give examples.

3. How is the center of mass concept related to the concept of geographic center of the country? To the population center of the country? What can you conclude from the fact that the geographic center differs from the population center?

4. A sculptor decides to portray a bird (Fig. 9–13). Luckily the final model is actually able to stand upright. The model is formed of a single sheet of metal of uniform thickness. Of the points shown, which is most likely to be the center of mass?

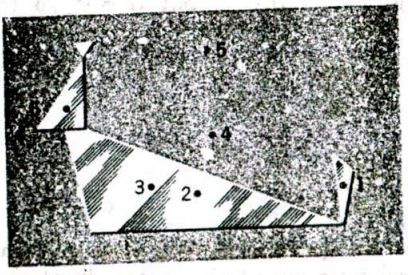


Fig. 9–13

5. If only an external force can change the state of motion of the center of mass of a body, how does it happen that the internal force of the brakes can bring a car to rest?

6. Can a body have energy without having momentum? Explain. Can a body have momentum without having energy? Explain.

7. A light and a heavy body have equal kinetic energies of translation. Which one has the larger momentum?

8. A bird is in a wire cage hanging from a spring balance. Is the reading of the balance when the bird is flying about greater than, less than, or the same as that when the bird sits in the cage?

9. Can a sailboat be propelled by air blown at the sails from a fan attached to the boat?

10. A man stands still on a large sheet of slick ice; in his hand he holds a lighted firecracker. He throws the firecracker into the air. Describe briefly, but as exactly as you can, the motion of the center of mass of the firecracker and the motion of the center of mass of the system consisting of man and firecracker. It will be most convenient to describe each motion during each of the following periods: (*a*) after he throws the firecracker, but before it explodes; (*b*) between the explosion and the first piece of firecracker hitting the ice; (*c*) between the first fragment hitting the ice and the last fragment landing; (*d*) during the time when all fragments have landed but none has reached the edge of the ice.

11. As stated in the text one cannot use the equation $\mathbf{F} = d(M\mathbf{v})/dt$ for a system of variable mass. To show this (*a*) put the equation in the equivalent form $(\mathbf{F} - M\, d\mathbf{v}/dt)/(dM/dt) = \mathbf{v}$ and (*b*) show that one side of this equation has the same value in all inertial frames, whereas the other side does not. Hence the equation cannot be generally valid. (*c*) Show that Eq. 9-20 leads to no such contradiction.

12. The final velocity of the final stage of a multistage rocket is much greater than the final velocity of a single-stage rocket of the same total weight and fuel supply. Explain this fact.

13. As a rocket expels burned fuel the location of the center of mass of the rocket (in a frame attached to the rocket) changes. Must one take this into account in an exact solution of the rocket problem?

14. Explain clearly the distinction between the origin of the varying mass of a classical system and that of a relativistic system.

15. Can you think of variable mass systems other than the examples given in the text?

PROBLEMS

1. Show that the ratio of the distances of two particles from their center of mass is the inverse ratio of their masses.

2. Show that the center of mass of two particles is on the line joining them at a point whose distance from each particle is inversely proportional to the mass of that particle.

3. Experiments using the diffraction of electrons show that the distance between the centers of the carbon (C) and oxygen (O) atoms in the carbon monoxide gas molecule is 1.130×10^{-10} meter. Locate the center of mass of a CO molecule relative to the carbon atom.

4. The mass of the moon is about 0.013 times the mass of the earth, and the distance from the center of the moon to the center of the earth is about 60 times the radius of the earth. How far is the center of mass of the earth-moon system from the center of the earth? Take the earth's radius to be 4000 miles.

5. In the ammonia (NH_3) molecule, the three hydrogen (H) atoms form an equilateral triangle, the distance between centers of the atoms being 1.628×10^{-10} meter,

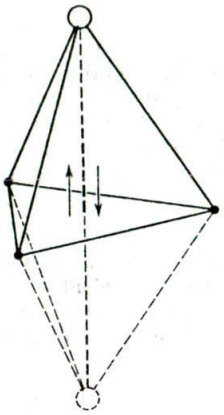


Fig. 9–14

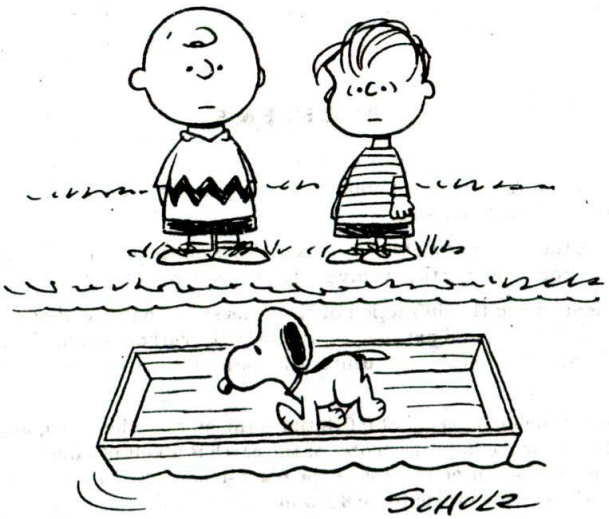
so that the center of the triangle is 9.39×10^{-11} meter from each hydrogen atom. The nitrogen (N) atom is at the apex of a pyramid, the three hydrogens constituting the base. (See Fig. 9–14.) The hydrogen-nitrogen distance is 1.014×10^{-10} meter. Locate the center of mass relative to the nitrogen atom. (The nitrogen atom actually oscillates up and down through the base as indicated by the arrows and dotted lines. Assume a static molecule for this problem, however.)

6. Find the center of mass of a homogeneous semicircular plate. Let a be the radius of the circle.

7. Two particles P and Q are initially at rest 1.0 meter apart. P has a mass of 0.10 kg and Q a mass of 0.30 kg. P and Q attract each other with a constant force of 1.0×10^{-2} nt. No external forces act on the system. Describe the motion of the center of mass. At what distance from P's original position do the particles collide?

8. Two bodies, each made up of weights from a set, are connected by a light cord which passes over a light, frictionless pulley with a diameter of 5.0 cm. The two bodies are at the same level. Each originally has a mass of 500 gm. (*a*) Locate their center of mass. (*b*) Twenty grams are transferred from one body to the other, but the bodies are prevented from moving. Locate the center of mass. (*c*) The two bodies are now released. Describe the motion of the center of mass and determine its acceleration.

9. What is the momentum of a 4000-lb car whose speed is 30 mph? At what speed would a 10-ton truck have the same momentum? The same kinetic energy?



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Fig. 9–15

10. A 200-lb man standing on a surface of negligible friction kicks forward a 0.10-lb stone lying at his feet so that it acquires a speed of 10 ft/sec. What velocity does the man acquire as a result?

11. A dog, weighing 10.0 lb is standing on a flatboat so that he is 20 ft from the shore. He walks 8.0 ft on the boat toward shore and then halts. The boat weighs 40 lb, and one can assume there is no friction between it and the water. How far is he from the shore at the end of this time? (*Hint:* The center of mass of boat + dog does not move. Why?) The shoreline is also to the left in Fig. 9–15.

12. A cannon and a supply of cannon balls are inside a sealed railroad car as in Fig. 9–16. The cannon fires to the right, the car recoiling to the left. The cannon balls remain in the car after hitting the far wall. Show that no matter how the cannon balls are fired the railroad car cannot travel more than its length L, assuming it starts from rest.

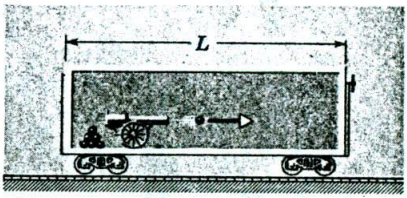


Fig. 9–16

13. A vessel at rest explodes, breaking into three pieces. Two pieces, having equal mass, fly off perpendicular to one another with the same speed of 30 meters/sec. The third piece has three times the mass of each other piece. What is the direction and magnitude of its velocity immediately after the explosion?

14. A radioactive nucleus, initially at rest, decays by emitting an electron and a neutrino at right angles to one another. The momentum of the electron is 1.2×10^{-22} kg-m/sec and that of the neutrino is 6.4×10^{-23} kg-m/sec. (*a*) Find the direction and magnitude of the momentum of the recoiling nucleus. (*b*) The mass of the residual nucleus is 5.8×10^{-26} kg. What is its kinetic energy of recoil?

15. A projectile is fired from a gun at an angle of 45° with the horizontal and with a muzzle speed of 1500 ft/sec. At the highest point in its flight the projectile explodes into two fragments of equal mass. One fragment, whose initial speed is zero, falls vertically. How far from the gun does the other fragment land, assuming a level terrain?

16. A body of mass 8.0 kg is traveling at 2.0 meters/sec under the influence of *no* external agency. At a certain instant an internal explosion occurs, splitting the body into two chunks of 4.0 kg mass each; 16 joules of translational kinetic energy are imparted to the two-chunk system by the explosion. Neither chunk leaves the line of the original motion. Determine the speed and direction of motion of each of the chunks after the explosion.

17. The last stage of a rocket is traveling at a speed of 17,000 miles/hr or 25,000 ft/sec. This last stage is made up of two parts which are clamped together, namely, a rocket case with a mass of 20 slugs and a passenger capsule with a mass of 10 slugs, including a chimpanzee. When the clamp is released, a compressed spring causes the two parts to separate with a relative speed of 3000 ft/sec. (*a*) What are the speeds of the two parts after they have separated? Assume that all velocities are along the same line. (*b*) Find the total kinetic energy of the two parts before and after they separate and account for the difference, if any.

18. A block of mass m rests on a wedge of mass M which, in turn, rests on a horizontal table, as shown in Fig. 9–17. All surfaces are frictionless. If the system starts at rest with point P of the block a distance h above the table, find the velocity of the wedge the instant point P touches the table.

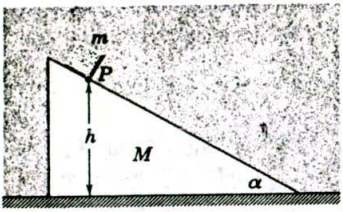


Fig. 9–17

19. A railroad flat car of weight W can roll without friction along a straight horizontal track as shown. Initially a man of weight w is standing on the car which is moving to the right with speed v_0. What is the change in velocity of the car if the man runs to the left (Fig. 9–18) so that his speed relative to the car is v_{rel} just before he jumps off at the left end?

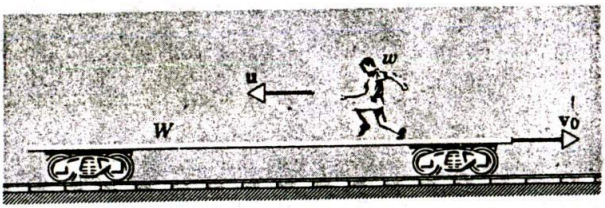


Fig. 9–18

20. Assume that the car in Problem 19 is initially at rest. It holds n men each of weight w. If each man in succession runs with a relative velocity v_{rel} and jumps off the end, do they impart to the car a greater velocity than if they all run and jump at the same time?

21. A machine gun fires 50-gm bullets at a speed of 1000 meters/sec. The gunner, holding the machine gun in his hands, can exert an average force of 180 nt against the gun. Determine the maximum number of bullets he can fire per minute.

22. A 6000-kg rocket is set for a vertical firing. If the exhaust speed is 1000 meters/sec, how much gas must be ejected per second to supply the thrust needed (*a*) to overcome the weight of the rocket, (*b*) to give the rocket an initial upward acceleration of 19.6 meters/sec^2?

23. Show that the rocket speed is equal to the exhaust speed when the ratio M_0/M is e (about 2.7). Specify the coordinate system within which this result holds. Show also that the rocket speed is twice the exhaust speed when M_0/M is e^2 (about 7.4).

24. A widely used rocket fuel is kerosene and liquid oxygen, capable of giving an exhaust velocity v_{rel} of 8000 ft/sec (about 1.5 miles/sec). (*a*) Neglect gravity and the weight of fuel tanks, pumps, etc., and find how many pounds of this fuel one needs for each pound of payload in order to get a rocket, starting from rest, to reach a velocity of 7.5 miles/sec (the velocity of escape from the earth is 7.0 miles/sec). (*b*) In the Mariner probe to Mars the initial weight was about 200,000 lb and the payload about 500 lb, a "fuel" to payload ratio of 400 to 1. Starting a rocket from rest, what final velocity is achievable under these circumstances? (*c*) The actual final rocket velocity was about 15 miles/sec, much greater than the value found in (*b*). Explain this, considering the following factors: the external forces and weight neglected in (*a*) must be taken into account; the rocket uses a number of stages; the initial rocket ... that of the earth's surface, in a reference frame attached to the sun.

25. A jet airplane is traveling 600 ft/sec. The engine takes in 2400 ft^3 of air having a mass of 4.8 slugs each second. The air is used to burn 0.20 slug of fuel each second. The energy is used to compress the products of combustion and to eject them at the rear of the plane at 1600 ft/sec relative to the plane. Find the thrust of the jet engine and the delivered horsepower.

26. A freight car, open at the top, weighing 10 tons, is coasting along a level track with negligible friction at 2.0 ft/sec when it begins to rain hard. The raindrops fall vertically with respect to the ground. What is the speed of the car when it has collected 0.50 ton of rain? What assumptions, if any, must you make to get your answer?

27. A freight car filled with sand has a hole so that sand leaks out through the bottom at a constant rate, $-dm/dt = \lambda$. A force **F** acts on the car in the direction of its motion. Call the instantaneous speed v and write the equation of motion of the freight car.

28. A flexible rope of length l and mass per unit length μ slides over the edge of a frictionless table. At time $t = 0$ let a length y_0 of it be hanging at rest over the edge and at time t let a length y, moving with speed dy/dt, be over the edge. (*a*) Show that in terms of a variable mass problem $v_{rel} = 0$, so that the equation of motion has the form $m\,dv/dt = \mathbf{F}_{ext}$. (*b*) Show that the specific equation of motion is $l(d^2y/dt^2) = gy$. (*c*) Show that conservation of mechanical energy leads to $l(dy/dt)^2 - gy^2 =$ a constant, and that this is consistent with (*b*). (*d*) Show that $y = (y_0/2)(e^{\sqrt{g/l}\,t} + e^{-\sqrt{g/l}\,t})$ is a solution to the equation of motion [by substitution into (*b*)] and discuss the solution.

29. Consider a particle acted on by a force having the same direction as its velocity. (*a*) Using the relativistic relation $F = d(mv)/dt$ for a single particle, show that

$$F\,ds = mv\,dv + v^2\,dm.$$

(*b*) Using the relativistic relation $v^2 = (1 - m_0^2/m^2)c^2$, show that

$$mv\,dv = \frac{m_0^2c^2}{m^2}\,dm.$$

(*c*) Substitute the relations for $mv\,dv$ and v^2 into result (*a*) and show that

$$W = \int F\,ds = (m - m_0)c^2.$$

Collisions

CHAPTER 10

10–1 What is a Collision?

We learn much about atomic, nuclear, and elementary particles experimentally by observing collisions between them. On a larger scale we can better interpret such things as the properties of gases in terms of particle collisions. In this chapter we apply the principles of conservation of energy and conservation of momentum to the collisions of particles.

In a collision a relatively large force acts on each colliding particle for a relatively short time. The basic idea of a "collision" is that the motion of the colliding particles (or of at least one of them) changes rather abruptly and that we can make a relatively clean separation of times that are "before the collision" and those that are "after the collision."

When a bat strikes a baseball for example, the beginning and the end of the collision can be determined fairly precisely. The bat is in contact with the ball for an interval that is quite short in comparison to the time during which we are watching the ball. During the collision the bat exerts a large force on the ball (Fig. 10–1). This force varies with time in a complex way that we can measure only with difficulty. Both the ball and the bat are deformed during the collision. Forces that act for a time that is short compared to the time of observation of the system are called *impulsive* forces.

When an alpha particle (He^4) "collides" with a nucleus of gold (Au^{197}), the force acting between them may be the well-known repulsive electrostatic force associated with the charges on the particles. The particles may not "touch," but we still may speak of a "collision" because a rela-

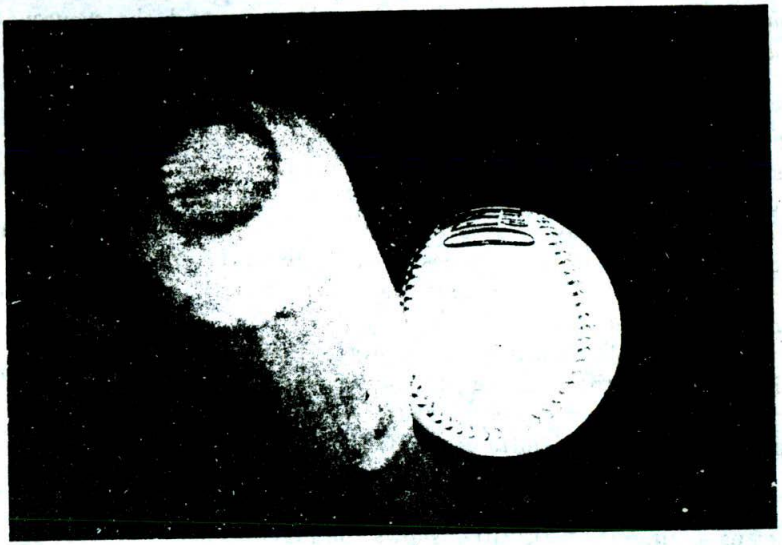


Fig. 10–1 A high-speed flash photograph of a bat striking a baseball. Notice the deformation of the ball, indicating the enormous magnitude of the impulsive force at this instant. (Courtesy Harold E. Edgerton, Massachusetts Institute of Technology, Cambridge, Mass.)

tively strong force, acting for a time that is short in comparison to the time that the alpha particle is under observation, has a marked effect on the motion of the alpha particle.

When a proton (H^1 or p) with energy of, say, 25 Mev, "collides" with a nucleus of, say, a silver isotope (perhaps Ag^{107}), the particles may actually "touch," the predominant force then acting between them being, not the electrostatic repulsive force, but the strong, short-range, attractive nuclear force (see page 120). The proton may enter the silver nucleus, forming a compound structure. A short time later—the "collision interval" may be 10^{-18} sec—this compound structure may break up into two *different* particles, according to a scheme such as

$$p + Ag^{107} \rightarrow \alpha + Pd^{104},$$

in which α (= He^4) is an alpha particle. Thus we may broaden the concept of collision to include events (usually called *reactions*) in which the identities of the interacting particles change during the event. The conservation principles are applicable to all these examples

We may, if we wish, broaden our definition of "collision" even further to include the spontaneous decay of a single particle into two or more other

particles. An example is the decay of the elementary particle called the *sigma particle* into two other particles, the *pion* and the *neutron* (see Appendix E) or

$$\Sigma^- \rightarrow \pi^- + n.$$

Although two bodies do not come in contact in this process (unless we consider it in reverse), it has many features in common with collisions: (1) there is a clean distinction between "before the event" and "after the event," and (2) the laws of conservation of momentum and energy permit us to learn much about such processes by studying the "before" and "after" situations, even though we may know little about the force laws that operate during the "event" itself.

In studying collisions in this chapter our aim will we this: given the initial motions of the colliding particles, what can we learn about their final motions from the principles of conservation of momentum and energy, assuming that we know nothing about the forces acting during the collision?

10–2 Impulse and Momentum

Let us assume that Fig. 10–2 shows the magnitude of the force exerted on a body during a collision. We assume that the force has a constant direction. The collision begins at time t_i and ends at time t_f, the force being zero before and after collision. From Eq. 9–12 we can write the change in momentum $d\mathbf{p}$ of a body in a time dt during which a force $\mathbf{F}$ acts on it as

$$d\mathbf{p} = \mathbf{F}\,dt. \qquad (10\text{–}1)$$

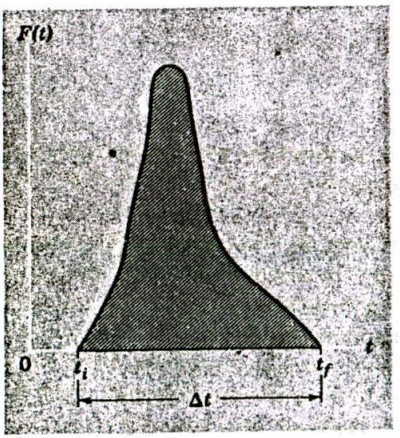


Fig. 10–2 How an impulsive force $F(t)$ might vary with time during a collision starting at time t_i and ending at t_f.

We can find the change in momentum of the body during a collision by integrating over the time of collision, that is,

$$\mathbf{p}_f - \mathbf{p}_i = \int_{\mathbf{p}_i}^{\mathbf{p}_f} d\mathbf{p} = \int_{t_i}^{t_f} \mathbf{F}\,dt \qquad (10\text{–}2)$$

in which the subscripts i (= *initial*) and f (= *final*) refer to the times before and after the collision, respectively. The integral of a force over the time interval during which the force acts is called the *impulse* $\mathbf{J}$ of the force. Hence the change in momentum of a body acted on by an impulsive force is equal to the impulse. Both impulse and momentum are vectors and both have the same units and dimensions.

The impulsive force represented in Fig. 10–2 is assumed to have a con-

stant direction. The impulse of this force, $\int_{t_i}^{t_f} \mathbf{F}\, dt$, is represented in magnitude by the area under the force-time curve.*

10–3 Conservation of Momentum during Collisions

Consider now a collision between two particles, such as those of masses m_1 and m_2, shown in Fig. 10–3. During the brief collision these particles exert large forces on one another. At any instant $\mathbf{F}_1$ is the force exerted on particle 1 by particle 2 and $\mathbf{F}_2$ is the force exerted on particle 2 by particle 1. By Newton's third law these forces at any instant are equal in magnitude but oppositely directed.

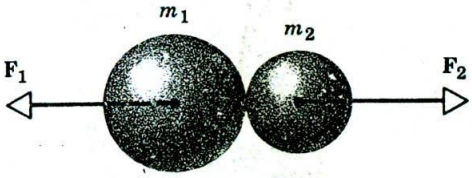


Fig. 10–3 Two "particles" m_1 and m_2, in collision, experience equal and opposite forces along their line of centers, according to Newton's third law; $\mathbf{F}_2(t) \equiv -\mathbf{F}_1(t)$.

The change in momentum of particle 1 resulting from the collision is

$$\Delta \mathbf{p}_1 = \int_{t_i}^{t_f} \mathbf{F}_1\, dt = \overline{\mathbf{F}_1}\, \Delta t$$

in which $\overline{\mathbf{F}_1}$ is the average value of the force $\mathbf{F}_1$ during the time interval of the collision $\Delta t = t_f - t_i$.

The change in momentum of particle 2 resulting from the collision is

$$\Delta \mathbf{p}_2 = \int_{t_i}^{t_f} \mathbf{F}_2\, dt = \overline{\mathbf{F}_2}\, \Delta t$$

in which $\overline{\mathbf{F}_2}$ is the average value of the force $\mathbf{F}_2$ during the time interval of the collision $\Delta t = t_f - t_i$.

If no other forces act on the particles, then $\Delta \mathbf{p}_1$ and $\Delta \mathbf{p}_2$ give the total change in momentum for each particle. But we have seen that at each instant $\mathbf{F}_1 = -\mathbf{F}_2$, so that $\overline{\mathbf{F}_1} = -\overline{\mathbf{F}_2}$, and therefore

$$\Delta \mathbf{p}_1 = -\Delta \mathbf{p}_2.$$

If we consider the two particles as an isolated system, the total momentum of the system is

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2,$$

and the total *change* in momentum of the system as a result of the collision is zero, that is,

$$\Delta \mathbf{P} = \Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 = 0.$$

* The impulse $\mathbf{J}$, defined from Eq. 10–2, does not depend critically on the precise values of t_i and t_f as long as these times are far enough apart to include the crosshatched area of Fig. 10–2. For reasons that will appear later we usually choose t_i and t_f with a separation that is *just large enough* to make a clean distinction between the "collision" and the "before and after intervals."

Hence, *if there are no external forces* the total momentum of the system is not changed by the collision. The impulsive forces acting during the collision are internal forces which have no effect on the total momentum of the system.

We have defined a collision as an interaction which occurs in a time Δt that is negligible compared to the time during which we are observing the system. We can also characterize a collision as an event in which the external forces that may act on the system are negligible compared to the impulsive collision forces. When a bat strikes a baseball, a golf club strikes a golf ball, or one billiard ball strikes another, external forces act on the system. Gravity or friction exerts forces on these bodies, for example; these external forces may not be the same on each colliding body nor are they necessarily cancelled by other external forces. Even so it is quite safe to neglect these external forces during the collision and to assume momentum conservation provided, as is almost always true, that the external forces are negligible compared to the impulsive forces of collision. As a result the change in momentum of a particle during a collision arising from an external force is negligible compared to the change in momentum of that particle arising from the impulsive collisional force (Fig. 10–4).

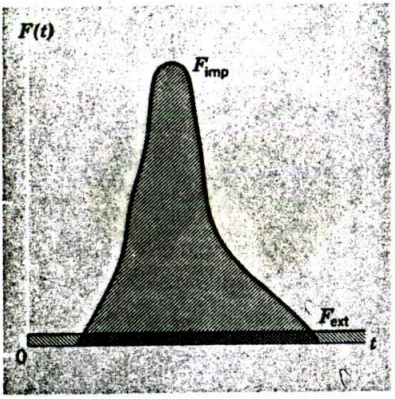


Fig. 10–4 During a collision, the impulsive force F_{imp} is generally much greater than any external forces F_{ext}, which may act on the system.

For example, when a bat strikes a baseball, the collision lasts only a small fraction of a second. Since the change in momentum is large and the time of collision is small, it follows from

$$\Delta \mathbf{p} = \bar{\mathbf{F}}\,\Delta t$$

that the average impulsive force $\bar{\mathbf{F}}$ is relatively large. Compared to this force, the external force of gravity is negligible. *During the collision* we can safely ignore this external force in determining the change in motion of the ball; the shorter the duration of the collision the more likely this is to be true.

In practice, therefore, *we can apply the principle of momentum conservation during collisions if the time of collision is small enough.* We can then say that the momentum of a system of particles just before the particles collide is equal to the momentum of the system just after the particles collide.

10–4 Collisions in One Dimension

We can always calculate the motions of bodies after collision from their motions before collision if we know the forces that act during the collision, and if we can solve the equations of motion. Often we do not know these forces. However, the principle of conservation of momentum must hold during the collision. We already know that the principle of conservation of total energy holds. Although we may not know the details of the interaction, we can use these principles in many cases to predict the results of the collision.

Collisions are usually classified according to whether or not *kinetic energy* is conserved in the collision. When kinetic energy is conserved, the collision is said to be *elastic*. Otherwise, the collision is said to be *inelastic*. Collisions between atomic, nuclear, and fundamental particles are sometimes elastic. These are, in fact, the only truly elastic collisions known. Collisions between gross bodies are always inelastic to some extent. We can often treat such collisions as approximately elastic, however, as, for example, collisions between ivory or glass balls. When two bodies stick together after collision, the collision is said to be *completely inelastic*. For example, the collision between a bullet and its target is completely inelastic when the bullet remains embedded in the target. The term completely inelastic does *not* mean that all the initial kinetic energy is lost; as we shall see, it means rather that the loss is as great as is consistent with momentum conservation.

Even if the forces of collision are not known, we can find the motions of the particles after collision from the motions before collision, provided the collision is completely inelastic, or, if the collision is elastic, provided the collision is a one-dimensional one. For a one-dimensional collision the relative motion after collision is along the same line as the relative motion before collision. We restrict ourselves to one-dimensional motion for the present.

Consider first an *elastic* one-dimensional collision. We can imagine two smooth nonrotating spheres moving initially along the line joining their centers, then colliding head-on and moving along the same straight line without rotation after collision (see Fig. 10–5). These bodies exert forces on each other during the collision that are along the initial line of motion, so that the final motion is also along this same line.

The masses of the spheres are m_1 and m_2, the (scalar) velocity components being v_{1i} and v_{2i} before collision and v_{1f} and v_{2f} after collision.* We take the positive direction of the momentum and velocity to be to the right. We assume, unless we specify otherwise, that the speeds of the colliding particles are low enough so that we need not use the relativistic

* The notation used is easy to interpret and to remember and reveals much information in a simple compact way. The number subscripts, such as 1 and 2, specify the particle and the letter subscripts, i and f, indicate initial value (before the collision) and final value (after the collision), respectively.

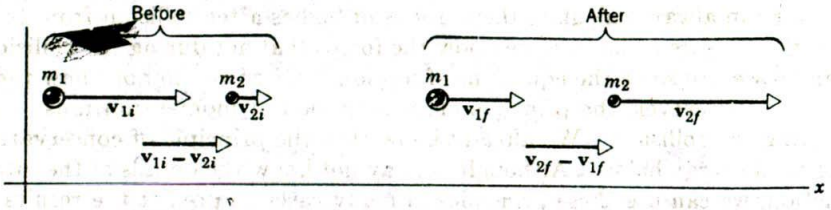


Fig. 10–5 Two spheres before and after an elastic collision. The velocity, $\mathbf{v}_{1i} - \mathbf{v}_{2i}$, of m_1 relative to m_2 before collision is equal to the velocity, $\mathbf{v}_{2f} - \mathbf{v}_{1f}$, of m_2 relative to m_1 after collision.

expressions for momentum and kinetic energy. Then from conservation of momentum we obtain

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}.$$

Because the collision is elastic, kinetic energy is conserved and we obtain

$$\tfrac{1}{2} m_1 v_{1i}^2 + \tfrac{1}{2} m_2 v_{2i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2.$$

It is clear at once that, if we know the masses and the initial velocities, we can calculate the two final velocities v_{1f} and v_{2f} from these two equations.

The momentum equation can be written as

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}), \tag{10–3}$$

and the energy equation can be written as

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2). \tag{10–4}$$

Dividing Eq. 10–4 by Eq. 10–3, and assuming $v_{2f} \neq v_{2i}$ and $v_{1f} \neq v_{1i}$ (see Question 5), we obtain

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

and, after rearrangement,

$$v_{1i} - v_{2i} = v_{2f} - v_{1f} \tag{10–5}$$

This tells us that in an elastic one-dimensional collision, the relative velocity of approach before collision is equal to the relative velocity of separation after collision.

To find the velocity components v_{1f} and v_{2f} after collision from the velocity components v_{1i} and v_{2i} before collision, we can use any two of the three previous numbered equations. Thus from Eq. 10–5

$$v_{2f} = v_{1i} + v_{1f} - v_{2i}.$$

Inserting this into Eq. 10–3 and solving for v_{1f}, we find that

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i}.$$

Likewise, inserting $v_{1f} = v_{2f} + v_{2i} - v_{1i}$ (from Eq. 10–5) into Eq. 10–3 and solving for v_{2f}, we obtain

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i}.$$

There are several cases of special interest. For example, when the colliding particles have the same mass, m_1 equals m_2 so that the two previous equations become simply

$$v_{1f} = v_{2i} \qquad \text{and} \qquad v_{2f} = v_{1i}.$$

That is, in a one-dimensional elastic collision of two particles of equal mass, the particles simply exchange velocities during collision.

Another case of interest is that in which one particle m_2 is initially at rest. Then v_{2i} equals zero and

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i}, \qquad v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i}.$$

Of course, if $m_1 = m_2$ also, then $v_{1f} = 0$ and $v_{2f} = v_{1i}$ as we expect. The first particle is "stopped cold" and the second one "takes off" with the velocity the first one originally had. If, however, m_2 is very much greater than m_1, we obtain

$$v_{1f} \cong -v_{1i} \qquad \text{and} \qquad v_{2f} \cong 0.$$

That is, when a light particle collides with a very much more massive particle at rest, the velocity of the light particle is approximately reversed and the massive particle remains approximately at rest. For example, suppose that we drop a ball vertically onto a horizontal surface attached to the earth. This is in effect a collision between the ball and the earth. If the collision is elastic, the ball will rebound with a reversed velocity and will reach the same height from which it fell.

If, finally, m_2 is very much smaller than m_1, we obtain

$$v_{1f} \cong v_{1i} \qquad v_{2f} \cong 2v_{1i}.$$

This means that the velocity of the massive incident particle is virtually unchanged by the collision with the light stationary particle, but that the light particle rebounds with approximately twice the velocity of the incident particle. The motion of a bowling ball is hardly affected by collision with an inflated beach ball of the same size, but the beach ball bounces away quickly.

Neutrons produced in a reactor from the fission of uranium atoms move very fast and must be slowed down if they are to produce more fissions. Assuming that they make elastic collisions with the nuclei at rest, what material should be picked to moderate the neutrons in the reactor? We can answer this from the considerations just discussed. If the stationary targets were massive nuclei, like lead, the neutrons would simply bounce back with practically the same speed they had initially. If the stationary

targets were very much lighter than the neutrons, like electrons, the neutrons would move on with practically the same velocity they had initially. However, if the stationary targets are particles of nearly the same mass, the neutrons will be brought almost to rest in a (head-on) collision with them. Hence, hydrogen, whose nucleus (proton) has nearly the same mass as a neutron, should be most effective. Other considerations affect the choice of a moderator for neutrons, but momentum and energy considerations alone limit the choice to the lighter elements.

If a collision is *inelastic* then, by definition, the kinetic energy is not conserved. The final kinetic energy may be less than the initial value, the difference being ultimately converted to heat or to potential energy of deformation in the collision, for example; or the final kinetic energy may exceed the initial value, as when potential energy is released in the collision. In any case, the conservation of momentum still holds, as does the conservation of *total* energy.

Let us consider finally a *completely inelastic* collision. The two particles stick together after collision, so that there will be a final common velocity $\mathbf{v}_f$. It is not necessary to restrict the discussion to one-dimensional motion. Using only the conservation of momentum principle, we find

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f. \tag{10–6}$$

This determines $\mathbf{v}_f$ when $\mathbf{v}_{1i}$ and $\mathbf{v}_{2i}$ are known.

▶ **Example 1.** A baseball weighing 0.35 lb is struck by a bat while it is in horizontal flight with a speed of 90 ft/sec. After leaving the bat the ball travels with a speed of 110 ft/sec in a direction opposite to its original motion. Determine the impulse of the collision.

We cannot calculate the impulse from the definition $\mathbf{J} = \int \mathbf{F}\,dt$ because we do not know the force exerted on the ball as a function of time. However, we have seen (Eq. 10–2) that the change in momentum of a particle acted on by an impulsive force is equal to the impulse. Hence

$$\mathbf{J} = \text{change in momentum} = \mathbf{p}_f - \mathbf{p}_i$$

$$= m\mathbf{v}_f - m\mathbf{v}_i = \left(\frac{W}{g}\right)(\mathbf{v}_f - \mathbf{v}_i).$$

Assuming arbitrarily that the direction of $\mathbf{v}_i$ is positive, the impulse is then

$$J = \left(\frac{0.35 \text{ lb}}{32 \text{ ft/sec}_2}\right)(-110 \text{ ft/sec} - 90 \text{ ft/sec}) = -2.2 \text{ lb-sec}.$$

The minus sign shows that the direction of the impulse acting on the ball is opposite that of the original velocity of the ball.

We cannot determine the force of the collision from the data we are given. Actually, any force whose impulse is −2.2 lb-sec will produce the same change in momentum. For example, if the bat and ball were in contact for 0.0010 sec, the average force during this time would be

$$\bar{F} = \frac{\Delta p}{\Delta t} = \frac{-2.2 \text{ lb-sec}}{0.0010 \text{ sec}} = -2200 \text{ lb}.$$

For a shorter contact time the average force would be greater. The actual force would have a maximum value greater than this average value.

How far would gravity cause the baseball to fall during its collision time?

Example 2. (*a*) By what fraction is the kinetic energy of a neutron (mass m_1) decreased in a head-on elastic collision with an atomic nucleus (mass m_2) initially at rest?

The initial kinetic energy of the neutron K_i is $\frac{1}{2}m_1v_{1i}^2$. Its final kinetic energy K_f is $\frac{1}{2}m_1v_{1f}^2$. The fractional decrease in kinetic energy is

$$\frac{K_i - K_f}{K_i} = \frac{v_{1i}^2 - v_{1f}^2}{v_{1i}^2} = 1 - \frac{v_{1f}^2}{v_{1i}^2}.$$

But, for such a collision,

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i},$$

so that

$$\frac{K_i - K_f}{K_i} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 = \frac{4m_1m_2}{(m_1 + m_2)^2}.$$

(*b*) Find the fractional decrease in the kinetic energy of a neutron when it collides in this way with a lead nucleus, a carbon nucleus, and a hydrogen nucleus. The ratio of nuclear mass to neutron mass ($= m_2/m_1$) is 206 for lead, 12 for carbon, and 1 for hydrogen.

For lead, $m_2 = 206m_1$,

$$\frac{K_i - K_f}{K_i} = \frac{4 \times 206}{(207)^2} = 0.02 \quad \text{or} \quad 2\%.$$

For carbon, $m_2 = 12m_1$,

$$\frac{K_i - K_f}{K_i} = \frac{4 \times 12}{(13)^2} = 0.28 \quad \text{or} \quad 28\%.$$

For hydrogen, $m_2 = m_1$,

$$\frac{K_i - K_f}{K_i} = \frac{4 \times 1}{(2)^2} = 1 \quad \text{or} \quad 100\%.$$

These results explain why paraffin, which is rich in hydrogen, is far more effective in slowing down neutrons than is lead.

Example 3. *The Ballistic Pendulum.* The ballistic pendulum is used to measure bullet speeds. The pendulum is a large wooden block of mass M hanging vertically by two cords. A bullet of mass m, traveling with a horizontal speed v_i, strikes the pendulum and remains embedded in it (Fig. 10–6). If the collision time (the time required for the bullet to come to rest with respect to the block) is very small compared to the time of swing of the pendulum, the supporting cords remain approximately vertical during the collision. Therefore, no external horizontal force acts on the system (bullet + pendulum) during collision, and the horizontal component of momentum is conserved. The speed of the system after collision v_f is much less than that of the bullet before collision. This final speed can be easily determined, so that the original speed of the bullet can be calculated from momentum conservation.

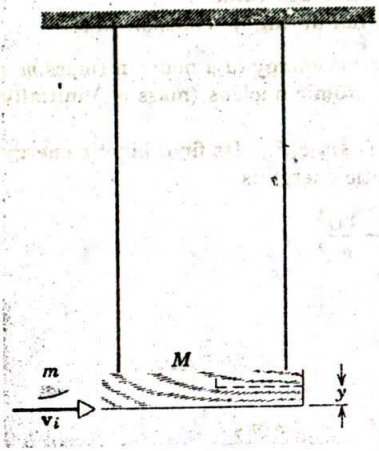


Fig. 10–6 Example 3. A ballistic pendulum consisting of a large wooden block of mass M suspended by cords. When a bullet of mass m and velocity v_i is fired into it, the block swings, rising a maximum distance y.

The initial momentum of the system is that of the bullet mv_i, and the momentum of the system just after collision is $(m + M)v_f$, so that

$$mv_i = (m + M)v_f.$$

After the collision is over, the pendulum and bullet swing up to a maximum height y, where the kinetic energy left after impact is converted into gravitational potential energy. Then, using the conservation of mechanical energy *for this part of the motion*, we obtain

$$\tfrac{1}{2}(m + M)v_f^2 = (m + M)gy.$$

Solving these two equations for v_i, we obtain

$$v_i = \frac{m + M}{m}\sqrt{2gy}.$$

Hence, we can find the initial speed of the bullet by measuring m, M, and y.

The kinetic energy of the bullet initially is $\frac{1}{2}mv_i^2$ and the kinetic energy of the system (bullet + pendulum) just after collision is $\frac{1}{2}(m + M)v_f^2$. The ratio is

$$\frac{\frac{1}{2}(m + M)v_f^2}{\frac{1}{2}mv_i^2} = \frac{m}{m + M}.$$

For example, if the bullet has a mass $m = 5$ gm and the block has a mass $M = 2000$ gm, only about one-fourth of 1% of the original kinetic energy remains; over 99% is converted to other forms of energy, such as heat. ◀

The velocity of the center of mass of two particles is not changed by their collision, for the collision does not change the total momentum of the system of two particles, only the distribution of momentum between the two particles. The momentum of the system can be written (Eq. 9–15) as $\mathbf{P} = (m_1 + m_2)\mathbf{v}_{cm}$. If no external forces act on the system, then $\mathbf{P}$ is constant before and after the collision, and the center of mass moves with uniform velocity throughout.

If we choose a reference frame attached to the center of mass, then in this center-of-mass reference frame, $\mathbf{v}_{cm} = 0$ and $\mathbf{P} = 0$. There is a great simplicity and

symmetry in describing collisions with respect to the center of mass, and it is customary to do so in nuclear physics. For whether collisions are elastic or inelastic, momentum is conserved, and in the center of mass reference frame the total momentum is zero. These results hold in two and three dimensions as well as in one because momentum is a vector quantity.

As an example, consider a head-on elastic collision between two particles m_1 and m_2. Let m_2 equal $3m_1$ and let m_2 be at rest, so that v_{2i} equals zero in the laboratory reference frame. The total momentum of the two particles is just that of the incident particle m_1v_{1i}, so that

$$m_1v_{1i} = (m_1 + m_2)v_{cm}$$

or

$$v_{cm} = \left(\frac{m_1}{m_1 + m_2}\right)v_{1i} = \tfrac{1}{4}v_{1i}.$$

After the collision, m_1 has a velocity $v_{1f} = -\frac{1}{2}v_{1i}$ and m_2 has a velocity $v_{2f} = \frac{1}{2}v_{1i}$. The total momentum of the two particles ($m_1v_{1f} + m_2v_{2f}$) is the same as before the collision, and the motion of the center of mass is unchanged (check this). In Fig. 10–7*a* we show a series of "snapshots" of the collision taken at equal time intervals as seen in the laboratory reference frame. In Fig. 10–7*b* we show the same situation as seen in the center-of-mass reference frame, where v_{cm} is zero. Notice the symmetry of the particles' motions when described in this way. The particle coming from the left has a speed $\frac{3}{4}v_{1i}$ with respect to the center of mass (where v_{1i} is the speed of m_1 in the laboratory frame) and recedes with this same speed. The particle coming from the right has a speed $\frac{1}{4}v_{1i}$ with respect to the center of mass and recedes with this same speed.

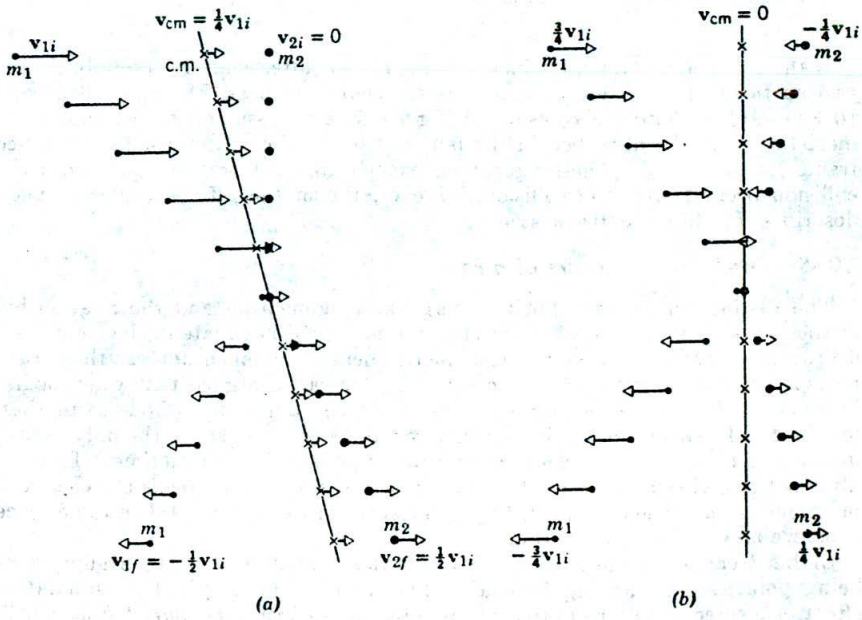


Fig. 10–7 (*a*) An elastic collision in the laboratory reference frame. (*b*) The same elastic collision in the center-of-mass reference frame.

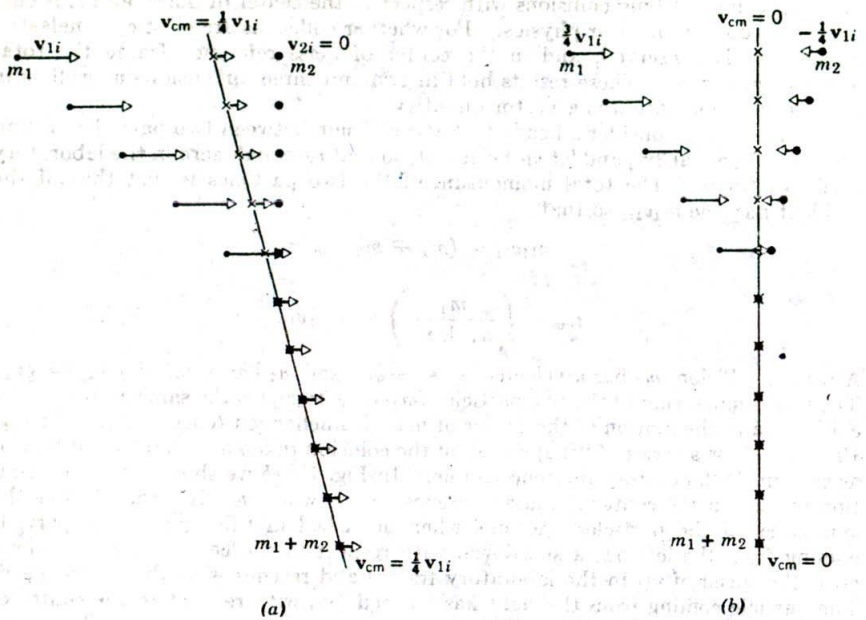


Fig. 10–8 (*a*) A completely inelastic collision in the laboratory reference frame. (*b*) The same completely inelastic collision in the center-of-mass reference frame. In each case the motion before collision is the same as that of Fig. 10–7.

If the collision is completely inelastic, the motion after collision is simply that of two particles moving along together at the center of mass. In Figs. 10–8*a* and 10–8*b* we show how the collision of Fig. 10–7, now assumed to be completely inelastic, would be described in the laboratory and the center-of-mass reference frames, respectively. These figures are exactly like the previous ones until the collision occurs; after the collision, however, the motion of the center of mass describes that of the entire system.

10–5 The "True" Measure of a Force

The distinction between kinetic energy and momentum and the relationship of these concepts to force were not clearly understood until late in the eighteenth century. Scientists argued whether kinetic energy or momentum was the "true" measure of the effect of a force on a body. Descartes argued that when bodies interact, all that can happen is the transfer of momentum from one body to another, for the total momentum of the universe remains constant; hence the only "true" measure of a force is the change in momentum it produces in a given time. Leibnitz attacked this view and argued that the "true" measure of a force is the change it produces in kinetic energy (called by him *vis viva* or living force, taken to be twice what we now call kinetic energy).

In his treatise on mechanics (1743), D'Alembert dismissed the argument as being pointless and arising from a confusion of terminology. The cumulative effect of a force can be measured by its integrated effect over *time*, $\int F\,dt$, which produces a change in momentum, *or* by its integrated effect over *space*, $\int F\,dx$, which produces a change in kinetic energy. Both concepts are useful and valid, although different. Which one we use depends on what we are interested in or

what is more convenient. As our present study of collisions illustrates, we frequently use both concepts in the same problem.

A more modern view is to look for quantities of the motion that are invariant, rather than focusing on the concept of force. The question as to whether the energy or the momentum is the "real" quantity of motion becomes pointless for there is no unique "quantity of motion." Instead, *both* energy *and* momentum may be regarded as invariant quantities of the motion in that for an isolated system the total of each of these quantities, summed up over all parts of the system, remains constant with time. There may be an exchange of energy, and of momentum, between different parts of an isolated system, but the total of each quantity is conserved.

10–6 Collisions in Two and Three Dimensions

In two or three dimensions (except for a completely inelastic collision) the conservation laws alone cannot tell us the motion of particles after a collision if we know the motion before the collision. For example, for a two-dimensional elastic collision, which is the simplest case, we have four unknowns, namely the two components of velocity for each of two particles after collision; but we have only three known relations between them, one for the conservation of kinetic energy and a conservation of momentum relation for each of the two dimensions. Hence we need more information than just the initial conditions. When we do not know the actual forces of interaction, as is often the case, the additional information must be obtained from experiment. It is simplest to specify the angle of recoil of one of the colliding particles.

Let us consider what happens when one particle is projected at a target particle which is at rest. This case is not as restrictive as it may seem, for we can always pick our reference frame to be one in which the target particle is at rest before the collision. Much experimental work in nuclear physics involves projecting nuclear particles at a target which is stationary in the laboratory reference frame. In such collisions, because of momentum conservation, the motion is in a plane determined by the lines of recoil of the colliding particles. The initial motion need not be along the line joining the centers of the two particles. The force of interaction may be electromagnetic (in which we include "contact" forces; see page 120), gravitational, or nuclear. The particles need not "touch"; strong forces, which act at relatively close distances of approach and for a time short compared to the observation time, deflect the particles from their initial courses.

A typical situation is shown in Fig. 10–9. The distance b between the initial line of motion and a line parallel to it through the center of the target particle is called the *impact parameter*. This is a measure of the directness of the collision, $b = 0$, corresponding to a head-on collision. The direction of motion of the incident particle m_1 after collision makes an angle θ_1 with the initial direction, and the target projectile m_2, initially at rest, moves in a direction after collision making an angle θ_2 with the initial direction of the incident projectile. Applying the conservation of momentum, which is a vector relation, we obtain two scalar equations; for

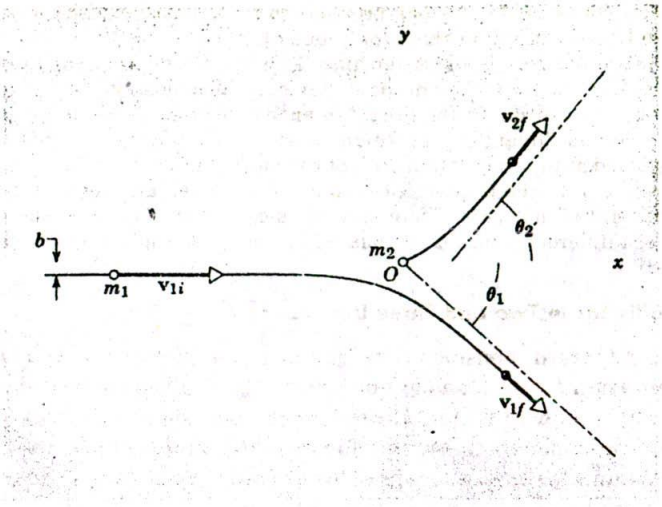


Fig. 10–9 Two particles, m_1 and m_2, undergoing a collision. The open circles indicate their positions before collision, the shaded ones after collision. Initially m_2 is at rest. The impact parameter b is the distance by which the collision misses being head-on.

the x-component of motion we have

$$m_1 v_{1i} = m_1 v_{1f} \cos \theta_1 + m_2 v_{2f} \cos \theta_2,$$

and for the y-component

$$0 = m_1 v_{1f} \sin \theta_1 - m_2 v_{2f} \sin \theta_2.$$

Let us now assume that the collision is *elastic*. Here the conservation of kinetic energy applies and we obtain a third equation,

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2.$$

If we know the initial conditions (m_1, m_2, and v_{1i}), we are left with four unknowns (v_{1f}, v_{2f}, θ_1, and θ_2) but only three equations relating them. We can determine the motion after collision only if we specify a value for one of these quantities, such as θ_1.

▶ **Example 4.** A gas molecule having a speed of 300 meters/sec collides elastically with another molecule of the same mass which is initially at rest. After the collision the first molecule moves at an angle of 30° to its initial direction. Find the speed of each molecule after collision and the angle made with the incident direction by the recoiling target molecule.

This example corresponds exactly to the situation just discussed, with $m_1 = m_2$, $v_{1i} = 300$ meters/sec, and $\theta_1 = 30°$. Setting m_1 equal to m_2, we have the relations

$$v_{1i} = v_{1f} \cos \theta_1 + v_{2f} \cos \theta_2,$$

$$v_{1f} \sin \theta_1 = v_{2f} \sin \theta_2,$$

and

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2.$$

We must solve for v_{1f}, v_{2f}, and θ_2. To do this we square the first equation (rewriting it as $v_{1i} - v_{1f}\cos\theta_1 = v_{2f}\cos\theta_2$), and add this to the square of the second equation (noting that $\sin^2\theta + \cos^2\theta = 1$); we obtain

$$v_{1i}^2 + v_{1f}^2 - 2v_{1i}v_{1f}\cos\theta_1 = v_{2f}^2.$$

Combining this with the third equation, we obtain

$$2v_{1f}^2 = 2v_{1i}v_{1f}\cos\theta_1$$

or (since $v_{1f} \neq 0$)

$$v_{1f} = v_{1i}\cos\theta_1 = (300 \text{ meters/sec})(\cos 30°)$$

or

$$v_{1f} = 260 \text{ meters/sec}.$$

From the third equation

$$v_{2f}^2 = v_{1i}^2 - v_{1f}^2 = (300 \text{ meters/sec})^2 - (260 \text{ meters/sec})^2,$$

or

$$v_{2f} = 150 \text{ meters/sec}.$$

Finally, from the second equation

$$\sin\theta_2 = (v_{1f}/v_{2f})\sin\theta_1$$
$$= (260/150)(\sin 30°) = 0.866$$

or

$$\theta_2 = 60°.$$

The two molecules move apart at right angles ($\theta_1 + \theta_2 = 90°$ in Fig. 10–9).

The student should be able to show that in an elastic collision between particles of equal mass, one of which is initially at rest, the recoiling particles always move off at right angles to one another. ◀

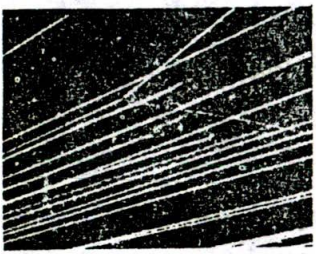
In Fig. 10–10, we show photographs of four elastic nuclear collisions that take place in a Wilson cloud chamber.* The tracks of the particles are made visible by the trail of droplets left in their wake. In each case the incident particle is an α-particle (He^4) and the target nucleus is essentially at rest before collision. Notice that as the target mass increases, the angle between the recoiling particles increases (see Problem 26). In case (*b*) where the target is also an α-particle, stereoscopic photos show that the recoiling particles move off at right angles; the angle is not quite a right angle in the figure because the particles do not lie in the plane of the figure.

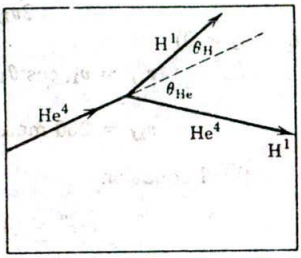
Figure 10–11 shows a series of four successive elastic collisions between protons caused when a high energy proton enters a bubble chamber† filled

* In 1927, the English physicist, C. T. R. Wilson, received the Nobel prize for inventing the cloud chamber; his investigations started along an entirely different line, namely, an attempt to produce in the laboratory a certain atmospheric phenomenon observed on the mountain Ben Nevis in Scotland.

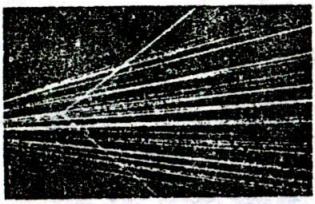
† In 1960, the American physicist, Donald Glaser, received the Nobel prize for inventing the bubble chamber; it is said that the concept occurred to him while watching bubbles form in a glass of beer.

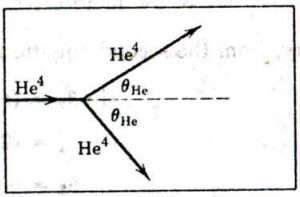
with liquid hydrogen, which supplies the target protons. The tracks of the particles are made visible in this case by the trail of bubbles left in their wake. Since the interacting particles are of equal mass and the collisions are elastic, the particles recoil at right angles to each other; this is apparent when the tracks of Fig. 10–11*a* are viewed stereoscopically.



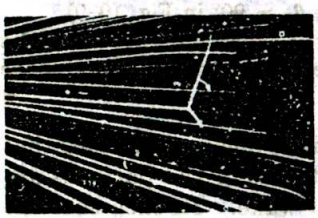


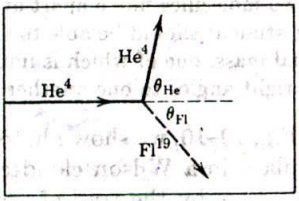
(*a*)



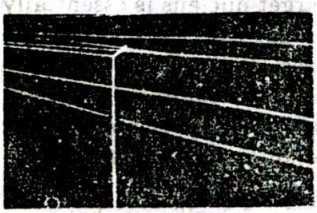


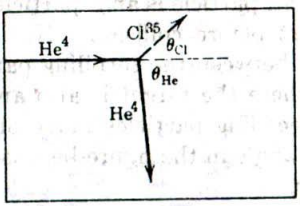
(*b*)





(*c*)





(*d*)

Fig. 10–10 Photographs of trajectories of particles undergoing collisions in a cloud chamber, a device that makes these paths visible. The chamber contains saturated water vapor. If the vapor is slightly compressed and then allowed to expand quickly, the water vapor will condense in droplets along the trajectory. The incident particle in all four cases is a helium nucleus (He^4, or α). In (*a*) the target is a hydrogen nucleus (H^1, or p). The other tracks are similar, except that in (*b*) the target is another He^4 nucleus, whereas in (*c*) and (*d*) the targets are fluorine and chlorine nuclei respectively. In general, the particles do not lie in the plane of the photograph. Stereoscopic photos are required for a complete analysis.

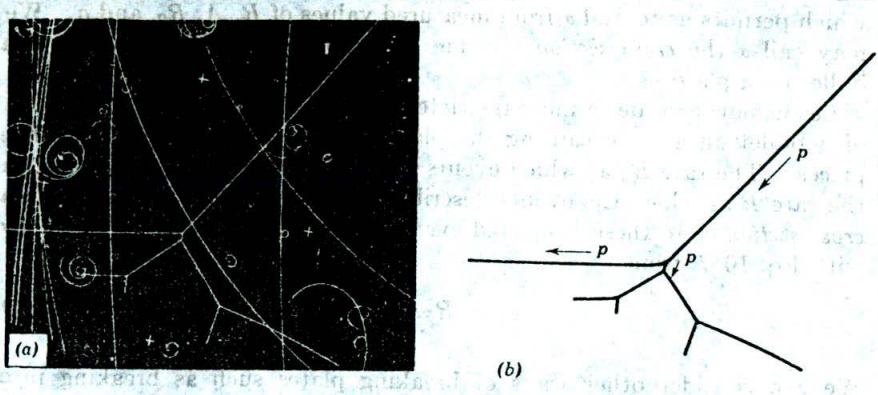


Fig. 10–11 (*a*) Four proton-proton collisions in a 10″-diameter bubble chamber. The original high-energy proton entered from the upper right. The spiral tracks are low energy electrons. The other tracks passing through the chamber are mesons of various kinds. This photo may be viewed stereoscopically using a device supplied with the explanatory booklet Introduction to the Detection of Nuclear Particles in a Bubble Chamber, Ealing Press, 1964 (courtesy Lawrence Radiation Laboratory). Such stereoscopic viewing shows that the angle between the outgoing tracks in each case is 90°. This is not apparent in the figure because the tracks do not lie in the plane of the figure. (*b*) A schematic representation of the proton tracks in *a*.

10–7 Cross Section

Although we have introduced the concept of the impact parameter b to describe collisions (see Fig. 10–9), it must be clear that, when we deal with particles of atomic or subatomic dimensions, we cannot define the track of the incident particle or the location of the target particle precisely enough. In practice, as when we bombard a thin target foil with a beam of deuterons from a cyclotron, we must deal in a statistical way with a large number of collisions between deuterons and the nuclei in the target; the impact parameters for individual collisions cannot be determined.

The situation is much the same as if we were firing a machine gun at random (in the dark, say) at the side of a distant barn of area A on which someone had hung a number of small dinner plates, each of area σ, in random (but not overlapping) positions. If the number of plates is q and if the rate at which bullets strike the barn is R_0, what is the rate R at which plates are broken? It is, on the basis of the random character of the events,

$$R = R_0(\sigma q/A), \tag{10–7a}$$

where σq is the total area of all the plates. We could, in fact, use this relation to measure σ, the geometrical area of a single plate. Solving for σ yields

$$\sigma = RA/R_0 q \tag{10–7b}$$

which permits us to find σ from measured values of R, A, R_0, and q. We may call σ the *cross section* for the event consisting of the impact of a bullet on a plate.

Let us now consider a more restricted class of event, namely the impact of a bullet on a plate causing the plate to break into (say) exactly five pieces. The rate R_5 at which events of this kind occur is much less than the rate R at which the events described above occur. We may assign a *cross section* σ_5 to these restricted events and may measure it, by analogy with Eq. 10–7, from

$$\sigma_5 = R_5 A / R_0 q. \tag{10–8}$$

We can consider other ways of breaking plates such as breaking into thirteen pieces, breaking so that one fragment has an area equal to half the plate or more, breaking so that one fragment flies vertically upward, etc. Each of these events can be assigned its own cross section σ_x by measuring the rate R_x at which the events occur. *None of these cross sections necessarily has anything to do with the geometrical area of the plate; all are measures of the probability of occurrence of the events to which they are assigned.* Cross sections are important because they are identified with single events and are independent of the details of particular experimental setups. In Eq. 10–8, for example, we would find the same value for σ_5 no matter how large the barn (A), how many plates (q), or how rapid the rate of fire of the machine gun (R_0); the measured value of R_5 would always be such as to yield the same measured value for σ_5.

Similarly, in nuclear physics we often bombard targets with nuclear projectiles, measure the rate at which events of a selected type occur, and assign a cross section to those events. For example, let us bombard a thin gold foil (Au^{197}) with deuterons (H^2, or d) whose energy is, say, 30 Mev. Many events can occur, among them (1) elastic scattering of the deuteron into the forward hemisphere, (2) elastic scattering of the deuteron into the backward hemisphere, (3) inelastic scattering of the deuteron between the angles of 30° and 60° with the direction of the incident beam, (4) the nuclear reaction $d + Au^{197} \rightarrow p + Au^{198}$, and (5) the nuclear reaction $d + Au^{197} \rightarrow n + Hg^{198}$, in which n represents a neutron. Each of these events (and many others that could be written down) has its own cross section σ_x which allows us to calculate the rate R_x at which these events occur if we know the details of the experimental arrangement. The ultimate goal of all the experiments is to understand the nature of nuclear forces.

Let the area of the foil exposed to the beam be A and the thickness of the foil be x. If there are n target particles per unit volume in the foil, the total number of available target particles is nAx. If the *effective area* (that is, the cross section) for the event we are concerned with is σ_x, the total *effective area* of all the nuclei is $nAx\sigma_x$. If R_0 is the rate at which projectiles strike the target and R_x is the rate at which the events in which we are interested occur, we have, because of the random nature of the

events (see Eq. 10–7*a*),

$$\frac{R_x}{R_0} = \frac{(nAx\sigma_x)}{A}$$

or

$$R_x = R_0 nx\sigma_x. \tag{10–9}$$

Thus we can measure σ_x for the event by measuring R_x, R_0, n, and x and substituting into Eq. 10–9. Cross sections are commonly expressed in *barns* or submultiples thereof; one barn = 10^{-28} meter^2.

Cross sections almost always depend on the energy of the incident particle, often exhibiting sharp peaks as the energy is varied. This simply means that at certain rather precisely defined characteristic energies the reaction is much more likely to "go" than at others. In much the same way a steel plate will vibrate with large amplitude at a series of rather precisely defined characteristic frequencies. We shall study such "resonances" in Chapter 20 and elsewhere.

Figure 10–12 shows one of the thousands of cross section curves that have been measured for various atomic and nuclear process. Let a highly collimated beam of neutrons of kinetic energy K fall on a thin foil of cadmium. The process to which the cross section σ in Fig. 10–12 refers is any process (absorption, elastic scattering, or inelastic scattering) that

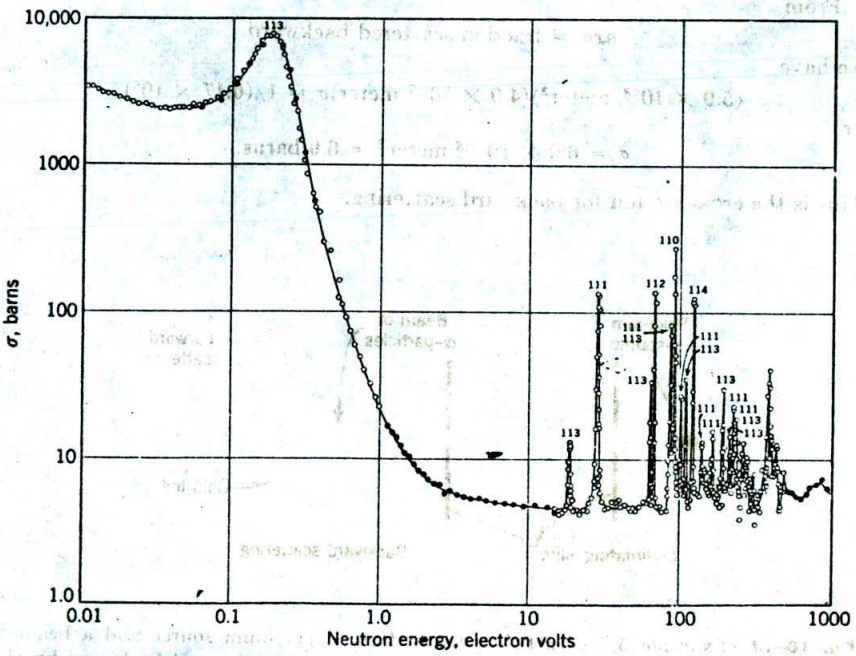


Fig. 10–12 The cross section as a function of energy for processes that remove neutrons from a collimated beam as they pass through cadmium. Note that both scales are logarithmic.

results in the removal of a neutron from the collimated beam. The numbers above the various peaks show the particular isotope responsible for that peak; this can be learned from other experiments using foils made of the separated isotopes. The strong peak labeled "113" that occurs at 0.17 electron volts is caused by the reaction

$$Cd^{113} + n \rightarrow Cd^{114} + \gamma$$

in which γ represents a gamma ray. This reaction, which has a peak cross section of 7600 barns, is responsible for the very large absorbing power of cadmium for slow neutrons. Note that both scales in Fig. 10–12 are logarithmic.

▶ **Example 5.** (*a*) About 1910, Geiger and Marsden, working under Lord Rutherford at Manchester University, performed a series of classic experiments that established the fact that atoms consisted of a small nucleus surrounded by a cloud of electrons rather than a sphere of distributed positive and negative charges, as Thomson had suggested earlier.

This experiment was in essence that shown in Fig. 10–13. Here α-particles from a polonium source are allowed to strike a gold foil 4.0×10^{-7} meter thick. It is found that although most of the α-particles pass through the foil (forward scattering), about 1 in 6.17×10^6 are scattered backward, that is, are deflected through an angle greater than 90°. The number of gold atoms per unit volume in the foil is 5.9×10^{28}/meter^3. What is the scattering cross section in barns for backward scattering (1 barn = 10^{-28} meter^2)?

From

$$nx\sigma = \text{fraction scattered backward}$$

we have

$$(5.9 \times 10^{28}/\text{meter}^3)(4.0 \times 10^{-7}\ \text{meter})\sigma = 1/(6.17 \times 10^6)$$

or

$$\sigma = 6.9 \times 10^{-28}\ \text{meter}^2 = 6.9\ \text{barns}.$$

This is the cross section for backward scattering.

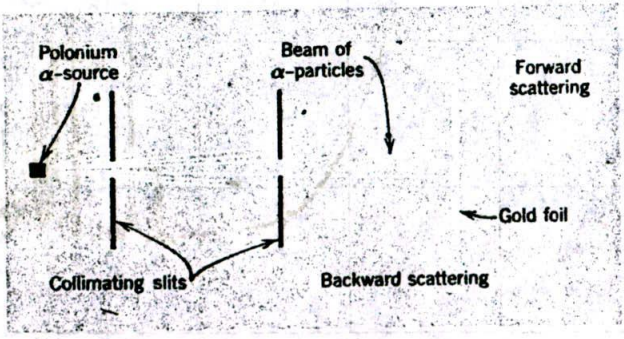


Fig. 10–13 Example 5. α-particles stream from a polonium source and a beam is formed by collimating slits. Some of the α-particles are scattered backward by the gold foil target; the rest pass through the foil.

(*b*) Rutherford reasoned that the backward scattering could not be caused by electrons in the atom; the α-particles are so much more massive than the electrons that they would hardly be deflected at all by them, let alone be scattered backward. He then suggested the nuclear model of the atom, attributing the scattering to collisions between α-particles and the massive positive core of the atom, the nucleus.

Assuming that the cross section for backward scattering is approximately equal to the area offered by a gold nucleus for direct collisions, estimate the *effective* size of a gold nucleus.

If the *effective* radius of the gold nucleus is taken to be r, we have

$$\sigma = \pi r^2,$$

$$r^2 = \sigma/\pi = 6.9 \times 10^{-28} \text{ meter}^2/\pi,$$

or

$$r = 1.5 \times 10^{-14} \text{ meter}.$$

This is the approximate radius of a gold *nucleus* which compares with the value of about 1.5×10^{-10} meter for the gold *atom*. Hence the massive nucleus is concentrated in a very small region of the atom (about 1 part in 10^{12} by volume). ◀

10–8 Reactions and Decay Processes

We stated in Section 10–1 that reactions and radioactive decay processes, for atoms, nuclei, and elementary particles, can be treated by the same methods used in collision studies, namely: We can apply the principles of conservation of linear momentum and energy to the (well-defined) periods "before the event" and "after the event." For these processes we must use the conservation of *total* energy because kinetic energy is *not* conserved. In this section we only consider examples in which the speeds of the particles are negligible with respect to the speed of light. This means that we may use the classical expressions for momentum and energy and need not use the relativistic expressions.

▶ **Example 6.** *Nuclear Reactions.* A thin film containing a fluorine (F^{19}) compound is bombarded by a beam of protons (p) which have been accelerated to an energy of 1.85 Mev (million electron volts; 1 Mev = 1.60×10^{-13} joule) in a Van de Graaff accelerator. Some of the protons interact with the fluorine nuclei to produce the following nuclear reaction:

$$F^{19} + p \rightarrow O^{16} + \alpha$$

It is observed that the α-particles (which are helium nuclei) that emerge at *right angles* to the incident proton beam (see Fig. 10–14) have speeds of 1.95×10^7 meters/sec. What can you learn about the reaction by applying the laws of conservation of linear momentum and of total energy? The masses involved are, to a precision good enough for our purposes,

$$m_p = 1.01 \text{ amu} \qquad m_O = 16.0 \text{ amu}$$

$$m_F = 19.0 \text{ amu} \qquad m_\alpha = 4.00 \text{ amu},$$

in which 1 amu (*atomic mass unit*) = 1.66×10^{-27} kg.

The x- and y-components of linear momentum are conserved, which means that they have the same values before and after the reaction. In the laboratory refer-

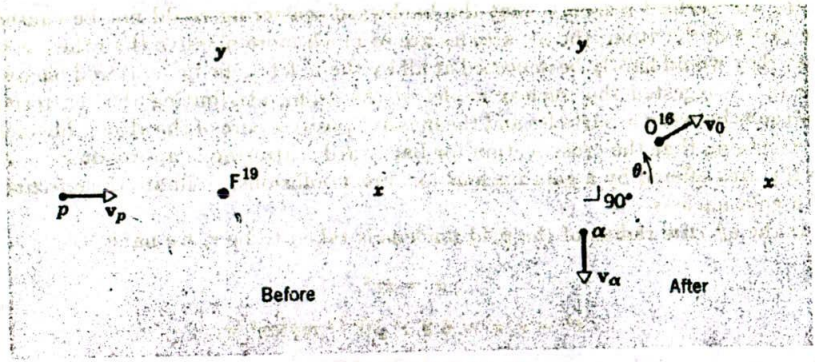


Fig. 10–14 The nuclear reaction $p + F^{19} \rightarrow \alpha + O^{16}$, showing the situation before and after the event in the laboratory reference frame.

ence frame of Fig. 10–14, then

$$m_p v_p = m_O v_O \cos\theta \qquad (x\text{-component}) \qquad (10\text{–}10)$$

and

$$0 = m_\alpha v_\alpha - m_O v_O \sin\theta \qquad (y\text{-component}) \qquad (10\text{–}11)$$

For the conservation of total energy we write

$$Q + \tfrac{1}{2}m_p v_p^2 = \tfrac{1}{2}m_O v_O^2 + \tfrac{1}{2}m_\alpha v_\alpha^2 \qquad (10\text{–}12)$$

in which it is clear that Q is the amount by which the kinetic energy of the system after the reaction exceeds the kinetic energy of the system before the reaction. Note that we have assumed that the particles are moving slowly enough so that we may use the classical expression for kinetic energy ($\frac{1}{2}mv^2$) rather than the relativistic one ($m_0c^2\sqrt{1 - v^2/c^2}$). If Q is positive, kinetic energy must be generated by the reaction.

The energy represented by Q can only come from differences in the rest energies of the particles before or after the reaction, according to Einstein's well-known relation $E = \Delta mc^2$ (see Section 8–9). Thus (if Q is positive) we expect that the rest mass of the system after the reaction would be slightly less than its rest mass before the reaction and that Q would indeed be given by the Einstein relation

$$\begin{aligned} Q &= \Delta mc^2 \\ &= [(m_p + m_F) - (m_\alpha + m_O)]c^2. \end{aligned} \qquad (10\text{–}13)$$

Note that Eqs. 10–12 and 10–13 are independent relations for Q, being connected through Einstein's mass-energy relation.

The three conservation equations contain just three unknowns, v_O, θ, and Q. To find Q from them let us first eliminate θ between the first two equations by squaring and adding (recalling that $\cos^2\theta + \sin^2\theta = 1$). We obtain

$$m_p^2 v_p^2 + m_\alpha^2 v_\alpha^2 = m_O^2 v_O^2.$$

We can now eliminate v_O between this relation and Eq. 10–12. The student can show that, after a little rearrangement, we obtain

$$Q = K_\alpha(1 + m_\alpha/m_O) - K_p(1 - m_p/m_O). \qquad (10\text{–}14)$$

From the data given we know that $K_p(=\frac{1}{2}m_p v_p^2) = 1.85$ Mev and

$$\begin{aligned} K_\alpha &= \tfrac{1}{2}m_\alpha v_\alpha^2 \\ &= \tfrac{1}{2}(4.00 \text{ amu} \times 1.66 \times 10^{-27} \text{ kg/amu})(1.95 \times 10^7 \text{ meters/sec})^2 \\ &= (1.26 \times 10^{-12} \text{ joule})(1 \text{ Mev}/1.60 \times 10^{-13} \text{ joule}) \\ &= 7.88 \text{ Mev}. \end{aligned}$$

We may now calculate Q from Eq. 10–14 as

$$Q = (7.88 \text{ Mev})(1 + 4.00/16.0) - (1.85 \text{ Mev})(1 - 1.01/16.0) = 8.13 \text{ Mev}.$$

Thus, by using the principles of conservation of linear momentum and total energy, we can calculate Q for the reaction without making any observations on the recoiling O^{16} nucleus. If we want to know v_O and θ for this nucleus we can easily calculate them from Eqs. 10–10 and 10–11.

The result $Q = 8.13$ Mev is an important bit of information about the reaction. From Eq. 10–13, which is a relation for Q independent of Eq. 10–14, we can now calculate that the decrease in rest mass during the reaction is given by

$$\begin{aligned} \Delta m &= Q/c^2 \\ &= (8.13 \text{ Mev} \times 1.60 \times 10^{-13} \text{ joule/Mev})/(3.00 \times 10^8 \text{ meters/sec})^2 \\ &= (1.44 \times 10^{-29} \text{ kg})(1 \text{ amu}/1.66 \times 10^{-27} \text{ kg}) \\ &= 0.00873 \text{ amu}. \end{aligned}$$

We can verify this result by calculating $\Delta m\ [= (m_p + m_F) - (m_\alpha + m_O)]$ from very precise measurements of the four separate masses made in a mass spectrometer (see Problem 37). The excellent agreement that we get shows once again the essential validity of Einstein's mass-energy relationship. ◀

QUESTIONS

1. Explain how conservation of momentum applies to a handball bouncing off a wall.

2. How can you reconcile the sailing of a sailboat into the wind with the conservation of momentum principle?

3. A sand glass is being weighed on a sensitive balance, first when sand is dropping in a steady stream from the upper to the lower part and then again after the upper part is empty. Are the two weights the same or not? Explain your answer.

4. The blades of a turbine are usually curved rather than flat in shape so that the fluid striking them follows a path resembling a u-turn. Convince yourself about the fluid's motion and explain the advantage of the curved shape over the flat one.

5. It is obvious from inspection of Eqs. 10–3 and 10–4 that a valid solution to the problem of finding the final velocities of two particles in a one-dimensional elastic collision is $v_{1f} = v_{1i}$ and $v_{2f} = v_{2i}$. What does this mean physically? Explain.

6. Consider a one-dimensional elastic collision between a given incoming body A and a body B initially at rest. How would you choose the mass of B, in comparison to the mass of A, in order that B should recoil with (*a*) the greatest speed, (*b*) the greatest momentum, and (*c*) the greatest kinetic energy?

7. In a two body collision *in the center-of-mass reference frame* the momenta of the particles are equal and opposite to one another both before and after the collision. Is

the line of relative motion necessarily the same after collision as before? Under what conditions would the magnitudes of the velocities of the bodies increase? Decrease? Remain the same as a result of the collision?

8. When dealing with atoms, nuclei, or elementary particles, what does it mean to say that two such bodies "touch" during a collision?

9. When the forces of interaction between two particles have an infinite range, such as the mutual gravitational attraction between two bodies, can the cross section for "collision" be finite? Is it at all useful to regard this interaction as a collision?

10. Why does the computation of the radius of the gold nucleus in Example 5 give only an approximate answer?

11. Could we determine in principle the cross section for a collision by using only one bombarding particle and one target particle? In practice?

PROBLEMS

1. A cue strikes a billiard ball, exerting an average force of 50 nt over a time of 10 milliseconds. If the ball has mass 0.20 kg, what speed does it have after impact?

2. A 1.0-kg ball drops vertically onto the floor with a speed of 25 meters/sec. It rebounds with an initial speed of 10 meters/second. (*a*) What impulse acts on the ball during contact? (*b*) If the ball is in contact for 0.020 sec, what is the average force exerted on the floor?

3. A croquet ball (mass 0.50 kg) is struck by a mallet, receiving the impulse shown in the graph (Fig. 10–15). What is the ball's velocity just after the force has become zero?

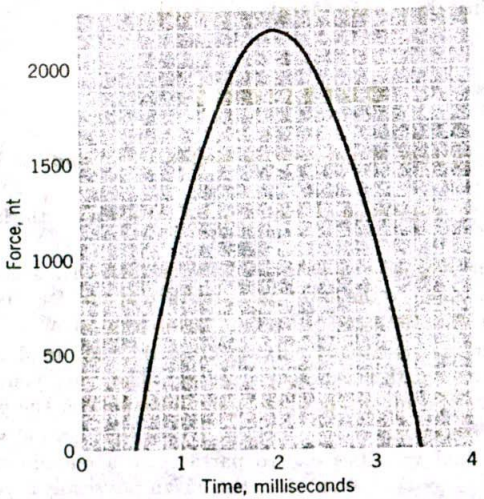


Fig. 10–15

ball; (*b*) the impulse imparted to the club; (*c*) the average force exerted on the ball by the club; (*d*) the work done on the ball.

5. A ball of mass m and speed v strikes a wall perpendicularly and rebounds with undiminished speed. If the time of collision is t, what is the average force exerted by the ball on the wall?

6. A stream of water impinges on a stationary "dished" turbine blade, as shown in Fig. 10–16. The speed of the water is u, both before and after it strikes the curved surface of the blade, and the mass of water striking the blade per unit time is constant at the value μ. Find the force exerted by the water on the blade.

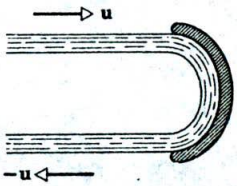


Fig. 10–16

7. A 6.0-kg box sled is traveling across the ice at a speed of 9.0 meters/sec when a 12-kg package is dropped into it vertically. Describe the subsequent motion of the sled.

8. A body of 2.0 kg mass makes an elastic collision with another body at rest and afterwards continues to move in the original direction but with one-fourth of its original speed. What is the mass of the struck body?

9. A block of mass $m_1 = 100$ kg is at rest on a very long frictionless table, one end of which is terminated in a wall. Another block of mass m_2 is placed between the first block and the wall and set in motion to the left with constant speed v_{2i}, as in Fig. 10–17. Assuming that all collisions are completely elastic, find the value of m_2 for which both blocks move with the same velocity after m_2 has collided once with m_1 and once with the wall. The wall has infinite mass effectively.

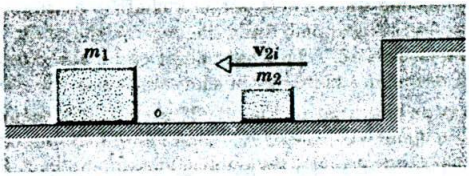


Fig. 10–17

10. A bullet weighing 1.0×10^{-2} lb is fired horizontally into a 4.0-lb wooden block at rest on a horizontal surface. The coefficient of kinetic friction between block and surface is 0.20. The bullet comes to rest in the block which moves 6.0 ft. Find the speed of the bullet.

11. A bullet of mass 10 gm strikes a ballistic pendulum of mass 2.0 kg. The center of mass of the pendulum rises a vertical distance of 12 cm. Assuming the bullet remains embedded in the pendulum, calculate its initial speed.

12. In a breech-loading automatic firearm of early vintage the reloading mechanism at the rear of the bore is activated when the breech-block, which recoils after the bullet is fired, compresses a spring by a predetermined amount d. (*a*) Show that the speed

v of the bullet of mass m must be at least $d\sqrt{kM}/m$ on firing, for automatic loading, where k is the force constant of the spring and M is the mass of the breech-block. (*b*) In what sense, if any, can this process be regarded as a collision?

13. A steel ball weighing 1.0 lb is fastened to a cord 27 in. long and is released when the cord is horizontal. At the bottom of its path the ball strikes a 5.0-lb steel block initially at rest on a frictionless surface (Fig. 10–18). The collision is elastic. Find the speed of the ball and the speed of the block just after the collision.

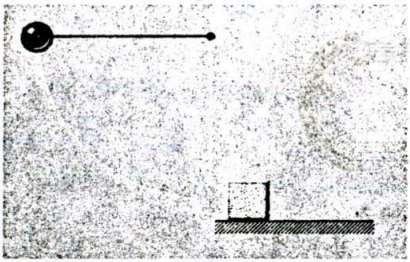


Fig. 10–18

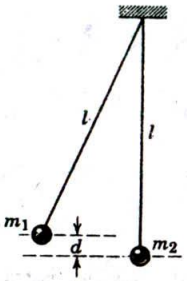


Fig. 10–19

14. Two pendulums each of length l are initially situated as in Fig. 10–19. The first pendulum is released and strikes the second. Assume that the collision is completely inelastic and neglect the mass of the strings and any frictional effects. How high does the center of mass rise after the collision?

15. A railroad freight car weighing 32 tons and traveling 5.0 ft/sec overtakes one weighing 24 tons traveling 3.0 ft/sec in the same direction. (*a*) Find the speed of the cars after collision and the loss of kinetic energy during collision if the cars couple together. (*b*) If the collision is elastic, the freight cars do not couple but separate after collision. What are their speeds?

16. An elevator is moving *up* at 6.0 ft/sec in a shaft. At the instant the elevator is 60 ft from the top, a ball is dropped from the top of the shaft. The ball rebounds elastically from the elevator roof. To what height can it rise relative to the top of the shaft? Do the same problem assuming the elevator is moving *down* at 6.0 ft/sec.

17. The two masses on the right of Fig. 10–20 are slightly separated and initially at rest; the left mass is incident with speed v_0. Assuming head-on elastic collisions, (*a*) if $M \leq m$, show that there are exactly two collisions and find all final velocities; (*b*) if $M > m$, show that there are three collisions and find all final velocities.

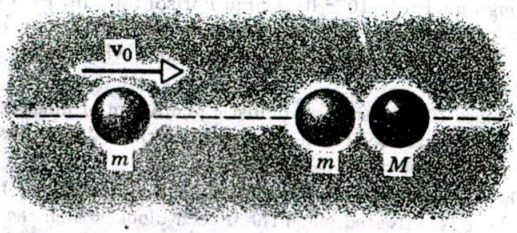


Fig. 10–20

18. A ball of mass m is projected with speed v_i into the barrel of a spring-gun of mass M initially at rest on a frictionless surface; see Fig. 10–21. The mass m sticks in the barrel at the point of maximum compression of the spring. No energy is lost in friction. What fraction of the initial kinetic energy of the ball is stored in the spring?

19. A block of mass $m_1 = 2.0$ kg slides along a frictionless table with a speed of 10 meters/sec. Directly in front of it, and moving in the same direction, is a block of mass $m_2 = 5.0$ kg moving at 3.0 meters/sec. A massless spring with a spring constant of $k = 1120$ nt/meter is attached to the backside of m_2 as shown in Fig. 10–22. When the blocks collide, what is the maximum compression of the spring? Assume that the spring does not bend and always obeys Hooke's law.



Fig. 10–21

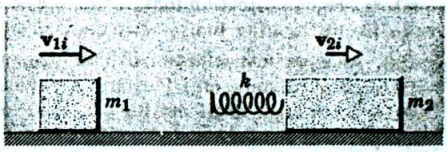


Fig. 10–22

20. A box is put on a scale which is adjusted to read zero when the box is empty. A stream of pebbles is then poured into the box from a height h above its bottom at a rate of μ (pebbles per second). Each pebble has a mass m. If the collisions between the pebbles and box are completely inelastic, find the scale reading at time t after the pebbles begin to fill the box. Determine a numerical answer when $\mu = 100$ sec^{-1}, $h = 25$ ft, $mg = 0.010$ lb, and $t = 10$ sec.

21. A scale is adjusted to read zero. Particles fall from a height of 9.0 ft before colliding with the balance pan on the scale; the collisions are elastic, i.e., the particles rebound upward with the same speed. If each particle has a mass of $\frac{1}{128}$ slug and collisions occur at the rate of 32 particles/sec, what is the scale reading in pounds?

22. An electron collides elastically with a hydrogen atom initially at rest. The initial and final motions are along the same straight line. What fraction of the electron's initial kinetic energy is transferred to the hydrogen atom? The mass of the hydrogen atom is 1840 times the mass of the electron.

23. An electron, mass m, collides head-on with an atom, mass M, initially at rest. As a result of the collision a characteristic amount of energy E is stored internally in the atom. What is the minimum initial velocity v_0 that the electron must have? (Hint: Conservation principles lead to a quadratic equation for the final electron velocity v and a quadratic equation for the final atom velocity V. The minimum value v_0 follows from the requirement that the radical in the solutions for v and V be real.)

24. (*a*) Show that in a one-dimensional elastic collision the speed of the center of mass of two particles, m_1 moving with initial speed v_{1i} and m_2 moving with initial speed v_{2i}, is

$$v_{cm} = \left(\frac{m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2}{m_1 + m_2}\right) v_{2i}$$

(*b*) Use the expressions obtained for v_{1f} and v_{2f}, the particles' speeds after collision, to derive the same result for v_{cm} *after* the collision.

25. Mass m_1 collides head-on with m_2, initially at rest, in a completely inelastic collision. (*a*) What is the kinetic energy of the system before collision? (*b*) What is the kinetic energy of the system after collision? (*c*) What fraction of the original kinetic energy was converted into heat? (*d*) Let $\mathbf{v}_{cm}$ be the velocity of the center of

mass of the system. View the collision from a primed reference frame moving with the center of mass so that $v_{1i}' = v_{1i} - v_{cm}$, $v_{2i}' = -v_{cm}$. Repeat parts (*a*), (*b*), and (*c*), as seen by an observer in this reference frame. Is the mechanical energy converted to heat the same in each case? Explain.

26. Show that, in the case of an elastic collision between a particle of mass m_1 with a particle of mass m_2 initially at rest, (*a*) the maximum angle θ_m through which m_1 can be deflected by the collision is given by $\cos^2 \theta_m = 1 - m_2^2/m_1^2$, so that $0 \leq \theta_m \leq \pi/2$, when $m_1 > m_2$; (*b*) $\theta_1 + \theta_2 = \pi/2$, when $m_1 = m_2$; (*c*) θ_1 can take on all values between 0 and π, when $m_1 < m_2$.

27. A billiard ball moving at a speed of 2.2 meters/sec strikes an identical stationary ball a glancing blow. After the collision one ball is found to be moving at a speed of 1.1 meters/sec in a direction making a 60° angle with the original line of motion. Find the velocity of the other ball. Can the collision be inelastic, given these data?

28. An α-particle collides with an oxygen nucleus, initially at rest. The α-particle is scattered at an angle of 64° from its initial direction of motion and the oxygen nucleus recoils at an angle of 51° on the other side of this initial direction. What is the ratio of the speeds of these particles? The mass of the oxygen nucleus is four times that of the α-particle.

29. Two balls A and B, having different but unknown masses, collide. A is initially at rest when B has a speed v. After collision B has a speed $v/2$ and moves at right angles to its original motion. Find the direction in which ball A moves after collision. Can you determine the speed of A from the information given? Explain.

30. Two vehicles A and B are traveling west and south, respectively, toward the same intersection where they collide and lock together. Before the collision A (total weight, 900 lb) is moving with a speed of 40 mph, and B (total weight, 1200 lb) has a speed of 60 mph. Find the magnitude and direction of the velocity of the (interlocked) vehicles immediately after collision.

31. A deuteron is a nuclear particle made up of one proton and one neutron. Its mass is approximately 4×10^{-24} gm. A deuteron, accelerated by a cyclotron to a speed of 10^9 cm/sec, collides with another deuteron at rest. (*a*) If the two particles stick together to form a helium nucleus, find the velocity of the nucleus. (*b*) The helium nucleus then breaks up into a neutron with a mass of about 2×10^{-24} gm and a helium isotope of mass 6×10^{-24} gm. If the neutron is given off at right angles to the direction of the original velocity with a speed of 5×10^8 cm/sec, find the magnitude and direction of the velocity of the helium isotope.

32. In 1932 Chadwick in England demonstrated the existence and properties of the neutron (one of the fundamental particles making up the atom) with the device shown in Fig. 10–23. In an evacuated chamber, a sample of radioactive polonium decays to yield α-rays (helium nuclei). These nuclei impinge on a block of beryllium inducing a process whereby neutrons are emitted. (The reaction is: He and Be combine to form radioactive carbon, which decays to stable carbon + neutrons.) The neutrons strike a film of paraffin (CH_4), releasing hydrogen nuclei which are detected in an ionization chamber. In other words, an elastic collision takes place in which the momentum of the neutron is partially transferred to the hydrogen nucleus.

(*a*) Find an expression for the maximum speed v_H that the hydrogen nucleus (mass m_H) can achieve. Let the incoming neutrons have mass m_n and speed v_n. (Hint: Will more energy be transferred in a head-on collision or in a glancing collision?)

(*b*) One of Chadwick's goals was to find the mass of his new particle. Inspection of expression (*a*) which contains this parameter, however, shows that *two* unknowns are present, v_n and m_n (v_H is known; it can be measured with the ionization chamber). To eliminate the unknown v_n, he substituted a paracyanogen (CN) block for the paraffin. The neutrons then underwent elastic collisions with nitrogen nuclei instead of hydrogen nuclei. Of course, expression (*a*) still holds if v_N is written for v_H and m_N for m_H.

Therefore if v_H and v_N are measured in separate experiments, v_n can be eliminated between the two expressions for hydrogen and nitrogen to yield a value for m_n. Chadwick's values were

$$v_H = 3.3 \times 10^9 \text{ cm/sec},$$

$$v_N = 0.47 \times 10^9 \text{ cm/sec}.$$

What was his value for m_n? How does this compare with the established value $m_n = 1.00867$ amu? (Take $m_H = 1.0$ amu, $m_N = 14$ amu.)

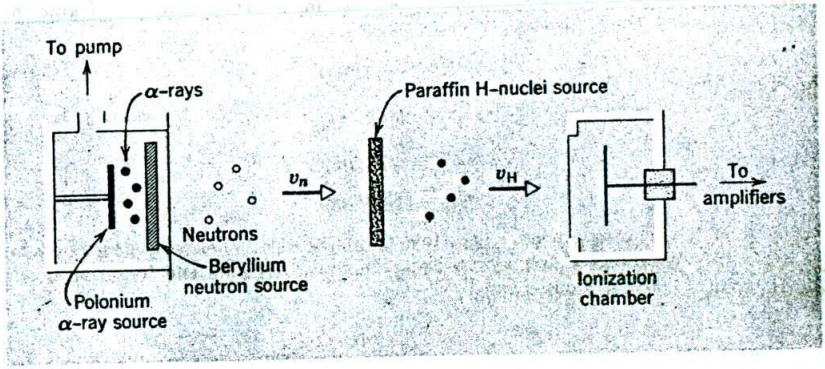


Fig. 10–23

33. A sphere of radius r_1 impinges on a sphere of radius r_2. What is the cross section for a contact collision?

34. A beam of fast neutrons impinges on a 5.0-mg sample of Cu^{65}, a stable isotope of copper. A possibility exists that the copper nucleus may capture a neutron to form Cu^{66}, which is radioactive and decays to Zn^{66}, which is again stable. If a study of the electron emission of the copper sample implies that 4.6×10^{11} neutron captures occur each second, what is the neutron capture cross section in barns for this process? The intensity of the neutron beam is 1.1×10^{18} neutrons/meter^2 sec.

35. A beam of slow neutrons strikes an aluminum foil 1.0×10^{-5} meter thick. Some neutrons are captured by the aluminum which becomes radioactive and decays by emitting an electron (β^-) and forming silicon:

$$n + Al^{27} \rightarrow Al^{28} \rightarrow Si^{28} + \beta^-.$$

Suppose that the neutron flux is 3.0×10^{16}/meter^2 sec, and the neutron capture cross section is 0.23 barn. How many transmutations will occur each second?

36. In a thick foil there are a great many layers of target particles so that the number of projectile particles reaching a layer will depend on how many have been scattered out by previous layers. Let the number of particles reaching a layer at a depth s be N and the number lost by scattering from that layer be $-dN$; then show that

$$-\frac{dN}{N} = n\sigma\, ds$$

and

$$N = N_0 e^{-n\sigma s}$$

where N_0 is the number of particles incident on the face of the foil ($s = 0$).

37. The precise masses in the reaction

$$p + F^{19} \rightarrow \alpha + O^{16}$$

have been determined by mass spectrometer measurements and are

$m_p = 1.00783$ amu $\quad m_\alpha = 4.00260$ amu

$m_F = 18.99840$ amu $\quad m_O = 15.99491$ amu

Calculate the Q of the reaction from these data and compare with the Q calculated in Example 6 from reaction studies.

38. An elementary particle called Σ^-, at rest in a certain reference frame, decays spontaneously into two other particles according to

$$\Sigma^- \rightarrow \pi^- + n$$

The masses are

$$m_{\Sigma^-} = 2340.5m_e$$

$$m_{\pi^-} = 273.2m_e$$

$$m_n = 1838.65m_e$$

where m_e is the electron mass. (*a*) How much kinetic energy is generated in this process? (*b*) Which of the decay products (π^- and n) gets the larger share of this kinetic energy? Of the momentum?

Rotational Kinematics

CHAPTER 11

11–1 Rotational Motion

So far we have dealt mostly with the translational motion of single particles or of rigid bodies, that is, of bodies whose parts all have a fixed relationship to each other. No real body is truly rigid, but many bodies, such as molecules, steel beams, and planets, are rigid enough so that, in many problems, we can ignore the fact that they warp, bend, or vibrate. As Fig. 3–1 shows, *a rigid body moves in pure translation if each particle of the body undergoes the same displacement as every other particle in any given time interval.*

In this chapter we are interested in *rotation.* For the time being we again restrict ourselves to single particles and to rigid bodies, which means that we shall not consider such rotational motions as those of the solar system or of water in a spinning beaker. We shall also deal only with rotation about axes that remain fixed in the reference frame in which we observe the rotation.

Figure 11–1 shows the rotational motion of a rigid body about a fixed axis, in this case the z-axis of our reference frame. Let P represent a particle in the rigid body, arbitrarily selected and described by the position vector $\mathbf{r}$. We then say that: *A rigid body moves in pure rotation if every particle of the body* (such as P in Fig. 11–1) *moves in a circle, the centers of which are on a straight line called the axis of rotation* (the z-axis in Fig. 11–1). If we draw a perpendicular from any point in the body to the axis, each such line will sweep through the same angle in any given time interval as another such line. Thus we can describe the pure rotation of a rigid body

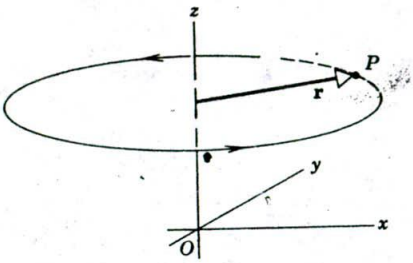


Fig. 11–1 A rigid body rotating about the z-axis. Each point in the body, such as P, describes a circle about this axis.

by considering the motion of any one of the particles (such as P) that make it up. (We must rule out, however, particles that are on the axis of rotation. Why?)

The general motion of a rigid body is a combination of translation and rotation however, rather than one of pure rotation. We can locate a rigid body that is moving in pure translation by giving the three coordinates x, y, z of any point in it (its center of mass, say) in a particular reference frame. For a body that rotates as it moves translationally we need, in the most general case, three more coordinates, such as angles, to specify the orientation of the body with respect to the reference frame. Figure 11–2 (see also Fig. 9–1) shows a special case of rigid body motion combining translation and rotation. The figure is an extension of Fig. 3–1 in which the body now rotates as it moves translationally. To locate this body we must not only locate point O in the body in the xy reference frame but we must also say how the $x'y'$ reference frame, which is fixed in the body, is oriented with respect to the xy frame.

As we saw in Chapter 9 we can describe the translational motion of any system of particles—whether rigid or not—whether rotating or not—by imagining that all of the mass M of the body is concentrated at the center

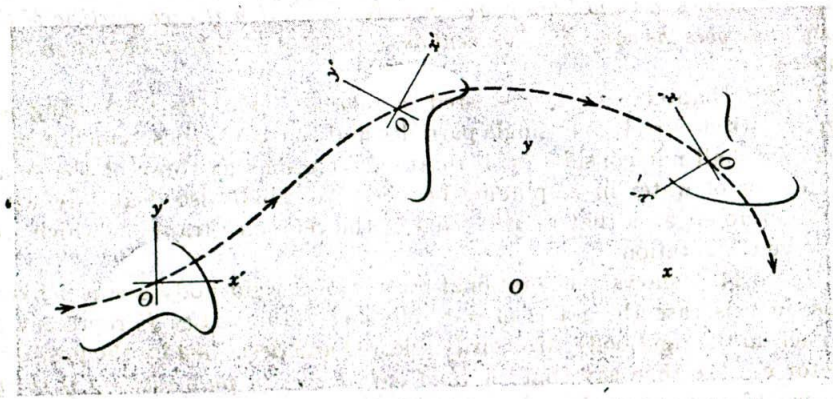


Fig. 11–2 A rigid body moving in combined translational and rotational motion as seen from reference frame x,y. Notice that the reference frame fixed on the body (x', y') changes its orientation with respect to x, y as the motion proceeds. Compare with Figs. 3–1 and 9–1. This figure represents a special case in that the translational motion occurs in two dimensions only (the xy plane) and the rotational motion occurs about an axis that maintains a fixed direction (the z'-axis).

of mass and that $\mathbf{F}_{ext}$, the resultant of the external forces acting on the body, acts at this point. The acceleration of the center of mass is then given by Eq. 9–10 or $\mathbf{F}_{ext} = M\mathbf{a}_{cm}$. It is very helpful to be able to represent the translational motion of a rigid body by the motion of a single point—its center of mass; all that is left is to determine its rotational motion. We shall discuss such combined translational and rotational motions in the next chapter. This will be simpler to do after we have studied pure rotation about a fixed axis.

We now return, therefore, to the pure rotation of a rigid body about a fixed axis (Fig. 11–1). First, we must *describe* the rotational motion. We call this description rotational kinematics; we must define the variables of angular motion and relate them to each other, just as in particle kinematics we defined the variables of translational motion and related them to each other. The next part of our program is to relate the rotational motion of a body to the properties of the body and of its environment. This is rotational dynamics. In this chapter we study the kinematics of rotation. We develop the dynamics of rotation in the next chapter.

11–2 Rotational Kinematics—the Variables

In Fig. 11–1 let us pass a plane through P at right angles to the axis of rotation. This plane, which cuts through the rotating body, contains the circle in which particle P moves. Figure 11–3 shows this plane, as we look downward on it from above, along the z-axis in Fig. 11–1.

We can tell exactly where the entire rotating body is in our reference frame if we know the location of any single particle (P) of the body in this frame. Thus, for the kinematics of this problem, we need only consider the (two-dimensional) motion of a particle in a circle.

The angle θ in Fig. 11–3 is the angular position of particle P with respect to the reference position. We arbitrarily choose the positive sense

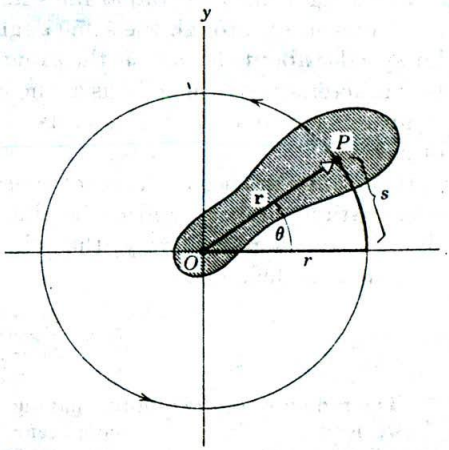


Fig. 11–3 A cross sectional view of the rigid body of Fig. 11–1, showing point P and vector $\mathbf{r}$ of that figure. Point P, which is fixed in the rotating body, rotates counterclockwise about the origin in a circle of radius r.

of rotation in Fig. 11–3 to be counterclockwise, so that θ increases for counterclockwise rotation and decreases for clockwise rotation.

It is convenient to measure θ in radians* rather than in degrees. By definition θ is given in radians by the relation

$$\theta = s/r,$$

in which s is the arc length shown in Fig. 11–3.

Let the body of Fig. 11–3 be rotating counterclockwise. At time t_1 the angular position of P is θ_1 and at a later time t_2 its angular position is θ_2. This is shown in Fig. 11–4, which gives the positions of P and of the position vector $\mathbf{r}$ at these times; the outline of the body itself has been omitted in that figure for simplicity. The *angular displacement* of P will be $\theta_2 - \theta_1 = \Delta\theta$ during the time interval $t_2 - t_1 = \Delta t$. The *average angular speed* $\bar{\omega}$ of particle P in this time interval is defined as

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}.$$

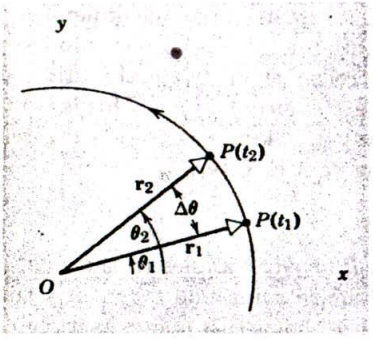


Fig. 11–4 The reference line r ($= OP$), fixed in the body of Figs. 11–1 and 11–3, is displaced through angle $\Delta\theta$ ($= \theta_2 - \theta_1$) in time Δt ($= t_2 - t_1$).

The *instantaneous angular speed* ω is defined as the limit approached by this ratio as Δt approaches zero:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}. \qquad (11\text{–}1)$$

For a rigid body all radial lines fixed in it perpendicular to the axis of rotation rotate through the same angle in the same time, so that the angular speed ω about this axis is the same for each particle in the body. Thus ω is characteristic of the body as a whole. Angular speed has the dimensions of an inverse time (T^{-1}); its units are commonly taken to be radians/sec or rev/sec.

If the angular speed of P is not constant, then the particle has an angular acceleration. Let ω_1 and ω_2 be the instantaneous angular speeds at the times t_1 and t_2 respectively; then the *average angular acceleration* $\bar{\alpha}$ of the particle P is defined as

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}.$$

* The radian is a pure number, having no physical dimension since it is the ratio of two lengths. Since the circumference of a circle of radius r is $2\pi r$, there are 2π radians in a complete circle, that is, $\theta = 2\pi r/r = 2\pi$. Therefore 2π radians = 360°, π radians = 180°, and 1 radian $\cong$ 57.3°.

The *instantaneous angular acceleration* is the limit of this ratio as Δt approaches zero, or

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}. \tag{11–2}$$

Because ω is the same for all particles in the rigid body, it follows from Eq. 11–2 that α must be the same for each particle and thus α, like ω, is a characteristic of the body as a whole. Angular acceleration has the dimensions of an inverse time squared (T^{-2}); its units are commonly taken to be radians/sec^2 or rev/sec^2.

The rotation of a particle (or a rigid body) *about a fixed axis* has a formal correspondence to the translational motion of a particle (or a rigid body) *along a fixed direction*. The kinematical variables are θ, ω, and α in the first case and x, v, and a in the second. These quantities correspond in pairs: θ to x, ω to v, and α to a. Note that the angular quantities differ dimensionally from the corresponding linear quantities by a length factor. Note, too, that all six quantities may be treated as scalars in this special case. For example, a particle at any instant can be moving in one direction or the other along its straight-line motion, corresponding to a positive or a negative value for v; similarly a particle at any instant can be rotating in one direction or another about its fixed axis, corresponding to a positive or a negative value for ω.

When, in translational motion, we remove the restriction that the motion be along a straight line and consider the general case of motion in three dimensions along a curved path, the linear variables x, v, and a reveal themselves as the scalar components of the kinematic vectors **r**, **v**, and **a**. In Section 11–4, we shall see to what extent the rotational kinematic variables reveal themselves as vectors when we remove the restriction of a fixed axis of rotation.

11–3 Rotation with Constant Angular Acceleration

For translational motion of a particle or a rigid body along a fixed direction, such as the x-axis, we have seen (in Chapter 3) that the simplest type of motion is that in which the acceleration a is zero. The next simplest type corresponds to a = a constant (other than zero); for this motion we derived the equations of Table 3–1, which connect the kinematic variables x, v, a, and t in all possible combinations.

For the rotational motion of a particle or a rigid body around a fixed axis the simplest type of motion is that in which the angular acceleration α is zero (such as uniform circular motion). The next simplest type of motion, in which α = a constant (other than zero), corresponds exactly to linear motion with a = a constant (other than zero). As before, we can derive four equations linking the four kinematic variables θ, ω, α, and t in all possible combinations. The student can either derive these angular equations by the methods used to derive the linear equations (see Example 2) or he may write them down at once by substituting corresponding

angular quantities for the linear quantities in the linear equations.

We list both sets of equations in Table 11–1, having chosen $x_0 = 0$ and $\theta_0 = 0$ in these relations for simplicity. Here ω_0 is the angular speed at the time $t = 0$. The student should check these equations dimensionally before verifying them. Both sets of equations hold not only for particles but also for rigid bodies.

Table 11-1

MOTION WITH CONSTANT LINEAR OR ANGULAR ACCELERATION

	Translational Motion (Fixed Direction)	Rotational Motion (Fixed Axis)	
(3–12)	$v = v_0 + at$	$\omega = \omega_0 + \alpha t$	(11–3)
(3–14)	$x = \frac{v_0 + v}{2} t$	$\theta = \frac{\omega_0 + \omega}{2} t$	(11–4)
(3–15)	$x = v_0 t + \frac{1}{2} a t^2$	$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$	(11–5)
(3–16)	$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha\theta$	(11–6)

For the angular quantities, we arbitrarily select one of the two possible directions of rotation about the fixed axis as the direction in which θ is increasing. From Eq. 11–1 ($\omega = d\theta/dt$) we see that if θ is increasing with time ω is positive. Similarly, from Eq. 11–2 ($\alpha = d\omega/dt$), we see that if ω is increasing with time α is positive. There are corresponding sign conventions for the linear quantities.

▶ **Example 1.** A grindstone has a constant angular acceleration α of 3.0 radians/sec^2. Starting from rest a line, such as OP in Fig. 11–5, is horizontal. Find (*a*) the angular displacement of the line OP (and hence of the grindstone) and (*b*) the angular speed of the grindstone 2.0 sec later.

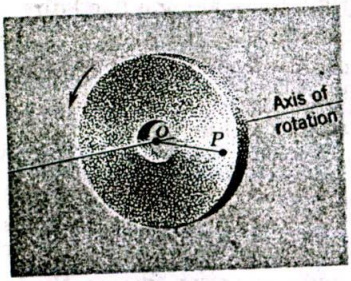


Fig. 11–5 Example 1. The line OP is attached to a grindstone rotating as shown about an axis through O that is fixed in the reference frame of the observer.

(*a*) α and t are given; we wish to find θ. Hence, we use Eq. 11–5,

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2.$$

At $t = 0$, we have $\omega = \omega_0 = 0$ and $\alpha = 3.0$ radians/sec^2. Therefore, after 2.0 sec,

$$\theta = (0)(2.0 \text{ sec}) + \tfrac{1}{2}(3.0 \text{ radians/sec}^2)(2.0 \text{ sec})^2 = 6.0 \text{ radians} = 0.96 \text{ rev}.$$

(*b*) α and t are given; we wish to find ω. Hence we use Eq. 11–3

$$\omega = \omega_0 + \alpha t,$$

and

$$\omega = 0 + (3.0 \text{ radians/sec}^2)(2.0 \text{ sec}) = 6.0 \text{ radians/sec}.$$

Using Eq. 11–6 as a check, we have

$$\omega^2 = \omega_0^2 + 2\alpha\theta,$$

$$\omega^2 = 0 + (2)(3.0 \text{ radians/sec}^2)(6.0 \text{ radians}) = 36 \text{ radians}^2/\text{sec}^2,$$

$$\omega = 6.0 \text{ radians/sec}.$$

Example 2. Derive the equation $\omega = \omega_0 + \alpha t$ for constant angular acceleration.

(*a*) Starting from the definition of angular acceleration,

$$\alpha = \frac{d\omega}{dt},$$

we have

$$\alpha\, dt = d\omega$$

or

$$\int \alpha\, dt = \int d\omega.$$

But α is a constant, so that

$$\alpha \int dt = \int d\omega.$$

If at $t = 0$ we call the angular speed ω_0, then

$$\alpha \int_0^t dt = \int_{\omega_0}^{\omega} d\omega$$

or

$$\alpha t = \omega - \omega_0$$

and

$$\omega = \omega_0 + \alpha t.$$

(*b*) We can also derive the result by making use of the fact that the average acceleration equals the instantaneous acceleration when the acceleration is constant. The average acceleration is

$$\bar{\alpha} = \frac{\omega - \omega_0}{t - t_0}.$$

For constant acceleration we have $\alpha = \bar{\alpha}$. Letting $t_0 = 0$, we obtain

$$\alpha = \frac{\omega - \omega_0}{t}$$

or

$$\omega = \omega_0 + \alpha t.$$

The student should compare this derivation with that of the corresponding linear relation $v = v_0 + at$ in Section 3–8. ◀

11–4 Rotational Quantities as Vectors

The linear displacement, velocity, and acceleration are vectors. The corresponding angular quantities *may* be vectors also, for in addition to a magnitude we must also specify a direction for them, namely the direction of the axis of rotation in space. Because we considered rotation only about a fixed axis, we were able to treat θ, ω, and α as scalar quantities. If

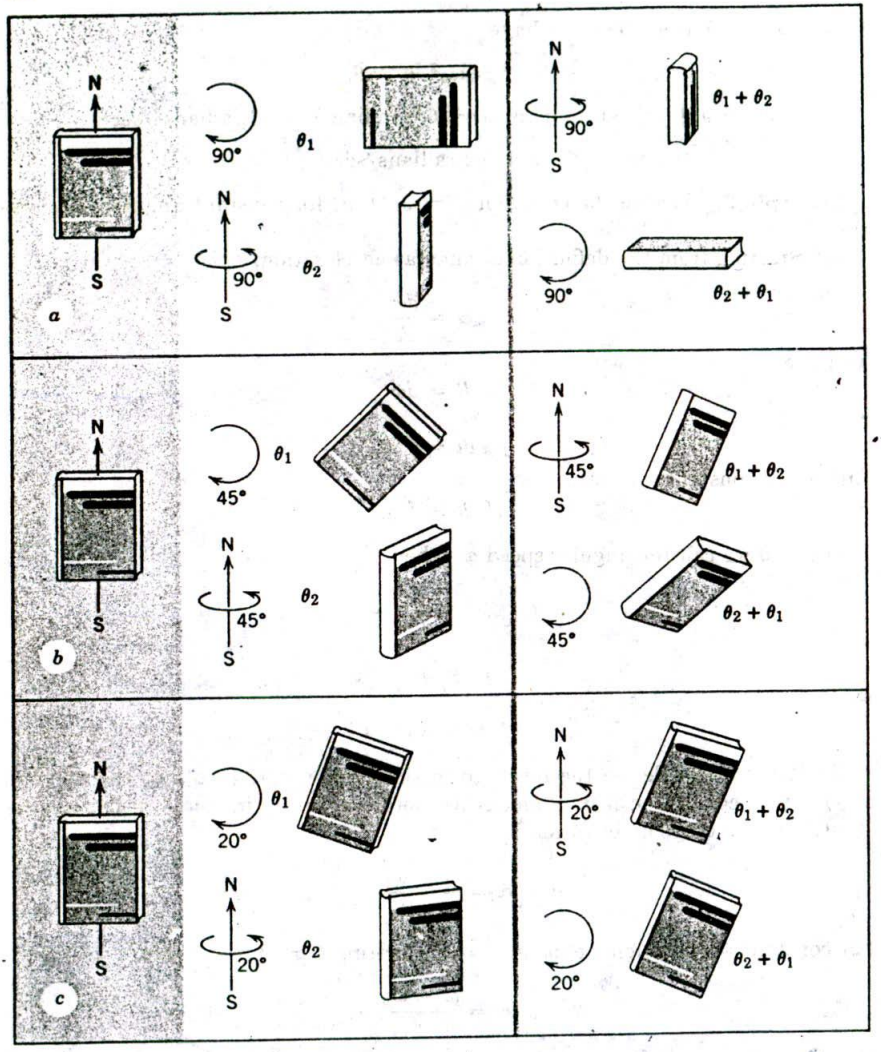


Fig. 11–6 (*a*) A book rotated θ_1 (90° as shown about a vertical axis) and then θ_2 (90° as shown about a north-south axis) has a different final orientation than if rotated first through θ_2 and then θ_1. This property is called the noncommutivity of finite angles under addition: $\theta_1 + \theta_2 \neq \theta_2 + \theta_1$. (*b*) The middle group is the same except that the angular displacements are smaller, being 45°. Although the final orientations still differ, they are much nearer each other. (*c*) The lower group repeats the experiment for 20° displacements. We see here that $\theta_1 + \theta_2 \cong \theta_2 + \theta_1$. As $\theta_1, \theta_2 \to 0$, the final positions approach each other. Finite angles under addition tend to commute as the angles become very small. Infinitesimal angles *do* commute under addition, making it possible to treat them as vectors.

the direction of the axis changes, however, we can no longer avoid the question "are rotational quantities vectors?" We can find out only by seeing whether or not they obey the laws of vector addition.

Let us discuss first the angular displacement θ. The magnitude of the angular displacement of a body is the angle through which the body turns. Angular displacements, however, are *not* vectors because they do *not* add like vectors. For example, give two successive rotations θ_1 and θ_2 to a book which initially lies flat on a table. Let rotation θ_1 be a 90° clockwise turn about a vertical axis through the center of the book as we view it from above. Let θ_2 be a 90° clockwise turn about a north-south axis through the center of the book as we view it looking north. In one case, apply operation θ_1 first and then θ_2. In the other case, apply operation θ_2 first and then θ_1. The student should try this for himself. Now, if angular displacements are vector quantities, they must add like vectors. In particular, they must obey the law of vector addition $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 = \boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$, which tells us that the order in which we add vectors does not affect their sum. This law fails for finite angular displacements (see exercise above and also Fig. 11–6*a*). Hence finite angular displacements are *not* vector quantities.

Suppose that instead of 90° rotations we had made 3° rotations. The result of $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$ would still differ from the result of $\boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$, but the difference would be much smaller. In fact, as the two angular displacements are made smaller, the difference between the two sums disappears rapidly (Fig. 11–6*b*,*c*). If the angular displacements are made infinitesimal, the order of addition no longer affects the result. Hence *infinitesimal angular displacements are vectors.*

Quantities defined in terms of infinitesimal angular displacements may themselves be vectors. For example, the angular velocity is $\boldsymbol{\omega} = d\boldsymbol{\theta}/dt$. Since $d\boldsymbol{\theta}$ is a vector and dt a scalar, the quotient is a vector. Therefore the angular velocity is a vector. In Fig. 11–7*a*, for example, we represent the

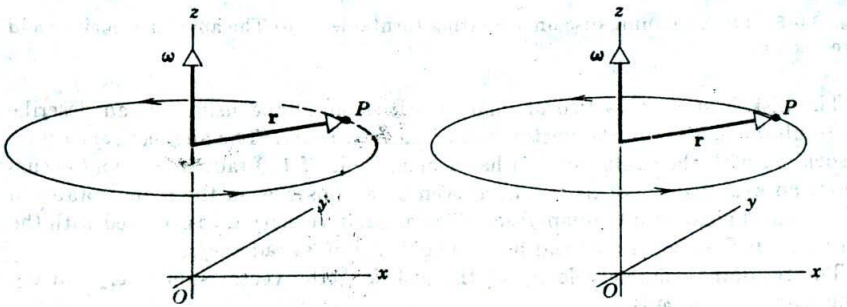


Fig. 11–7 The angular velocity $\boldsymbol{\omega}$ of (*a*) a rotating rigid body and (*b*) a rotating particle, about a fixed axis.

angular velocity $\boldsymbol{\omega}$ of the rotating rigid body by an arrow drawn along the axis of rotation; in Fig. 11–7*b* we represent the rotation of a particle (such as P in Fig. 11–7*a*) about a fixed axis in just the same way. The length of the arrow is made proportional to the magnitude of the angular velocity. The sense of the rotation determines the direction in which the arrow points along the axis. By convention, if the fingers of the *right-hand* curl around the axis in the direction of rotation of the body, the extended thumb points along the direction of the angular velocity vector. For the rigid body of Fig. 11–1, therefore, the angular velocity vector will be in the positive z-direction. In Fig. 11–3, $\boldsymbol{\omega}$ will be perpendicular to the page pointing up out of the page, corresponding to the counterclockwise rotation. The angular velocity of the turntable of a phonograph is a vector pointing down. Notice that nothing moves in the direction of the angular velocity vector. The vector represents the angular velocity of the rotational motion taking place in the plane perpendicular to it.

Angular acceleration is also a vector quantity. This follows from the definition $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$, in which $d\boldsymbol{\omega}$ is a vector and dt a scalar. Later we shall encounter other rotational quantities that are vectors, such as torque and angular momentum.

▶ **Example 3.** A disk spins on a horizontal shaft mounted in bearings, with an angular speed ω_1 of 100 radians/sec as in Fig. 11–8*a*. The entire disk and shaft assembly are placed on a turntable rotating about a vertical axis at $\omega_2 = 30.0$ radians/sec, counterclockwise as we view it from above. Describe the rotation of the disk as seen by an observer in the room.

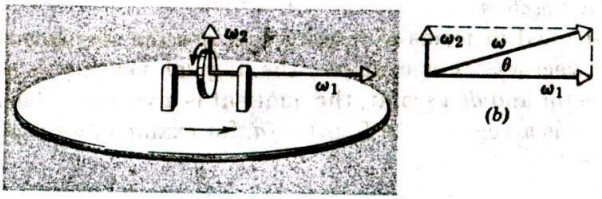


Fig. 11–8 (*a*) A spinning disc on a rotating turntable. (*b*) The angular velocities add like vectors.

The disk is subject to two angular velocities simultaneously; we can describe its resultant motion by the vector sum of these vectors. The angular velocity $\boldsymbol{\omega}_1$ associated with the shaft rotation has a magnitude of 100 radians/sec and occurs about an axis that is not fixed but, as seen by an observer in the room, rotates in a horizontal plane at 30 radians/sec. The angular velocity $\boldsymbol{\omega}_2$ associated with the turntable is fixed vertically and has a magnitude of 30 radians/sec.

The resultant angular velocity of the disk $\boldsymbol{\omega}$ is the vector sum of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. The magnitude of $\boldsymbol{\omega}$ is

$$\omega = \sqrt{\omega_1^2 + \omega_2^2} = \sqrt{(100 \text{ radians/sec})^2 + (30.0 \text{ radians/sec})^2}$$
$$= 104 \text{ radians/sec}.$$

The direction of $\boldsymbol{\omega}$ is not fixed in our observer's reference frame but rotates at the same angular rate as the turntable. The vector $\boldsymbol{\omega}$ does not lie in the horizontal plane but points above it by an angle θ (see Fig. 11–8*b*), where

$$\theta = \tan^{-1} \omega_2/\omega_1 = \tan^{-1} (30.0 \text{ radians/sec})/(100 \text{ radians/sec})$$

$$= \tan^{-1} 0.300 = 16.7°$$

We can describe the motion of the disk as a simple rotation about this new axis (whose direction in our observer's reference frame is changing with time as described above) at an angular rate of 104 radians/sec. How would the situation change if the direction of rotation of the disk, or of the turntable, were changed? ◀

11–5 Relation between Linear and Angular Kinematics for a Particle in Circular Motion—Scalar Form

In Sections 4–4 and 4–5 we discussed the linear velocity and acceleration of a particle moving in a circle. When a rigid body rotates about a fixed axis, every particle in the body moves in a circle. Hence we can describe the motion of such a particle either in linear variables or in angular variables. The relation between the linear and angular variables enables us to pass back and forth from one description to another and is very useful.

Consider a particle at P in the rigid body, a distance r from the axis through O. This particle moves in a circle of radius r as the body rotates, as in Fig. 11–9*a*. The reference position is Ox. The particle moves through a distance s along the arc when the body rotates through an angle θ, such that

$$s = \theta r \tag{11–7}$$

where θ is in radians.

Differentiating both sides of this equation with respect to the time, and

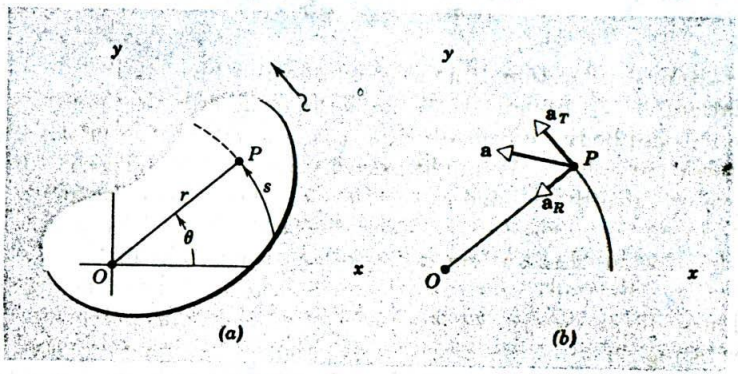


Fig. 11–9 (*a*) A rigid body rotates about a fixed axis through O perpendicular to the page. The point P sweeps out an arc s which subtends an angle θ. (*b*) The acceleration $\mathbf{a}$ of point P has components $\mathbf{a}_T$ (tangential) where $a_T = \alpha r$ and $\mathbf{a}_R$ (radial) where $a_R = v^2/r = \omega^2 r$ (ω = angular speed).

noting that r is constant, we obtain

$$\frac{ds}{dt} = \frac{d\theta}{dt} r.$$

But ds/dt is the linear speed of the particle at P and $d\theta/dt$ is the angular speed ω of the rotating body so that

$$v = \omega r. \tag{11-8}$$

This is a relation between the *magnitudes* of the linear velocity and the angular velocity; the linear speed of a particle in circular motion is the product of the angular speed and the distance r of the particle from the axis of rotation.

Differentiating Eq. 11–8 with respect to the time, we have

$$\frac{dv}{dt} = \frac{d\omega}{dt} r.$$

But dv/dt is the magnitude of the *tangential* component of acceleration of the particle (see Section 4–5) and $d\omega/dt$ is the magnitude of the angular acceleration of the rotating body, so that

$$a_T = \alpha r. \tag{11-9}$$

Hence the magnitude of the tangential component of the linear acceleration of a particle in circular motion is the product of the magnitude of the angular acceleration and the distance r of the particle from the axis of rotation.

We have seen that the *radial* component of acceleration is v^2/r for a particle moving in a circle. This can be expressed in terms of angular speed by use of Eq. 11–8. We have

$$a_R = \frac{v^2}{r} = \omega^2 r. \tag{11-10}$$

The resultant acceleration of point P is shown in Fig. 11–9*b*.

Equations 11–7 through 11–10 enable us to describe the motion of one point on a rigid body rotating about a fixed axis *either* in angular variables *or* in linear variables. We might ask why we need the angular variables when we are already familiar with the equivalent linear variables. The answer is that the angular description offers a distinct advantage over the linear description when various points on the same rotating body must be considered. Different points on the same rotating body do not have the same linear displacement, speed, or acceleration, but *all* points on a rigid body rotating about a fixed axis do have the same *angular* displacement, speed, or acceleration at any instant. By the use of angular variables we can describe the motion of the body as a whole in a simple way.

▶ **Example 4.** If the radius of the grindstone of Example 1 is 0.50 meter, calculate (*a*) the linear or tangential speed of a particle on the rim, (*b*) the tangential

acceleration of a particle on the rim, and (*c*) the centripetal acceleration of a particle on the rim, at the end of 2.0 sec.

We have $\alpha = 3.0$ radians/sec^2, $\omega = 6.0$ radians/sec after 2.0 sec, and $r = 0.50$ meter. Then,

(*a*)
$$\begin{aligned} v &= \omega r \\ &= (6.0 \text{ radians/sec})(0.50 \text{ meter}) \\ &= 3.0 \text{ meter/sec} \qquad \text{(linear speed)}; \end{aligned}$$

(*b*)
$$\begin{aligned} a_T &= \alpha r \\ &= (3.0 \text{ radians/sec}^2)(0.50 \text{ meter}) \\ &= 1.5 \text{ meters/sec}^2 \qquad \text{(tangential acceleration)}; \end{aligned}$$

(*c*)
$$\begin{aligned} a_R &= v^2/r = \omega^2 r \\ &= (6.0 \text{ radians/sec})^2(0.50 \text{ meter}) \\ &= 18 \text{ meters/sec}^2 \qquad \text{(centripetal acceleration)}. \end{aligned}$$

(*d*) Are the results the same for a particle halfway in from the rim, that is, at $r = 0.25$ meter?

The *angular* variables are the same for this point as for a point on the rim. That is, once again

$$\alpha = 3.0 \text{ radians/sec}^2, \qquad \omega = 6.0 \text{ radians/sec}.$$

But now $r = 0.25$ meter, so that for this particle

$$v = 1.5 \text{ meters/sec}, \qquad a_T = 0.75 \text{ meter/sec}^2, \qquad a_R = 9.0 \text{ meters/sec}^2. \blacktriangleleft$$

11–6 Relation between Linear and Angular Kinematics for a Particle in Circular Motion—Vector Form

The student should notice that the relations deduced in the previous section are relations between *scalar* quantities, both the linear and angular variables being expressed in scalar form. Let us now use vector methods, making an analysis essentially like that of Section 4–5 except that we now introduce the angular variables. This will illustrate, for a familiar case, the more general approach and prepare the way for situations in which vector methods are essential.

Figure 11–10*a* shows a particle P, rotating about a fixed axis through the origin, at times t and $t + \Delta t$. The particle moves in a circle of constant radius r; beyond

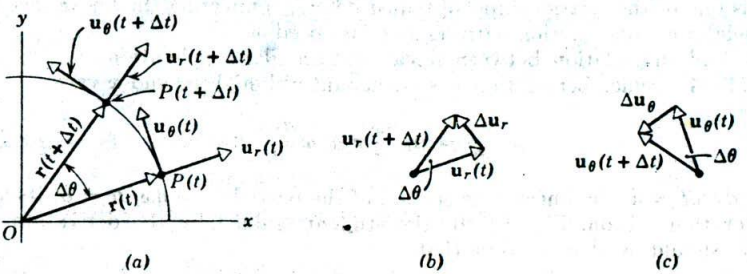


Fig. 11–10 (*a*) The particle P rotates through an angle $\Delta\theta$ in time Δt. The unit vectors, in polar coordinates, are shown at each point. (*b*) The change in $\mathbf{u}_r$; note that $\Delta\mathbf{u}_r$, as $\Delta\theta \to 0$, points in the direction of $\mathbf{u}_\theta$. (*c*) The change in $\mathbf{u}_\theta$; note that $\Delta\mathbf{u}_\theta$, as $\Delta\theta \to 0$, points in the direction of $-\mathbf{u}_r$.

this there are no restrictions on its motion and in general $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ may have values that vary as the particle moves. We can express the restriction to a constant radius by

$$\mathbf{r} = \mathbf{u}_r r, \tag{11–11}$$

in which $\mathbf{u}_r$ is a unit vector in the direction of $\mathbf{r}$.

Differentiating Eq. 11–11, remembering that r (but not $\mathbf{r}$ or $\mathbf{u}_r$, since their directions change) is a constant, we have

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{u}_r}{dt} r. \tag{11–12}$$

Now $d\mathbf{r}/dt$ is $\mathbf{v}$, the linear velocity of the particle. To evaluate $d\mathbf{u}_r/dt$, consider Fig. 11–10*b*, which shows the unit vector $\mathbf{u}_r$ for two different positions of P, corresponding to a rotation through a (small) angle $\Delta\theta$. Using the definition of angular measure in radians we obtain the *magnitude* of the (vector) change $\Delta\mathbf{u}_r$ in $\mathbf{u}_r$ from

$$\Delta u_r = (1)\,\Delta\theta,$$

in which the factor (1) reminds us that the two unit vectors in Fig. 11–10*b* have unit length. The above equation will be correct if $\Delta\theta$ is small enough so that we can neglect the difference between the chord and the arc in the small triangle in Fig. 11–10*b*. The change in $\mathbf{u}_r$ is a vector, $\Delta\mathbf{u}_r$, whose magnitude is given by the above equation; its *direction*, again assuming that $\Delta\theta$ is small enough, is given by the unit vector $\mathbf{u}_\theta$. This follows because, if $\Delta\mathbf{u}_r$ in Fig. 11–10*b* is translated to point P in Fig. 11–10*a*, we see that, as $\Delta\theta \to 0$, it points in the direction of $\mathbf{u}_\theta$. Thus we find

$$\Delta\mathbf{u}_r \cong \mathbf{u}_\theta \Delta\theta.$$

Dividing by Δt and allowing Δt to approach zero, we have

$$\frac{d\mathbf{u}_r}{dt} = \mathbf{u}_\theta \frac{d\theta}{dt} = \mathbf{u}_\theta \dot{\omega}. \tag{11–13}$$

Substituting these results into Eq. 11–12 yields, then,

$$\mathbf{v} = \mathbf{u}_\theta \omega r. \tag{11–14a}$$

The scalar relationship that corresponds to this is

$$v = \omega r \tag{11–14b}$$

and is one of the relationships, obtained before, connecting the linear speed v of a particle in circular motion with its angular speed ω.

To find the relation between linear and angular acceleration we differentiate Eq. 11–14*a*, remembering that r is a constant although $\mathbf{u}_\theta$ and ω vary. We have

$$\frac{d\mathbf{v}}{dt} = \mathbf{u}_\theta \frac{d\omega}{dt} r + \omega \frac{d\mathbf{u}_\theta}{dt} r. \tag{11–15}$$

Now $d\mathbf{v}/dt = \mathbf{a}$, the linear acceleration of the particle and $d\omega/dt = \alpha$, its angular acceleration. From Fig. 11–10*c* the student, guided by the derivation of Eq. 11–13, should be able to prove that

$$\frac{d\mathbf{u}_\theta}{dt} = -\mathbf{u}_r \omega. \tag{11–16}$$

The minus sign comes in because when we translate $\Delta\mathbf{u}_\theta$ in Fig. 11–10*c* to point P, we see that, as $\Delta\theta \to 0$, it points radially inward, in the direction opposite to $\mathbf{u}_r$.

Making these substitutions into Eq. 11–15 yields

$$\mathbf{a} = \mathbf{u}_\theta \alpha r - \mathbf{u}_r \omega^2 r. \tag{11–17}$$

Thus, as we know from Section 4–5, $\mathbf{a}$ has a radial (or centripetal) component $\mathbf{a}_R$ and a tangential component $\mathbf{a}_T$. Their magnitudes, from Eq. 11–17, are

$$a_T = \alpha r \tag{11–18a}$$

and (using Eq. 11–14*b*)

$$a_R = \omega^2 r = v^2/r. \tag{11–18b}$$

This last is the familiar result derived in Section 4–4. In Supplementary Topic I we derive the relations between the linear and angular kinematic variables for a particle free to move in a plane but not restricted to circular motion. Equations 11–14*a* and 11–17 will prove to be special cases of the more general relationships derived there.

Equations 11–14*a* and 11–17 are relations between the linear kinematic variables in vector form and the angular kinematic variables in scalar form. We should be able to derive relationships in which *each* set of variables is expressed in vector form. Let us do so now. This will be especially useful in cases where the axis of rotation is not fixed.

Figure 11–11 shows the vectors $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}_T$, $\mathbf{a}_R$, $\boldsymbol{\omega}$, and $\boldsymbol{\alpha}$ for the rotating particle of Fig. 11–7*b*. The angular quantities are on the axis of rotation, pointing in the direction given by the right-hand rule of page 25. We declare—and shall prove—that the relationships we seek are

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{11–19}$$

and

$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_R, \tag{11–20a}$$

in which

$$\mathbf{a}_T = \boldsymbol{\alpha} \times \mathbf{r} \quad \text{and} \quad \mathbf{a}_R = \boldsymbol{\omega} \times \mathbf{v} \tag{11–20b}$$

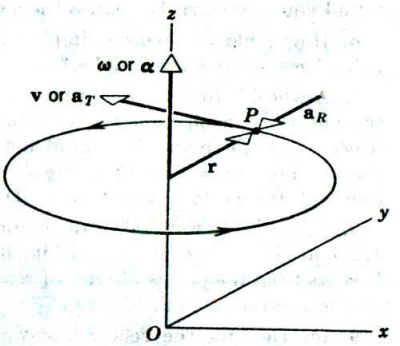


Fig. 11–11 The directions of the vectors $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}_T$, $\mathbf{a}_R$, $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ for a particle rotating in a circle about the z-axis.

In Section 2–4 (which the student should now reread) we defined the vector product of two vectors. If $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, then the *magnitude* of $\mathbf{c}$ is $ab \sin \phi$, where ϕ is the angle between $\mathbf{a}$ and $\mathbf{b}$. In applying this part of the definition to Eqs. 11–19 and 11–20 we note (see Fig. 11–11) that $\boldsymbol{\omega}$ and $\mathbf{r}$, $\boldsymbol{\omega}$ and $\mathbf{v}$, and $\boldsymbol{\alpha}$ and $\mathbf{r}$ are each mutually perpendicular; thus the angle ϕ for each of these three pairs of vectors is 90°. In Eq. 11–19 we have, for magnitudes

$$v = \omega r \sin 90° = \omega r,$$

which is exactly Eq. 11–14*b*. In Eqs. 11–20*b* we have, for magnitudes

$$a_R = \omega v = \omega(\omega r) = \omega^2 r$$

and

$$a_T = \alpha r.$$

These relations agree with Eqs. 11–18*b* and *a* exactly.

It remains to be seen whether *directions* are correctly given by Eqs. 11–19 and 11–20*b*. For the vector product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, Fig. 2–12 shows that the direction of

$\mathbf{c}$ is found by sweeping $\mathbf{a}$ into $\mathbf{b}$ through the (smaller) angle between them with the fingers of the right hand; the extended right thumb then points in the direction of $\mathbf{c}$. The student can readily check that, in Fig. 11–11, the directions of the vectors $\mathbf{v}$, $\mathbf{a}_T$ and $\mathbf{a}_R$ are indeed correctly given by Eqs. 11–19 and 11–20*b*.

QUESTIONS

1. In Section 11–1 it was stated that, in general, six variables are required to locate a rigid body with respect to a particular reference frame. How many variables are required to locate the body of Fig. 11–2 with respect to the x-y frame shown in that figure? If this number is not six, account for the difference.

2. An irregular body is free to rotate about its center of mass which is placed at the origin of a reference frame. How would you specify its orientation?

3. Could the angular quantities θ, ω, and α be expressed in terms of degrees instead of radians in the kinematical equations?

4. Explain why the radian measure of angle is equally satisfactory for all systems of units. Is the same true for degrees?

5. What is the angular speed of the second hand of a watch? Of the minute hand? Could you conveniently express the motion of these hands in terms of linear variables?

6. How could you express simply the relationship between the angular velocities of a pair of gears which are coupled?

7. A wheel is rotating about an axis through its center perpendicular to the plane of the wheel. Consider a point on the rim. When the wheel rotates with *constant angular velocity*, does the point have a radial acceleration? A tangential acceleration? When the wheel rotates with *constant angular acceleration*, does the point have a radial acceleration? A tangential acceleration? Do the magnitudes of these accelerations change?

8. Suppose you were asked to determine the equivalent distance traveled by a phonograph needle in playing, say, a 12-in., $33\frac{1}{3}$ rpm record. What information do you need? Discuss from the points of view of reference frames (*a*) fixed in the room, and (*b*) fixed on the rotating record.

9. (*a*) Describe the vector that would represent the angular velocity of the earth rotating about its axis. (*b*) Describe the vector that would represent the angular velocity of the earth rotating about the sun.

10. It is convenient to picture rotational vectors as lying along the axis of rotation. Is there any reason why they could not be pictured as merely parallel to the axis, but located anywhere? Recall that we are free to slide a displacement vector along its own direction or translate it sideways without changing its value.

11. In a centrifuge particles will tend to separate from the fluid in which they are suspended if their density (mass/volume) differs from that of the fluid. Discuss the dynamical principles upon which the operation of a centrifuge depends. View the situation from both an inertial (laboratory) frame and a noninertial (rotating) frame.

12.* A marksman stands at the center of a merry-go-round firing at a target fixed to a post on its outer perimeter. How, if at all, must the man take into account the (constant) angular velocity of the merry-go-round if he is to hit the target? What if the positions of marksman and target were reversed?

13.* A man on a merry-go-round rotating at constant angular velocity ω releases a cake of ice that he had been holding fixed to the merry-go-round at a radial distance r_0 from the center. Describe the motion of the ice in the reference frame of (*a*) a ground observer and (*b*) the man on the merry-go-round. Neglect frictional forces but describe all other forces.

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* See Supplementary Topic 1.

14.* A man on a rotating merry-go-round kicks a cake of ice outward along a radial line. What is its subsequent motion as seen by an observer (*a*) on the merry-go-round, and (*b*) on the ground? Assume that frictional forces may be neglected.

PROBLEMS

1. A phonograph record on a turntable rotates at 33 rev/min. What is the linear speed of a point on the record at the needle at (*a*) the beginning and (*b*) the end of the recording? The distances of the needle from the turntable axis are $5\frac{7}{8}$ and $2\frac{7}{8}$ in., respectively, at these two positions.

2. A solar day is the time interval between two successive appearances of the sun overhead at a given longitude, that is, the time for one complete rotation of the earth relative to the sun. A sidereal day is the time for one complete rotation of the earth relative to the fixed stars, that is, the time interval between two successive overhead observations of a fixed direction in the heavens called the vernal equinox. (*a*) Show that there is exactly one less (mean) solar day in a year than there are (mean) sidereal days in a year. (*b*) If the (mean) solar day is exactly 24 hr, how long is a (mean) sidereal day?

3. One method of measuring the speed of light makes use of a rotating toothed wheel. A beam of light passes through a slot at the outside edge of the wheel, as in Fig. 11–12, travels to a distant mirror, and returns to the wheel just in time to pass through the next slot in the wheel. One such toothed wheel has a radius of 5.0 cm and 500 teeth at its edge. Measurements taken when the mirror was 500 meters from the wheel indicated a speed of light of 3.0×10^5 km/sec. (*a*) What was the (constant) angular speed of the wheel? (*b*) What was the linear speed of a point on its edge?

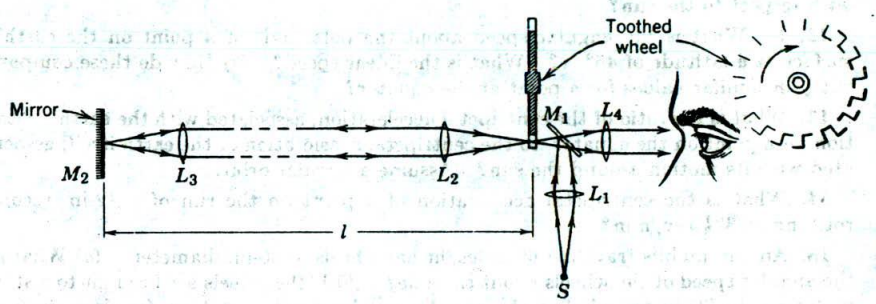


Fig. 11–12

4. The angular speed of an automobile engine is increased from 1200 rpm to 3000 rpm in 12 sec. (*a*) What is its angular acceleration, assuming it to be uniform? (*b*) How many revolutions does the engine make during this time?

5. A phonograph turntable rotating at 78 rev/min slows down and stops in 30 sec after the motor is turned off. (*a*) Find its (uniform) angular acceleration. (*b*) How many revolutions did it make in this time?

6. A wheel has a constant angular acceleration of 3.0 radians/sec^2. In a 4.0-sec interval it turns through an angle of 120 radians. Assuming the wheel started from rest, how long had it been in motion at the start of this 4.0-sec interval?

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* See Supplementary Topic I.

7. A heavy flywheel rotating on its axis is slowing down because of friction in its bearings. At the end of the first minute its angular velocity is 0.90 of its angular velocity ω_0 at the start. Assuming constant frictional forces, find its angular velocity at the end of the second minute.

8. The angle turned through by the flywheel of a generator during a time interval t is given by

$$\theta = at + bt^3 - ct^4,$$

where a, b, and c are constants. What is the expression for its angular acceleration?

9. A body moves in the x–y plane such that $x = R\cos\omega t$ and $y = R\sin\omega t$. Here x and y are the coordinates of the body, t is the time, and R and ω are constants. (*a*) Eliminate t between these equations to find the equation of the curve in which the body moves. What is this curve? What is the meaning of the constant ω? (*b*) Differentiate the equations for x and y with respect to the time to find the x and y components of the velocity of the body, v_x and v_y. Combine v_x and v_y to find the magnitude and direction of v. Describe the motion of the body. (*c*) Differentiate v_x and v_y with respect to the time to obtain the magnitude and direction of the resultant acceleration.

10. A wheel rotates with an angular acceleration α given by

$$\alpha = 4at^3 - 3bt^2,$$

where t is the time and a and b are constants. If the wheel has an initial angular speed ω_0, write the equations for (*a*) the angular speed and (*b*) the angle turned through as functions of time.

11. The earth's orbit about the sun, although elliptical, is almost a circle. (*a*) Calculate the angular velocity of the earth (regarded as a particle) about the sun and its average linear speed in its orbit. (*b*) What is the centripetal acceleration of the earth with respect to the sun?

12. (*a*) What is the angular speed about the polar axis of a point on the earth's surface at a latitude of 45° N? What is the linear speed? (*b*) How do these compare with the similar values for a point at the equator?

13. What is the ratio of the centripetal acceleration, associated with the earth's rotation, of a point on the equator, to the centripetal acceleration of the earth itself, associated with its motion around the sun? Assume a circular orbit.

14. What is the centripetal acceleration of a point on the rim of a 12-in. record rotating at $33\frac{1}{3}$ rev/min?

15. An automobile traveling 60 miles/hr has wheels of 30-in. diameter. (*a*) What is the angular speed of the wheels about the axle? (*b*) If the wheels are brought to a stop uniformly in 30 turns, what is the angular acceleration? (*c*) How far does the car advance during this braking period?

16. (*a*) If an airplane propeller of radius 5.0 ft rotates at 2000 rpm and the airplane is propelled at a ground speed of 100 miles/hr, what is the speed of a point on the tip of the propeller? (*b*) What kind of path does this point traverse?

17. What is the angular speed of a car rounding a circular turn of radius 360 ft at 30 miles/hr?

18. A rigid body, starting at rest, rotates about a fixed axis with constant angular acceleration α. Consider a particle a distance r from the axis. (*a*) Express the radial acceleration and the tangential acceleration of this particle in the body in terms of α, r and the time t. (*b*) If the resultant acceleration of the particle at some instant makes an angle of 60° with the tangential acceleration, what total angle has the body turned through to that instant?

19. Derive Eq. 11–20 by differentiation of Eq. 11–19.

20.* An insect of mass 8.0×10^{-2} gm walks out with a constant speed of 1.6 cm/sec along a radial line marked on a phonograph turntable rotating at a constant angular velocity of $33\frac{1}{3}$ rev/min. (*a*) Find the velocity and acceleration of the insect as seen by the ground observer when the insect is 12 cm from the axis of rotation. (*b*) What must the minimum coefficient of friction be to allow the insect to get all the way to the edge of the turntable (radius = 16 cm) without slipping? (*c*) In what way are the results of (*a*) and (*b*) changed if the turntable is given an angular acceleration that increases its angular velocity?

21.* A virus particle in solution in a centrifuge is, at a particular moment, at a distance of 6.5 cm from the axis of rotation and moving radially outward at a relatively constant speed. The centrifuge is rotating at 55,000 rev/min. Discuss the motion quantitatively, giving the magnitude of all forces and accelerations as viewed from a reference frame (*a*) rotating with the centrifuge tube and (*b*) fixed in the laboratory.

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* See Supplementary Topic I.

Rotational Dynamics I

CHAPTER 12

12–1 Introduction

In Chapter 11 we considered the kinematics of rotation. In this chapter, following the pattern of our study of translational motion, we study the causes of rotation, a subject called *rotational dynamics*. Rotating systems are made up of particles and we have already learned how to apply the laws of classical mechanics to the motion of particles. For this reason rotational dynamics should contain no features that are fundamentally new. In the same way rotational kinematics contained no basic new features, the rotational parameters θ, ω, and α being related to corresponding translational parameters x, v, and a for the particles that make up the rotating system. As in Chapter 11, however, it is very useful to recast the concepts of translational motion into a new form, especially chosen for its convenience in describing rotating systems.

We restricted our kinematical studies in Chapter 11 to a single but important special case, the rotation of a rigid body about an axis that is fixed in the reference frame in which we make our measurements. In studying rotational dynamics we start from a more fundamental point of view, that of a single particle viewed from an inertial reference frame. Later we shall generalize to systems of many particles, including the special case of a rigid body rotating about a fixed axis. In Chapter 13 we shall discuss the rotation of rigid bodies about axes that are *not* fixed in an inertial reference frame.

12–2 Torque Acting on a Particle

In translational motion we associate a *force* with the *linear acceleration* of a body. In rotational motion, what quantity shall we associate with

the *angular acceleration* of a body? It cannot be simply force because, as experiment with a heavy revolving door teaches us, a given force (vector) can produce various angular accelerations of the door depending on where the force is applied and how it is directed; a force applied to the hinge line cannot produce any angular acceleration, whereas a force of given magnitude applied at right angles to the door at its outer edge produces a maximum acceleration.

We shall call the rotational analogue of force *torque* and shall now define it for the special case of a single particle observed from an inertial reference frame. Later we shall extend the torque concept to systems of particles (including rigid bodies) and shall show that torque is intimately associated with angular acceleration.

If a force $\mathbf{F}$ acts on a single particle at a point P whose position with respect to the origin O of the inertial reference frame is given by the displacement vector $\mathbf{r}$ (Fig. 12–1), the *torque* $\boldsymbol{\tau}$ acting on the particle *with respect to the origin* O is defined as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \tag{12–1}$$

Torque is a vector quantity. Its magnitude is given by

$$\tau = rF \sin \theta, \tag{12–2a}$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$; its direction is normal to the plane formed by $\mathbf{r}$ and $\mathbf{F}$. The sense is given by the right-hand rule for the vector product of two vectors, namely, one swings $\mathbf{r}$ into $\mathbf{F}$ through the smaller angle between them with the curled fingers of the right hand; the direction of the extended thumb then gives the direction of $\boldsymbol{\tau}$.

Torque has the same dimensions as force times distance, or in terms of our assumed fundamental dimensions, M, L, T, it has the dimensions

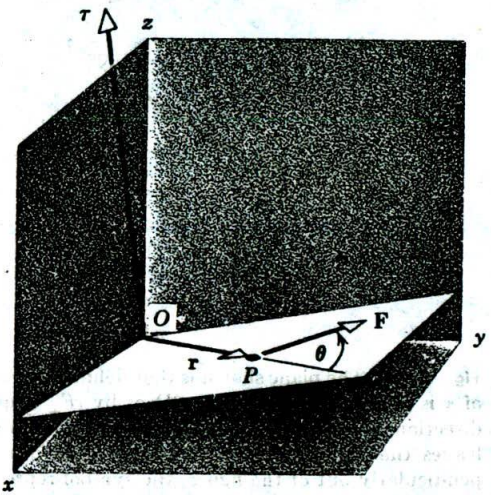


Fig. 12–1 A force $\mathbf{F}$ is applied to a particle P, displaced $\mathbf{r}$ relative to the origin. The force vector makes an angle θ with the radius vector $\mathbf{r}$. The torque $\boldsymbol{\tau}$ about O is shown. Its direction is perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{F}$ with the sense given by the right-hand rule.

ML^2T^{-2}. These are the same as the dimensions of work. However, torque and work are very different physical quantities. Torque is a vector and work is a scalar, for example. The unit of torque may be the nt-meter or lb-ft, among other possibilities.

Notice (Eq. 12–1) that the torque produced by a force depends not only on the magnitude and on the direction of the force but also on the point of application of the force relative to the origin, that is, on the vector **r**. In particular, when particle P is at the origin, so that the line of action of **F** passes through the origin, **r** is zero and the torque $\boldsymbol{\tau}$ about the origin is zero.

We can also write the magnitude of $\boldsymbol{\tau}$ (Eq. 12–2*a*) either as

$$\tau = (r \sin \theta) F = Fr_\perp, \tag{12–2b}$$

or as

$$\tau = r(F \sin \theta) = rF_\perp, \tag{12–2c}$$

in which, as Fig. 12–2*a* shows, $r_\perp$ ($= r \sin \theta$) is the component of **r** at right angles to the line of action of **F**, and $F_\perp$ ($= F \sin \theta$) is the component of **F**

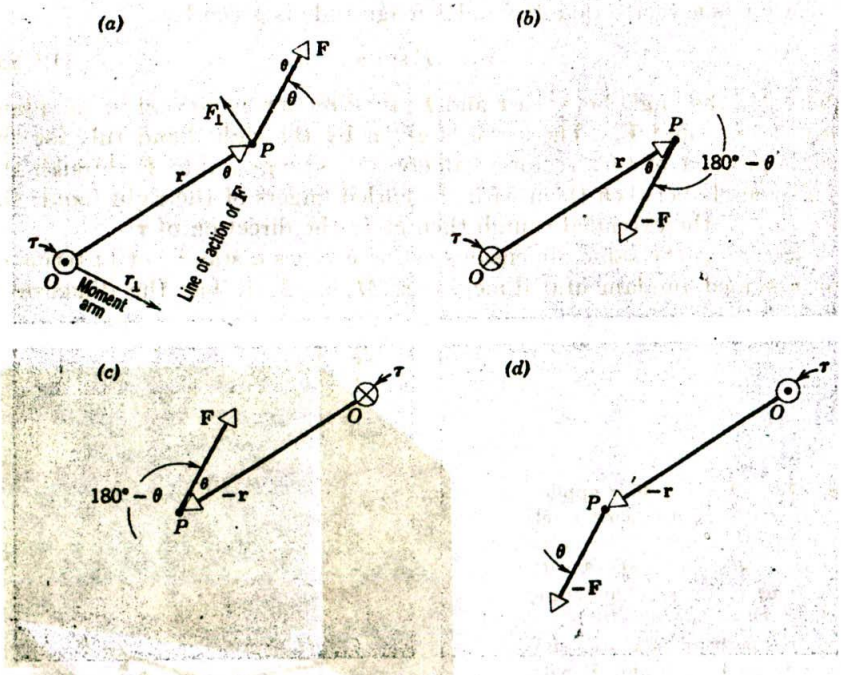


Fig. 12–2 The plane shown is that defined by **r** and **F** in Fig. 12–1. (*a*) The magnitude of $\boldsymbol{\tau}$ is given by $Fr_\perp$ (Eq. 12–2*b*) or by $rF_\perp$ (Eq. 12–2*c*). (*b*) Reversing **F** reverses the direction of $\boldsymbol{\tau}$. (*c*) Reversing **r** reverses the direction of $\boldsymbol{\tau}$. (*d*) Reversing **F** and **r** leaves the direction of $\boldsymbol{\tau}$ unchanged. The directions of $\boldsymbol{\tau}$ are represented by $\odot$ (perpendicularly out of the figure, the symbol representing the tip of an arrow) and by $\otimes$ (perpendicularly into the figure, the symbol representing the tail of an arrow).

at right angles to **r**. Torque is often called the *moment of force* and $r_\perp$ in Eq. 12–2*b* is called the *moment arm*. Equation 12–2*c* shows that only the component of **F** perpendicular to **r** contributes to the torque. In particular, when θ equals 0 or 180°, there is no perpendicular component ($F_\perp = F \sin\theta = 0$); then the line of action of the force passes through the origin and the moment arm $r_\perp$ about the origin is also zero. In this case both Eq. 12–2*b* and Eq. 12–2*c* show that the torque τ is zero.

If, as in Fig. 12–2*b*, we reverse the direction of **F**, the magnitude of τ remains unchanged but the direction of τ is reversed. Similarly, if, as in Fig. 12–2*c*, we reverse **r**, thereby changing the point of application of **F**, the magnitude of τ remains unchanged but the direction of τ is again reversed.

If, as in Fig. 12–2*d*, we reverse *both* **r** and **F**, then both the magnitude and the direction of τ remain unchanged. These results follow formally from the facts that: (1) $\sin\theta = \sin(180° - \theta)$, so that Eq. 12–2*a* for the magnitude of τ is unaffected; (2) reversing the direction of *one* vector in a vector product (either **r** *or* **F**) reverses the direction of the product; and (3) reversing the directions of *both* vectors in a vector product (both **r** *and* **F**) leaves the direction of the product unchanged. The student should verify the directions of τ shown in Fig. 12–2, using the right-hand rule.

12–3 Angular Momentum of a Particle

We have found *linear momentum* to be useful in dealing with the translational motion of single particles or of systems of particles (including rigid bodies). For example, linear momentum is conserved in collisions. For a single particle the linear momentum is $\mathbf{p} = m\mathbf{v}$ (Eq. 9–11); for a system of particles it is $\mathbf{P} = M\mathbf{v}_{cm}$ (Eq. 9–15) in which M is the total system mass and $\mathbf{v}_{cm}$ is the velocity of the center of mass. In rotational motion, what is the analog of linear momentum? We call it *angular momentum* and we define it below for the special case of a single particle. Later we shall broaden the definition to include systems of particles and shall show that angular momentum, as we define it, is as useful a concept in rotational motion as linear momentum is in translational motion.

Consider a particle of mass m and linear momentum **p** at a position **r** relative to the origin O of an inertial reference frame (Fig. 12–3). We define the *angular momentum* **l** of the particle *with respect to the origin O* to be

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}. \tag{12–3}$$

Angular momentum is a vector. Its magnitude is given by

$$l = rp \sin\theta, \tag{12–4a}$$

where θ is the angle between **r** and **p**; its direction is normal to the plane formed by **r** and **p**. The sense is given by the right-hand rule, namely, one swings **r** into **p**, through the smaller angle between them, with the curled fingers of the right hand; the extended right thumb then points in the direction of **l**.

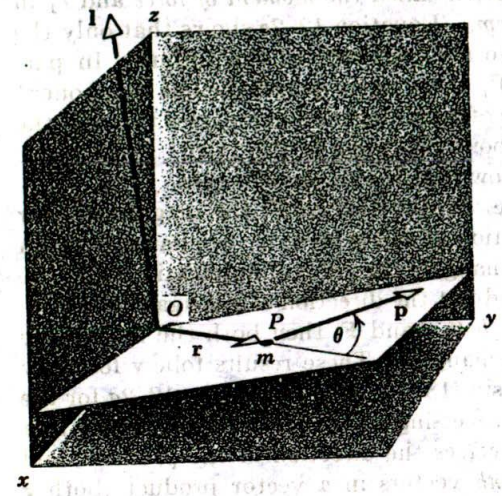


Fig. 12–3 A particle of mass m is at point P displaced $\mathbf{r}$ relative to the origin, and has linear momentum $\mathbf{p}$. The vector $\mathbf{p}$ makes an angle θ with the radius vector $\mathbf{r}$. The angular momentum $\mathbf{l}$ of the particle with respect to origin O is shown. Its direction is perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{p}$ with the sense given by the right-hand rule.

We can also write the magnitude of $\mathbf{l}$ either as

$$l = (r \sin \theta)\, p = p r_\perp, \tag{12–4b}$$

or as

$$l = r(p \sin \theta) = r p_\perp, \tag{12–4c}$$

in which $r_\perp$ ($= r \sin \theta$) is the component of $\mathbf{r}$ at right angles to the line of action of $\mathbf{p}$ and $p_\perp$ ($= p \sin \theta$) is the component of $\mathbf{p}$ at right angles to $\mathbf{r}$. Angular momentum is often called *moment of* (linear) *momentum* and $r_\perp$ in Eq. 12–4*b* is often called the *moment arm*. Equation 12–4*c* shows that only the component of $\mathbf{p}$ perpendicular to $\mathbf{r}$ contributes to the angular momentum. When the angle θ between $\mathbf{r}$ and $\mathbf{p}$ is 0 or 180°, there is no perpendicular component ($p_\perp = p \sin \theta = 0$); then the line of action of $\mathbf{p}$ passes through the origin and $r_\perp$ is also zero. In this case both Eqs. 12–4*b* and 12–4*c* show that the angular momentum l is zero.

We now derive an important relation between torque and angular momentum. We have seen that $\mathbf{F} = d(m\mathbf{v})/dt = d\mathbf{p}/dt$ for a particle. Let us take the vector product of $\mathbf{r}$ with both sides of this equation, obtaining

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}.$$

But $\mathbf{r} \times \mathbf{F}$ is the torque, or moment of a force, about O. We can then write

$$\boldsymbol{\tau} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}. \tag{12–5}$$

Next we differentiate Eq. 12–3 and obtain

$$\frac{d\mathbf{l}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}).$$

Now the derivative of a vector product is taken in the same way as the derivative of an ordinary product, except that we must not change the order of the terms. We have

$$\frac{d\mathbf{l}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}.$$

But $d\mathbf{r}$ is the vector displacement of the particle in the time dt so that $d\mathbf{r}/dt$ is the instantaneous velocity $\mathbf{v}$ of the particle. Also, $\mathbf{p}$ equals $m\mathbf{v}$, so that the equation can be rewritten as

$$\frac{d\mathbf{l}}{dt} = (\mathbf{v} \times m\mathbf{v}) + \mathbf{r} \times \frac{d\mathbf{p}}{dt}.$$

Now $\mathbf{v} \times m\mathbf{v} = 0$, because the vector product of two parallel vectors is zero. Therefore,

$$\frac{d\mathbf{l}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}. \tag{12–6}$$

Inspection of Eqs. 12–5 and 12–6 shows that

$$\boldsymbol{\tau} = d\mathbf{l}/dt, \tag{12–7}$$

which states that *the time rate of change of the angular momentum of a particle is equal to the torque acting on it.* This result is the rotational analog of Eq. 9–12, which stated that the time rate of change of the linear momentum of a particle is equal to the force acting on it, that is, that $\mathbf{F} = d\mathbf{p}/dt$.

Equation 12–7, like all vector equations, is equivalent to three scalar equations, namely

$$\tau_x = (dl/dt)_x \qquad \tau_y = (dl/dt)_y, \qquad \tau_z = (dl/dt)_z. \tag{12–8}$$

Hence, the x-component of the applied torque is given by the x-component of the change with time of the angular momentum. Similar results hold for the y- and z-directions.

▶ **Example 1.** A particle of mass m is released from rest at point a in Fig. 12–4, falling parallel to the (vertical) y-axis. (*a*) Find the torque acting on m at any time t, with respect to origin O. (*b*) Find the angular momentum of m at any time t, with respect to this same origin. (*c*) Show that the relation $\boldsymbol{\tau} = d\mathbf{l}/dt$ (Eq. 12–7) yields a correct result when applied to this familiar problem.

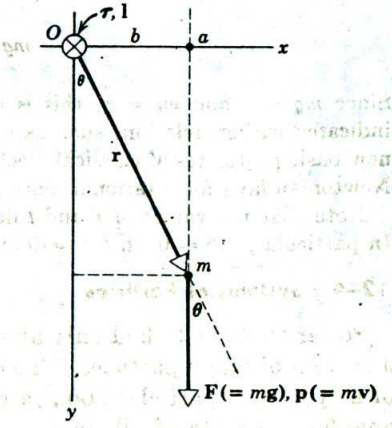


Fig. 12–4 A particle of mass m drops vertically from point a. The torque and the angular momentum about O are directed perpendicularly into the figure, as shown by the symbol ⊗ at O.

(*a*) The torque is given by Eq. 12–1 or $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, its magnitude being given by

$$\tau = rF \sin \theta.$$

In this example $r \sin \theta = b$ and $F = mg$ so that

$$\tau = mgb = \text{a constant}.$$

Note that the torque is simply the product of the force (mg) times the moment arm (b). The right-hand rule shows that $\boldsymbol{\tau}$ is directed perpendicularly into the figure.

(*b*) The angular momentum is given by Eq. 12–3 or $\mathbf{l} = \mathbf{r} \times \mathbf{p}$, its magnitude being given by

$$l = rp \sin \theta.$$

In this example $r \sin \theta = b$ and $p = mv = m(gt)$ so that

$$l = mgbt.$$

The right-hand rule shows that $\mathbf{l}$ is directed perpendicularly into the figure, which means that $\mathbf{l}$ and $\boldsymbol{\tau}$ are parallel vectors. The vector $\mathbf{l}$ changes with time in magnitude only, its direction always remaining the same in this case.

(*c*) Since $d\mathbf{l}$, the change in $\mathbf{l}$, and $\boldsymbol{\tau}$ are parallel we can replace the vector relation $\boldsymbol{\tau} = d\mathbf{l}/dt$ by the scalar relation

$$\tau = dl/dt.$$

Using the expressions for τ and l from (*a*) and (*b*) above we have

$$mgb = \frac{d}{dt}(mgbt) = mgb,$$

which is an identity. Thus the relation $\boldsymbol{\tau} = d\mathbf{l}/dt$ yields correct results in this simple case. Indeed, if we cancel the constant b out of the first two terms above and if we substitute for gt the equivalent quantity v, we have

$$mg = \frac{d}{dt}(mv).$$

Since $mg = F$ and $mv = p$, this is the familiar result $F = dp/dt$. Thus, as we indicated earlier, relations such as $\boldsymbol{\tau} = d\mathbf{l}/dt$, though often vastly useful, are not new basic postulates of classical mechanics but are rather the reformulation of the Newtonian laws for rotational motion.

Note that the values of τ and l depends on our choice of origin, that is, on b. In particular, if $b = 0$, then $\tau = 0$ and $l = 0$. ◀

12–4 Systems of Particles

So far we have talked only about single particles. Let us now consider a system of many particles. To calculate the total angular momentum $\mathbf{L}$ of a system of particles about a given point, we must add vectorially the angular momenta of all the individual particles of the system about this same point. For a system containing n particles we have, then,

$$\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2 + \cdots + \mathbf{l}_n = \sum_{i=1}^{i=n} \mathbf{l}_i,$$

in which the (vector) sum is taken over all particles in the system.

As time goes on, the total angular momentum **L** of the system about a fixed reference point (which we choose, as in our basic definition of **l** in Eq. 12–3, to be the origin of an inertial reference frame) may change. This change, $d\mathbf{L}/dt$, can arise from two sources: (1) torques exerted on the particles of the system by internal forces between the particles and (2) torques exerted on the particles of the system by external forces.

If Newton's third law holds in its so-called strong form, that is, if the forces between any two particles not only are equal and opposite but are also directed along the line joining the two particles, then the total internal torque is zero because the torque resulting from each internal action-reaction force pair is zero.

Hence the first source contributes nothing. For our reference point, therefore, only the second source remains, and we can write

$$\boldsymbol{\tau}_{\text{ext}} = d\mathbf{L}/dt, \tag{12–9}$$

where $\boldsymbol{\tau}_{\text{ext}}$ stands for the *sum* of *all* the *external* torques acting on the system. In words, *the time rate of change of the total angular momentum of a system of particles about the origin of an inertial reference frame is equal to the sum of the* external *torques acting on the system.* Later, for convenience, in situations in which no confusion is likely to arise, we shall drop the subscript on $\boldsymbol{\tau}_{\text{ext}}$.

Equation 12–9 is the generalization of Eq. 12–7 to many particles. When we have only one particle, there are no internal forces or torques. This relation (Eq. 12–9) holds whether the particles that make up the system are in motion relative to each other or whether they have fixed spatial relationships, as in a rigid body.

Equation 12–9 is the rotational analog of Eq. 9–17

$$\mathbf{F}_{\text{ext}} = d\mathbf{P}/dt \tag{9–17}$$

which tells us that for a system of particles (rigid body or not) the resultant external *force* acting on the system equals the time rate of change of the *linear momentum* of the system.

As we have derived it, Eq. 12–9 holds when $\boldsymbol{\tau}$ and **L** are measured with respect to the origin of an inertial reference frame. We may well ask whether it still holds if we measure these two vectors with respect to an arbitrary point (a particular particle, say) in the moving system. In general, such a point would move in a complicated way as the body or system of particles translated, tumbled, and changed its configuration and Eq. 12–9 would *not* apply to such a reference point. However, if the reference point is chosen to be the center of mass of the system, even though this point is not fixed in our reference frame, then Eq. 12–9 *does* hold.* This is another remarkable property of the center of mass. Thus we can separate the general motion of a system of particles into the trans-

* See Problem 8 of this chapter and Robert A. Becker, *Introduction to Theoretical Mechanics*, McGraw-Hill Book Co., 1954, Section 8–4.

lational motion of its center of mass (Eq. 9–17) and rotational motion about its center of mass (Eq. 12–9).

12–5 Kinetic Energy of Rotation and Rotational Inertia

We shall now confine our attention to an important special case of a system of particles, a rigid body. In a rigid body the particles in the system always maintain the same positions with respect to one another. In studying the rotation of a rigid body we shall consider first the special case, often encountered, in which the axis of rotation is fixed* in an inertial reference frame. Later we shall investigate more general systems and motions.

Let us now imagine a rigid body rotating with angular speed ω about an axis that is fixed in a particular inertial frame, as in Fig. 11–1. Each particle in such a rotating body has a certain amount of kinetic energy. A particle of mass m at a distance r from the axis of rotation moves in a circle of radius r with an angular speed ω about this axis and has a linear speed $v = \omega r$. Its kinetic energy therefore is $\frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$. The total kinetic energy of the body is the sum of the kinetic energies of its particles.

If the body is rigid, as we assume in this section, ω is the same for all particles. The radius r may be different for different particles. Hence the total kinetic energy K of the rotating body can be written as

$$K = \tfrac{1}{2}(m_1r_1^2 + m_2r_2^2 + \cdots)\omega^2 = \tfrac{1}{2}(\Sigma\, m_ir_i^2)\omega^2.$$

The term $\Sigma\, m_ir_i^2$ is the sum of the products of the masses of the particles by the squares of their respective distances from the axis of rotation. If we denote this quantity by I, then

$$I = \Sigma\, m_ir_i^2 \tag{12–10}$$

is called the *rotational inertia*, or moment of inertia, of the body with respect to the particular axis of rotation. Note that *the rotational inertia of a body depends on the particular axis about which it is rotating* as well as on the shape of the body and the manner in which its mass is distributed. Rotational inertia has the dimensions ML^2 and is usually expressed in kg-m^2 or slug-ft^2.

In terms of rotational inertia we can now write the kinetic energy of the rotating rigid body as

$$K = \tfrac{1}{2}I\omega^2. \tag{12–11}$$

This is analogous to the expression for the kinetic energy of translation of a

* As stated in Section 12–4, we can separate the general motion of a system of particles into translational motion of its center of mass and rotational motion about its center of mass. Hence the considerations of this chapter apply also to rotations about an axis that is *not* fixed in an inertial reference frame, provided (1) the axis passes through the center of mass and (2) the moving axis always has the same direction in space, that is, the axis at one instant is parallel to the axis at any other instant. Although we shall often refer to a "fixed axis" in what follows we shall always mean to include this special case of a moving axis.

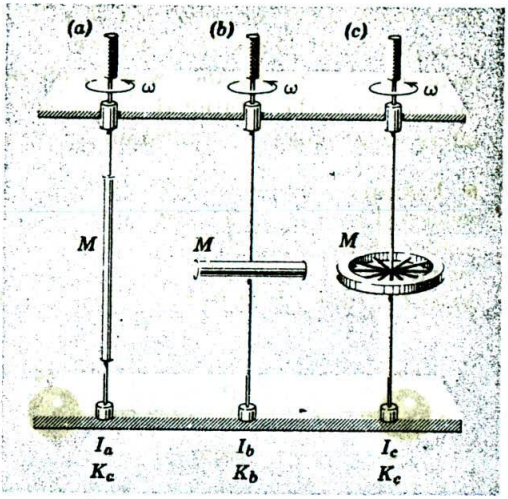


Fig. 12–5 An experiment to show that $I_a < I_b < I_c$. The three lead bodies have the same mass M but the mass is distributed differently about the axis of rotation.

body, $K = \frac{1}{2}Mv^2$. We have already seen that the angular speed ω is analogous to the linear speed v. Now we see that the rotational inertia I is analogous to the mass, or the translational inertia M. Although the mass of a body does not depend on its location, the rotational inertia of a body does depend on the axis about which it is rotating.

We should understand that the rotational kinetic energy given by Eq. 12–11 is simply the sum of the ordinary translational kinetic energy of all the parts of the body and not a new kind of energy. Rotational kinetic energy is simply a convenient way of expressing the kinetic energy for a rotating rigid body.

Equations 12–10 and 12–11 show that the rotational energy of a body, for a given angular speed ω, depends not only on the mass of the body but also on the way that mass is distributed around the axis of rotation. The experiment shown in Fig. 12–5 makes this convincing. The figure shows three identical aluminum shafts, to each of which is attached a body of mass M, made of lead. In (*a*) the mass is very close to the shaft so that the quantities r_i in Eq. 12–10 for the particles that make up the body are relatively small; in (*b*) the particles are, on the average, farther from the shaft and in (*c*), in which the body is a flywheel, they are still farther, corresponding to still larger values of r_i.

Now let us twist each handle until each shaft, starting from rest, is spinning at the same measured angular speed ω. We know from experience that we shall need to do relatively little work on shaft (*a*), somewhat more work on shaft (*b*), and still more on shaft (*c*). In fact, if we were not certain which body was attached to which shaft we could label the shafts with confidence using this technique. Since the work done on each shaft is equal to the kinetic energy $\frac{1}{2}I\omega^2$ imparted to each shaft, the experimental result, that $K_a < K_b < K_c$ when each shaft has the same angular speed ω,

leads to the conclusion that $I_a < I_b < I_c$. This is just what we expect from the defining equation for I (Eq. 12–10). We shall see in Section 12–6 that just as the mass M, which we may call the translational inertia, is a measure of the resistance a body offers to a change in its translational motion, so I, the rotational inertia, is a measure of the resistance a body offers to a change in its rotational motion about a given axis.

▶ **Example 2.** Consider a body consisting of two spherical masses of 5.0 kg each connected by a light rigid rod 1.0 meter long (Fig. 12–6). Treat the spheres as point particles and neglect the mass of the rod. Determine the rotational inertia (or moment of inertia) of the body (*a*) about an axis normal to it through its center at C, and (*b*) about an axis normal to it through one sphere.

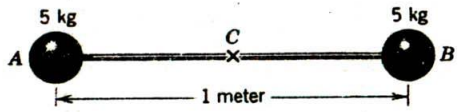


Fig. 12–6 Example 2. Calculating the rotational inertia of a dumbell.

(*a*) If the axis is normal to the page through C, we have

$$I_C = \Sigma m_i r_i^2 = m_a r_a^2 + m_b r_b^2$$
$$= (5.0 \text{ kg})(0.50 \text{ meter})^2 + (5.0 \text{ kg})(0.50 \text{ meter})^2 = 2.5 \text{ kg-m}^2.$$

(*b*) If the axis is normal to the page through A or B, we have

$$I_A = m_a r_a^2 + m_b r_b^2 = (5.0 \text{ kg})(0 \text{ meter})^2 + (5.0 \text{ kg})(1.0 \text{ meter})^2 = 5.0 \text{ kg-m}^2,$$

$$I_B = m_a r_a^2 + m_b r_b^2 = (5.0 \text{ kg})(1.0 \text{ meter})^2 + (5.0 \text{ kg})(0 \text{ meter})^2 = 5.0 \text{ kg-m}^2.$$

Hence the rotational inertia of this rigid dumbbell model is twice as great about an axis through an end as it is about an axis through the center. ◀

For a body that is not composed of discrete point masses but is instead a continuous distribution of matter, the summation in $I = \Sigma m_i r_i^2$ becomes an integration. We imagine the body to be subdivided into infinitesimal elements, each of mass dm. We let r be the distance from such an element to the axis of rotation. Then the rotational inertia is obtained from

$$I = \int r^2 \, dm, \tag{12–12}$$

where the integral is taken over the whole body. The procedure by which the summation Σ of a discrete distribution is replaced by the integral $\int$ for a continuous distribution is the same as that discussed for the center of mass in Section 9–1.

For bodies of irregular shape the integrals may be hard to evaluate. For bodies of simple geometrical shape the integrals are relatively easy when an axis of symmetry is chosen as the axis of rotation.

Let us illustrate the procedure for an annular cylinder (or ring) about the cylinder axis (Fig. 12–7). The most convenient mass element is an infinitesimally thin cylindrical shell of radius r, thickness dr, and length L.

If the density of the material, that is, the mass per unit volume, is called ρ, then

$$dm = \rho\, dV,$$

where dV is the volume of the cylindrical shell of mass dm. We have

$$dV = (2\pi r\, dr)L,$$

so that

$$dm = 2\pi L\rho r\, dr.$$

Then the rotational inertia about the cylinder axis is

$$I = \int r^2\, dm = 2\pi L \int_{R_1}^{R_2} \rho r^3\, dr.$$

Here R_1 is the radius of the inner cylindrical wall and R_2 is the radius of the outer cylindrical wall.

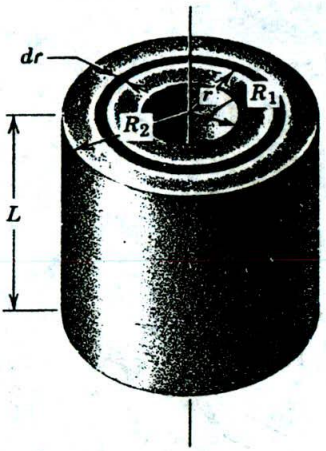


Fig. 12–7 Calculating the rotational inertia of an annular cylinder.

If this body did not have a uniform constant density, we would have to know how ρ depends on r before we could carry out the integration. Let us assume for simplicity that the density is uniform. Then

$$I = 2\pi L\rho \int_{R_1}^{R_2} r^3\, dr = 2\pi L\rho\, \frac{R_2^4 - R_1^4}{4} = \rho\pi(R_2^2 - R_1^2)L\, \frac{R_2^2 + R_1^2}{2}.$$

The mass M of the annular cylinder is the product of its density ρ by its volume $\pi(R_2^2 - R_1^2)L$, or

$$M = \rho\pi(R_2^2 - R_1^2)L.$$

The rotational inertia of the *annular cylinder* (or ring) of mass M, inner radius R_1 and outer radius R_2, is therefore

$$I = \tfrac{1}{2}M(R_1^2 + R_2^2)$$

about the cylinder axis.

If the inner radius is zero, R_1 equals zero, and we have a solid cylinder (or disk). Then

$$I = \tfrac{1}{2}MR^2$$

about the cylinder axis, where R is the radius of the solid cylinder of mass M.

A *hoop* can be thought of as a very thin-walled hollow cylinder. In that case

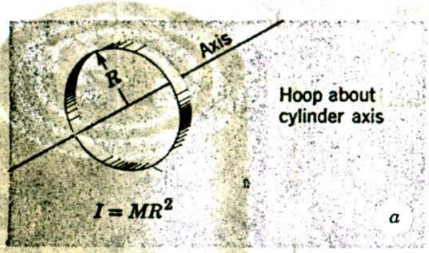
$$R_1 \cong R_2 \cong R,$$

and

$$I = MR^2$$

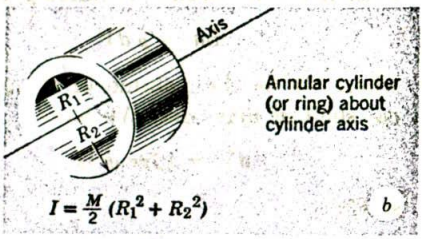
is the moment of a hoop of mass M and radius R about the cylinder axis.

Table 12–1



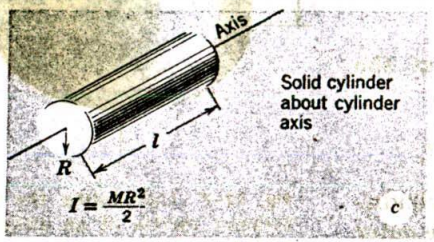
Hoop about cylinder axis

$I = MR^2$ *a*



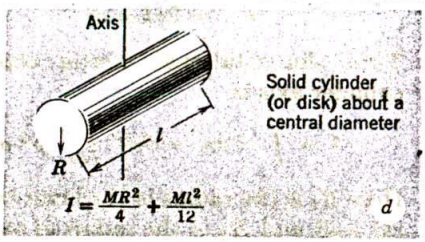
Annular cylinder (or ring) about cylinder axis

$I = \frac{M}{2}(R_1^2 + R_2^2)$ *b*



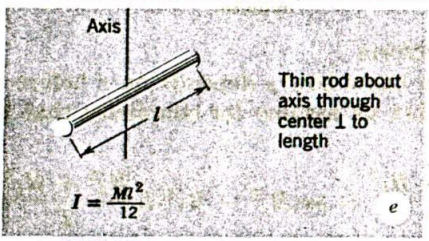
Solid cylinder about cylinder axis

$I = \frac{MR^2}{2}$ *c*



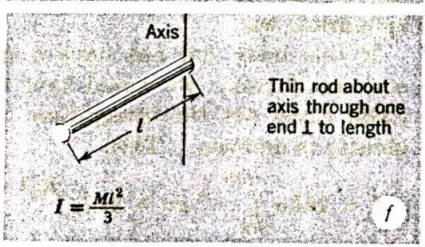
Solid cylinder (or disk) about a central diameter

$I = \frac{MR^2}{4} + \frac{Ml^2}{12}$ *d*



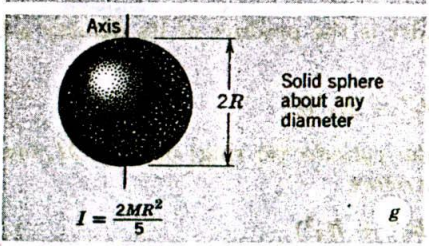
Thin rod about axis through center ⊥ to length

$I = \frac{Ml^2}{12}$ *e*



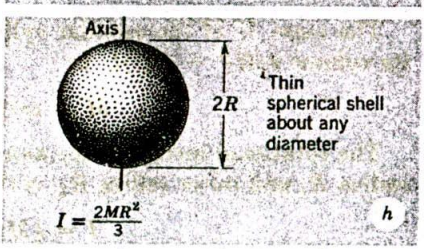
Thin rod about axis through one end ⊥ to length

$I = \frac{Ml^2}{3}$ *f*



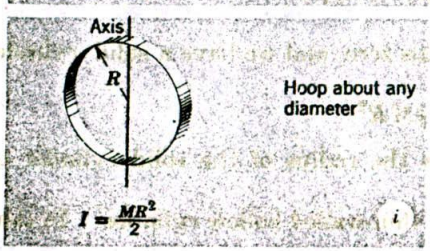
Solid sphere about any diameter

$I = \frac{2MR^2}{5}$ *g*



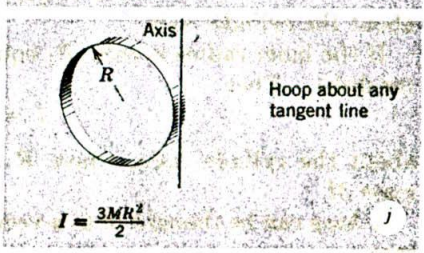
Thin spherical shell about any diameter

$I = \frac{2MR^2}{3}$ *h*



Hoop about any diameter

$I = \frac{MR^2}{2}$ *i*



Hoop about any tangent line

$I = \frac{3MR^2}{2}$ *j*

This result for the thin hoop is obvious since every mass point in the hoop is the same distance R from the central axis. For the solid cylinder (or disk) having the *same mass* as the hoop, the rotational inertia (or moment of inertia) would naturally be less than that of the hoop, because most of the cylinder (or disk) is less than a distance R from the axis.

The rotational inertias about certain axes of some common solids (of uniform density) are listed in Table 12–1. Each of these results can be derived by integration in a manner similar to that of our illustration. The total mass of the body is denoted by M in each equation.

There is a simple and very useful relation between the rotational inertia I of a body about any axis and its rotational inertia I_{cm} with respect to a parallel axis *through the center of mass*. If M is the total mass of the body and h the distance between the two axes, the relation is

$$I = I_{cm} + Mh^2. \tag{12–13}$$

The proof of this relation (parallel-axis theorem) follows. Let C be the center of mass of the arbitrarily shaped body whose cross section is shown in Fig. 12–8. The center of mass has coordinates x_{cm} and y_{cm}. We choose the x–y plane to include C, so that z_{cm} equals zero. Consider an axis through C at right angles to the plane of the paper and another axis parallel to it through P at $(x_{cm} + a)$ and $(y_{cm} + b)$. The distance between the axes is $h = \sqrt{a^2 + b^2}$. Then the square of the distance of a particle from the axis through C is $x_i^2 + y_i^2$, where x_i and y_i measure the coordinates of a mass element m_i relative to the axis through C. The square of its distance from an axis through P is $(x_i - a)^2 + (y_i - b)^2$. Hence the rotational inertia about an axis through P is

$$\begin{aligned} I &= \Sigma\, m_i[(x_i - a)^2 + (y_i - b)^2] \\ &= \Sigma\, m_i(x_i^2 + y_i^2) - 2a\,\Sigma\, m_i x_i - 2b\,\Sigma\, m_i y_i + (a^2 + b^2)\,\Sigma\, m_i. \end{aligned}$$

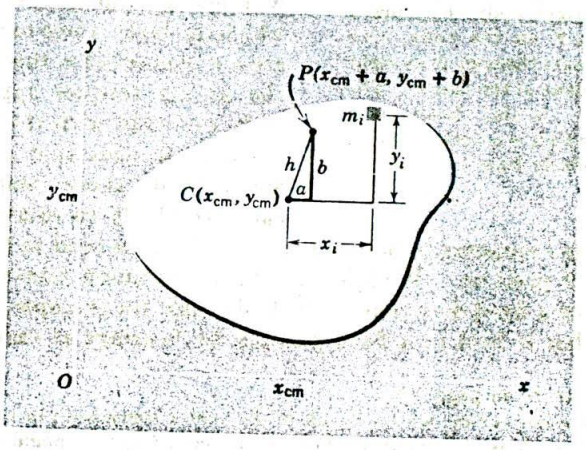


Fig. 12–8 Derivation of the parallel-axis theorem. Knowing the rotational inertia about an axis through C, we can find its value about a parallel axis through P.

From the definition of center of mass,

$$\Sigma m_i x_i = \Sigma m_i y_i = 0,$$

so that the two middle terms are zero. The first term is simply the rotational inertia about an axis through the center of mass I_{cm} and the last term is Mh^2 Hence it follows that $I = I_{cm} + Mh^2$.

With the aid of this formula several of the results of Table 12–1 can be deduced from previous results. For example, (*f*) follows from (*e*), and (*j*) follows from (*i*) with the aid of Eq. 12–13. The formula will prove to be especially useful in problems that combine rotational and translational motion.

12–6 Rotational Dynamics of a Rigid Body

In this section we continue to study the special case of a rigid body confined to rotate about an axis that is fixed* in an inertial reference frame. First we shall review the concept of torque as applied to such a rigid body; then we shall show how the torque is related to the angular acceleration of the body about this axis.

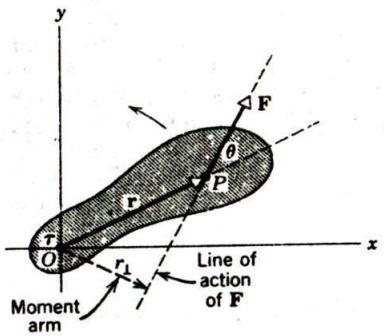


Fig. 12–9 A force **F** acts on the particle *P* in a rigid body, exerting a torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ on the body, with respect to an axis through *O* at right angles to the plane of the figure. The moment arm $r_\perp$ is also shown, as is the torque $\boldsymbol{\tau}$, which is a vector emerging perpendicularly from the page.

Suppose that we apply a torque $\boldsymbol{\tau}$ to one of the particles in a rigid body. Since all the particles of a truly rigid body maintain a fixed spatial relationship to all the other particles that make up the body, the torque may be said to act on the rigid body as a whole. In general, the vector $\boldsymbol{\tau}$ will not lie along the axis around which the body is free to rotate. We are not concerned in this section with the actual torques applied to the body but only with the components of these torques that lie along the axis.† Only these components can cause the body to rotate about this axis. Torque components perpendicular to the axis tend to turn the axis from its fixed position. We have specifically assumed, however, that the axis maintains a fixed direction. The body may, for example, be attached to a shaft that is held in a fixed position by bearings at each end; if an

* See the footnote on page 268.

† As for any other vector, we can speak of the vector component of a torque in any given direction, such as a given axis. For torque—and for other angular quantities—we also often speak of the component *around* a given direction or axis. The meaning is the same.

applied torque has a component at right angles to the shaft, tending to turn it, the bearings will automatically apply an equal and opposite counter-torque to the shaft, canceling the effect of this component.

In Fig. 12–9 (compare Fig. 11–3), we show a section through a rigid body that is free to rotate about the z-axis of an inertial reference frame. A force $\mathbf{F}$, taken for convenience to be in (or parallel to) the x-y plane of the section, acts on a particle at point P in the body, the position of P with respect to the rotational axis (the z-axis) being defined by the vector $\mathbf{r}$. The torque acting on the particle at P may be said to act on the rigid body as a whole and is given by Eq. 12–1, or

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

Because we have chosen $\mathbf{r}$ and $\mathbf{F}$ to lie in a plane parallel to the x-y plane, the torque $\boldsymbol{\tau}$ will point along the z-axis. The right-hand rule shows that it points perpendicularly *out of* the plane of Fig. 12–9. If $\mathbf{r}$ and $\mathbf{F}$ did *not* lie in the plane of the figure, $\boldsymbol{\tau}$ would not be parallel to the z-axis and we would concern ourselves here only with the component of $\boldsymbol{\tau}$ along this axis. The magnitude of $\boldsymbol{\tau}$ is given by Eq. 12–2 or

$$\tau = rF \sin \theta$$

which, as we have seen, can also be written as $\tau = rF_\perp$ or $\tau = Fr_\perp$.

▶ **Example 3.** A wagon wheel is free to rotate about a horizontal axis through O. A force of 10 lb is applied to a spoke at the point P, 1.0 ft from the center. OP makes an angle of 30° with the horizontal (x-axis) and the force is in the plane of the wheel making an angle of 45° with the horizontal (x-axis). What is the torque on the wheel?

The angle between the displacement vector $\mathbf{r}$ from O to P and the applied force $\mathbf{F}$ (Fig. 12–10) is θ, where

$$\theta = 45° - 30° = 15°.$$

Then the magnitude of the torque is

$$\tau = rF \sin \theta$$

$$= (1.0 \text{ ft})(10 \text{ lb})(\sin 15°) = 2.6 \text{ lb-ft}.$$

It is clear that we can obtain this same result from $\tau = rF_\perp$ or $\tau = Fr_\perp$ as well (see Eqs. 12–2). The torque ($\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$) is a vector pointing out ⊙ along the axis through O having a magnitude 2.6 lb-ft. ◀

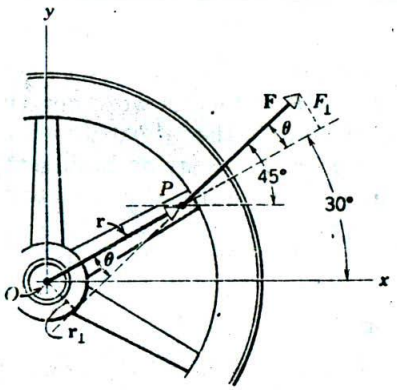


Fig. 12–10 Example 3.

We now investigate the relationship between the torque applied to the rigid body of Fig. 12–9 and the angular acceleration of this body. Let us observe the rigid body for an infinitesimal time dt, during which it will

rotate through an infinitesimal angle $d\theta$. We have seen earlier that we can describe the rotation of a rigid body about a fixed axis by examining the motion of any single point fixed in the body, such as P in Fig. 12–9. For convenience, then, we ignore the body itself in Fig. 12–11 and focus our attention on the representative point P and on the vector $\mathbf{r}$ which locates point P with respect to the axis of rotation.

During the time dt, the point P will move an infinitesimal distance ds along a circular path of radius r as the body rotates through an infinitesimal angle $d\theta$, where

$$ds = r\,d\theta.$$

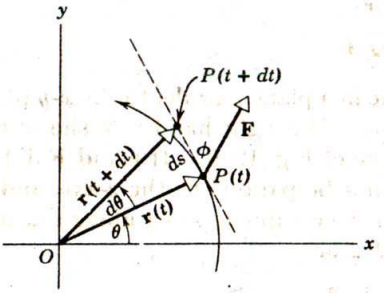


Fig. 12–11 In time dt point P in the rigid body of Fig. 12–9 moves a distance ds along the arc of a circle of radius r. The rigid body (not shown) and the vector $\mathbf{r}$ that locates point P in it each rotate through an angle $d\theta$ during this interval.

The work dW done by this force during this small rotation is

$$dW = \mathbf{F} \cdot d\mathbf{s} = F \cos \phi\, ds$$
$$= (F \cos \phi)(r\,d\theta),$$

where $F \cos \phi$ is the component of $\mathbf{F}$ in the direction of $d\mathbf{s}$.

The term $(F \cos \phi)r$, however, is the magnitude of the instantaneous torque exerted by $\mathbf{F}$ on the rigid body about the axis perpendicular to the page through O, so that

$$dW = \tau\,d\theta. \qquad (12\text{–}14)$$

This differential expression for the work done in rotation (about a fixed axis) is equivalent to the expression $dW = F\,dx$ for the work done in translation (along a straight line).

To obtain the rate at which work is done in rotational motion (about a fixed axis), we divide both sides of Eq. 12–14 by the infinitesimal time interval dt during which the body is displaced through $d\theta$, obtaining

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

or

$$P = \tau\omega,$$

giving the instantaneous power P. This last expression is the rotational analog of $P = Fv$ for translational motion (along a straight line).

If now a number of forces $\mathbf{F}_1$, $\mathbf{F}_2$, etc., are applied to the body in the plane normal to its axis of rotation, the work done by these forces on the body in a small rotation $d\theta$ will be

$$dW = F_1 \cos \phi_1 r_1\,d\theta + F_2 \cos \phi_2 r_2\,d\theta + \cdots,$$
$$= (\tau_1 + \tau_2 + \cdots)\,d\theta = \tau\,d\theta,$$

where $r_1\,d\theta$ equals ds_1, the displacement of the point at which $\mathbf{F}_1$ is applied,

and ϕ_1 is the angle between $\mathbf{F}_1$ and $d\mathbf{s}_1$, etc., and where τ is now the magnitude of the component of the *resultant* torque along the axis through O. In computing this sum each torque is considered positive or negative according to the sense in which it alone would tend to rotate the body about its axis. We can arbitrarily call the torque associated with a force positive if the effect of the force, acting alone, is to produce a counterclockwise rotation; then the torque is negative if the effect is to produce a clockwise rotation.

There is no internal motion of particles within a truly rigid body. The particles always maintain a fixed position relative to one another and move only with the body as a whole. Hence there can be no dissipation of energy within a truly rigid body. We can therefore equate the rate at which work is being done on the body to the rate at which its kinetic energy is increasing. The rate at which work is being done on the rigid body is

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega. \tag{12–15}$$

The rate at which the kinetic energy of the rigid body is increasing is

$$\frac{d}{dt}(\tfrac{1}{2}I\omega^2).$$

But I is constant because the body is rigid and the axis is fixed. Hence

$$\frac{d}{dt}(\tfrac{1}{2}I\omega^2) = \tfrac{1}{2}I\frac{d}{dt}(\omega^2) = I\omega\frac{d\omega}{dt} = I\omega\alpha. \tag{12–16}$$

Equating the right-hand members of Eqs. 12–15 and 12–16, we obtain

$$\tau\omega = I\alpha\omega,$$

or

$$\tau = I\alpha. \tag{12–17}$$

Equation 12–17 refers to the rotational motion of a rigid body about a fixed axis. The torque τ, the angular velocity ω, and the angular acceleration α are all constrained to point along this axis, in one direction or the other. The equivalent translational case is that in which the force $\mathbf{F}$ acting on a body, its velocity $\mathbf{v}$, and its acceleration $\mathbf{a}$ all point along a given straight line, in one direction or the other.

The above six quantities are vectors, but when they are directed along a fixed line, they can have only two directions. By taking one of these directions as $+$ and the other as $-$, we can treat these vectors algebraically and deal with their magnitudes only. Thus, in deriving Eq. 12–17 ($\tau = I\alpha$), we have simply transformed Newton's second law ($F = Ma$), written in scalar form suitable to describe rectilinear motion, into rotational terms. This suggests that just as we associate a force with the linear acceleration of a body, so we may associate a torque with the angular acceleration of a body about a given axis. The rotational inertia I is a

measure of the resistance a body offers to having its rotational motion changed by a given torque just as the translational inertia, or mass, M is a measure of the resistance a body offers to having its translational motion changed by a given force.

In Table 12–2 we compare the translational motion of a rigid body along a straight line with the rotational motion of a rigid body about a fixed axis.

Table 12–2

Rectilinear Motion		Rotation about a Fixed Axis	
Displacement	x	Angular displacement	θ
Velocity	$v = \dfrac{dx}{dt}$	Angular velocity	$\omega = \dfrac{d\theta}{dt}$
Acceleration	$a = \dfrac{dv}{dt}$	Angular acceleration	$\alpha = \dfrac{d\omega}{dt}$
Mass	M	Rotational inertia	I
Force	$F = Ma$	Torque	$\tau = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$\frac{1}{2}Mv^2$	Kinetic energy	$\frac{1}{2}I\omega^2$
Power	$P = Fv$	Power	$P = \tau\omega$
Linear momentum	Mv	Angular momentum	$I\omega$

The rotation of a rigid body about a fixed axis (to which $\tau = I\alpha$ applies) is not the most general kind of rotary motion in that the body may not be rigid and the axis may not be fixed in an inertial reference frame. In this general case Eq. 12–9, or $\boldsymbol{\tau}_{ext} = d\mathbf{L}/dt$, applies. As we have already pointed out, this is equivalent to Newton's second law for the general translational motion of a system of particles, namely, Eq. 9–17, or $\mathbf{F}_{ext} = d\mathbf{P}/dt$.

In the rest of this chapter we confine ourselves to the rotations of rigid bodies about fixed axes. In Chapter 13 we shall consider some more general kinds of rotary motion.

▶ **Example 4.** A uniform disk of radius R and mass M is mounted on an axle supported in fixed frictionless bearings, as in Fig. 12–12. A light cord is wrapped around the rim of the wheel and a steady downward pull $\mathbf{T}$ is exerted on the cord. Find the angular acceleration of the wheel and the tangential acceleration of a point on the rim.

The torque about the central axis is $\tau = TR$, and the rotational inertia of the disk about its central axis is $I = \frac{1}{2}MR^2$. From

$$\tau = I\alpha,$$

we have

$$TR = (\tfrac{1}{2}MR^2)\alpha,$$

or

$$\alpha = \frac{2T}{MR}.$$

If the mass of the disk is taken to be $M = 0.20$ slug, its radius $R = 0.50$ ft, and the force $T = 1.0$ lb, then

$$\alpha = \frac{(2)(1.0 \text{ lb})}{(0.20 \text{ slug})(0.50 \text{ ft})}$$

$$= 20 \text{ radians/sec}^2.$$

The tangential acceleration of a point on the rim is given by

$$a = R\alpha = (20 \text{ radians/sec}^2)(0.50 \text{ ft})$$

$$= 10 \text{ ft/sec}^2.$$

Example 5. Suppose that we hang a body of mass m from the cord in the previous problem. Find the angular acceleration of the disk and the tangential acceleration of a point on the rim in this case.

Now, let T be the tension in the cord. Since the suspended body will accelerate downward, the magnitude of the downward pull of gravity on it, mg, must exceed the magnitude of the upward pull of the cord on it, T. The acceleration a of the suspended body is the same as the tangential acceleration of a point on the rim of the disk. From Newton's second law

$$mg - T = ma.$$

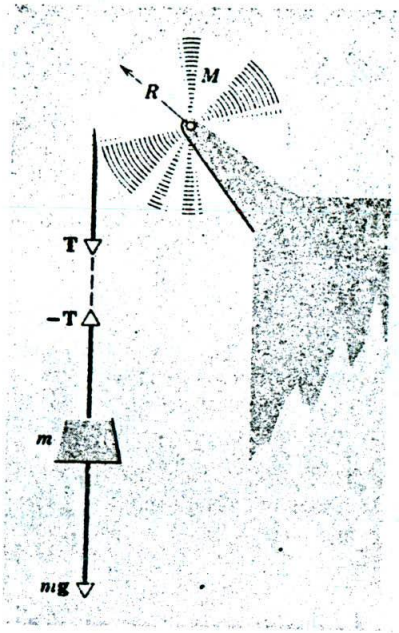


Fig. 12–12 Example 4. A steady downward force T produces rotation of the disk. Example 5. Here T is supplied by the falling mass m.

The resultant torque on the disk is TR and its rotational inertia is $\frac{1}{2}MR^2$, so that from

$$\tau = I\alpha$$

we obtain

$$TR = \tfrac{1}{2}MR^2\alpha.$$

Using the relation $a = R\alpha$, we can write this last equation as

$$2T = Ma.$$

Solving the first and last equations simultaneously leads to

$$a = \left(\frac{2m}{M + 2m}\right) g,$$

and

$$T = \left(\frac{Mm}{M + 2m}\right) g.$$

If now we let the disk have a mass $M = 0.20$ slug and a radius $R = 0.50$ ft as

before, and we let the suspended body weigh 1.0 lb, we obtain

$$a = \frac{2mg}{M + 2m} = \frac{(2)(1.0 \text{ lb})}{(0.20 \text{ slug}) + (2)(\frac{1}{32} \text{ slug})} = 7.6 \text{ ft/sec}^2,$$

$$\alpha = \frac{a}{R} = \frac{(7.6 \text{ ft/sec}^2)}{0.50 \text{ ft}} = 15 \text{ radians/sec}^2.$$

Notice that the accelerations are less for a suspended 1.0-lb body than they were for a steady 1.0-lb pull on the string (Example 4). This corresponds to the fact that the tension in the string supplying the torque is now less than 1.0 lb, namely

$$T = \frac{Mmg}{M + 2m} = \frac{(0.20 \text{ slug})(1.0 \text{ lb})}{(0.20 + 2.0/32) \text{ slug}} = 0.76 \text{ lb}.$$

The tension in the string must be less than the weight of the suspended body if the body is to accelerate downward.

Example 6. Assuming that the disk of Example 5 starts from rest, compute the work done by the applied torque on the disk in 2.1 sec. Compute also the increase in rotational kinetic energy of the disk.

Since the applied torque is constant, the resulting angular acceleration is constant. The total angular displacement in constant angular acceleration is obtained from Eq. 11–5,

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2,$$

in which

$$\omega_0 = 0, \qquad \alpha = 15 \text{ radians/sec}^2, \qquad t = 2.1 \text{ sec},$$

so that

$$\theta = 0 + (\tfrac{1}{2})(15 \text{ radians/sec}^2)(2.1 \text{ sec})^2 = 34 \text{ radians}.$$

For constant torque the work done in a finite angular displacement is

$$W = \tau(\theta_2 - \theta_1),$$

in which

$$\tau = TR = (0.76 \text{ lb})(0.50 \text{ ft}) = 0.38 \text{ lb-ft},$$

and

$$\theta_2 - \theta_1 = \theta = 34 \text{ radians}.$$

Therefore

$$W = (0.38 \text{ lb-ft})(34 \text{ radians}) = 13 \text{ ft-lb}.$$

This work must result in an increase in rotational kinetic energy of the disk. Starting from rest the disk acquires an angular speed ω.

The rotational energy is $\frac{1}{2}I\omega^2 = \frac{1}{2}(\frac{1}{2}MR^2)\omega^2$.

To obtain ω we use Eq. 11–3,

$$\omega = \omega_0 + \alpha t,$$

in which

$$\omega_0 = 0, \qquad t = 2.1 \text{ sec}, \qquad \alpha = 15 \text{ radians/sec}^2,$$

so that

$$\omega = 0 + (15 \text{ radians/sec}^2)(2.1 \text{ sec}) = 32 \text{ radians/sec}.$$

Then

$$\tfrac{1}{2}I\omega^2 = (\tfrac{1}{4})(0.20 \text{ slug})(0.50 \text{ ft})^2(32 \text{ radians/sec})^2 = 13 \text{ ft-lb},$$

as before. Hence the increase in kinetic energy of the disk is equal to the work done by the resultant force on the disk, as it must be.

Example 7. Show that the conservation of mechanical energy holds for the system of Example 5.

The resultant force acting on the system is the force of gravity on the suspended body. This is a conservative force. Viewing the system as a whole, we see that the suspended body loses potential energy U as it descends,

$$U = mgy,$$

where y is the vertical distance through which the block descends. At the same time the suspended body gains kinetic energy of translation and the disk gains kinetic energy of rotation. The total gain in kinetic energy is

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2,$$

where v is the linear speed of the suspended mass. We must show then that

$$mgy = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2.$$

For the linear motion starting from rest we have $v^2 = 2ay$. From Example 5, we obtained $a = 2\,mg/(M + 2m)$. Hence

$$mgy = \frac{mgv^2}{2a} = \tfrac{1}{2}\,mv^2\left(\frac{g}{a}\right) = \tfrac{1}{2}\,mv^2\left(\frac{M + 2m}{2m}\right) = \tfrac{1}{4}\,(M + 2m)v^2.$$

We also know that $\omega = v/R$ and $I = \tfrac{1}{2}MR^2$. Substituting these relations into the right-hand side of the conservation equation, we obtain

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}(\tfrac{1}{2}MR^2)(v^2/R^2) = \tfrac{1}{4}(M + 2m)v^2.$$

The mechanical energy is therefore conserved.

Example 8. Derive the relation $L = I\omega$, shown in Table 12–2, for the angular momentum of a rigid body confined to rotate about a fixed axis.

Starting from the scalar relation $\tau = I\alpha$ and the definition of α $(= d\omega/dt)$, we may write

$$\tau = I\alpha = I(d\omega/dt) = d(I\omega)/dt,$$

in which the last step is justified by the fact that I is a constant for a given rigid body and a specified (fixed) axis of rotation.

Next we use the vector relation $\boldsymbol{\tau}_{ext} = d\mathbf{L}/dt$ (Eq. 12–9) and write the corresponding relation for the *scalar components*, τ and dL, of $\boldsymbol{\tau}_{ext}$ and $d\mathbf{L}$ along the fixed axis of rotation, obtaining

$$\tau = dL/dt.$$

Simply by comparing the two equations above we obtain the relation sought, namely

$$L = I\omega. \tag{12–18}$$

Like Eq. 12–17 $(\tau = I\alpha)$, this is a scalar relation holding for the rotation of a rigid body about a fixed axis. L is the component along that axis of the vector angular momentum $\mathbf{L}$ of the rigid body and I must refer to that same axis.

Equation 12–18 is the rotational analog of the expression $P = Mv$ for the *linear* momentum of a rigid body of mass M in pure translational motion with linear speed v. It gives the *angular* momentum about a fixed axis of a rigid body having rotational inertia I and angular speed ω about that same axis. ◀

12–7 The Combined Translational and Rotational Motion of a Rigid Body

Up until now we have considered only bodies rotating about some fixed axis. If a body is rolling, however, it is rotating about an axis and also moving translationally. Therefore it would seem that the motion of rolling bodies must be treated as a combination of translational and rotational motion. It is also possible, however, to treat a rolling body as though its motion is one of pure rotation. We wish to illustrate the equivalence of the two approaches.

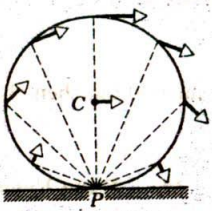


Fig. 12–13 A rolling body may at any instant be thought of as rotating about a perpendicular axis through its point of contact P.

Consider, for example, a cylinder rolling along a level surface, as in Fig. 12–13. At any instant the bottom of the cylinder is at rest on the surface, since it does not slide. The axis normal to the diagram through the point of contact P is called the *instantaneous axis of rotation*. At that instant the linear velocity of every particle of the cylinder is directed at right angles to the line joining the particle and P and its magnitude is proportional to this distance. This is the same as saying that the cylinder is rotating about a fixed axis through P with a certain angular speed ω, *at that instant*. Hence, at a given instant the motion of the body is equivalent to a pure rotation. The total kinetic energy can, therefore, be written as

$$K = \tfrac{1}{2} I_P \omega^2, \tag{12–19}$$

where I_P is the rotational inertia about the axis through P.

Let us now use the parallel axis theorem, which tells us that

$$I_P = I_{cm} + MR^2,$$

where I_{cm} is the rotational inertia of the cylinder of mass M and radius R about a parallel axis through the center of mass. Equation 12–19 now becomes

$$K = \tfrac{1}{2} I_{cm} \omega^2 + \tfrac{1}{2} MR^2 \omega^2. \tag{12–20}$$

The quantity $R\omega$ is the speed with which the center of mass of the cylinder is moving with respect to the fixed point P. Let $R\omega = v_{cm}$. Equation 12–20 then becomes

$$K = \tfrac{1}{2} I_{cm} \omega^2 + \tfrac{1}{2} M v_{cm}^2. \tag{12–21}$$

Now notice that the speed of the center of mass with respect to P is the same as the speed of P with respect to the center of mass. Hence, the angular speed ω of the center of mass about P as seen by someone at P is the same as the angular speed of a particle at P about C as seen by someone at C (moving along with the cylinder). This is equivalent to saying that any reference line in the cylinder turns through the same angle in a given

time whether it is observed from a reference frame fixed with respect to the surface on which the cylinder is rolling or from a frame moving translationally with respect to this fixed frame. We can therefore interpret Eq. 12–21, which was derived on the basis of a pure rotational motion, in another way; that is, the first term, $\frac{1}{2}I_{cm}\omega^2$, is the kinetic energy the cylinder would have if it were merely rotating about an axis through its center of mass, without translational motion; and the second term, $\frac{1}{2}Mv_{cm}^2$, is the kinetic energy the cylinder would have if it were moving translationally with the speed of its center of mass, without rotating. Notice that there is now no reference at all to the instantaneous axis of rotation. In fact, Eq. 12–21 applies to any body that is moving and rotating about an axis perpendicular to its motion whether or not it is rolling on a surface.

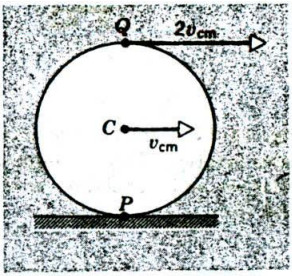


Fig. 12–14 Since Q and C have the same angular velocity about P, therefore Q, being twice as far from P, moves with twice the linear velocity of C.

The combined effects of translation of the center of mass and rotation about an axis through the center of mass are equivalent to a pure rotation with the same angular speed about an axis through the point of contact of a rolling body.

To illustrate this result simply, let us consider the instantaneous speed of various points on the rolling cylinder. If the speed of the center of mass (as seen by an observer fixed with respect to the surface) is v_{cm}, the instantaneous angular speed about an axis through P is $\omega = v_{cm}/R$. A point Q at the top of the cylinder will therefore have a speed $\omega 2R = 2v_{cm}$ at that instant. The point of contact P is instantaneously at rest. Hence, from the point of view of pure rotation about P, the situation is as shown in Fig. 12–14.

Now let us regard the rolling as a combination of translation of the center of mass and rotation about the cylinder axis through C. If we consider translation only, all points on the cylinder have the same speed v_{cm} of the center of mass. This is shown in Fig. 12–15*a*. If we consider the rotation only, the center is at rest, whereas the point Q at the top has a speed $+\omega R$ in the x-direction and the point P at the bottom of the cylinder has a speed $-\omega R$ in the $-x$-direction. This is shown in Fig. 12–15*b*. Now let us combine these two results. Recalling that $\omega = v_{cm}/R$, we obtain

$$\text{for the point } Q \qquad v = v_{cm} + \omega R = v_{cm} + \frac{v_{cm}}{R} R = 2v_{cm},$$

$$\text{for the point } C \qquad v = v_{cm} + 0 = v_{cm},$$

$$\text{for the point } P \qquad v = v_{cm} - \omega R = v_{cm} - \frac{v_{cm}}{R} R = 0.$$

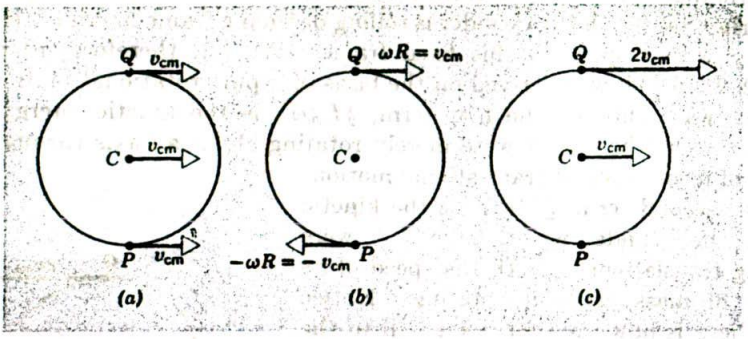


Fig. 12–15 (*a*) For pure translation, all points move with the same velocity. (*b*) For pure rotation about C, opposite points move with opposite velocities. (*c*) Combined rotation and translation is obtained by adding together corresponding vectors in (*a*) and (*b*).

This result, shown in Fig. 12–15*c* is exactly the same as that obtained from the purely rotational point of view, Fig. 12–14.

▶ **Example 9.** Consider a solid cylinder of mass M and radius R rolling down an inclined plane without slipping. Find the speed of its center of mass when the cylinder reaches the bottom.

The situation is illustrated in Fig. 12–16. We can use the conservation of energy to solve this problem. The cylinder is initially at rest. In rolling down the incline the cylinder loses potential energy of an amount Mgh, where h is the height of the incline. It gains kinetic energy equal to

$$\tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}Mv^2,$$

where v is the linear speed of the center of mass and ω is the angular speed about the center of mass at the bottom.

We have then the relation

$$Mgh = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}Mv^2,$$

in which

$$I_{cm} = \tfrac{1}{2}MR^2 \quad \text{and} \quad \omega = \frac{v}{R}.$$

Hence

$$Mgh = \tfrac{1}{2}(\tfrac{1}{2}MR^2)\left(\frac{v}{R}\right)^2 + \tfrac{1}{2}Mv^2 = (\tfrac{1}{4} + \tfrac{1}{2})Mv^2,$$

$$v^2 = \tfrac{4}{3}gh \quad \text{or} \quad v = \sqrt{\tfrac{4}{3}gh}.$$

The speed of the center of mass would have been $v = \sqrt{2gh}$ if the cylinder had *slid* down a *frictionless* incline. The speed of the rolling cylinder is, therefore, less than the speed of the sliding cylinder, because for the rolling cylinder, part of the lost potential energy has been transformed into rotational kinetic energy, leaving less available for the translational part of the kinetic energy. Although the rolling cylinder arrives later at the bottom of the incline than an identical sliding cylinder started at the same time down a frictionless, but otherwise identical, incline, both

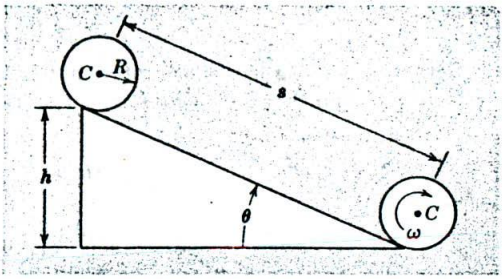


Fig. 12–16 Example 9. A cylinder rolling down an incline.

arrive at the bottom with the same amount of energy; the rolling cylinder happens to be rotating as it moves, whereas the sliding one does not rotate as it moves.

Notice that static friction is needed to cause the cylinder to rotate. Remembering that friction is a dissipative force, how can you justify using the conservation of mechanical energy in this problem?

Example 10. The previous result was derived by use of energy methods. Solve the same problem using only dynamical methods.

The force diagram is shown in Fig. 12–17. $M\mathbf{g}$ is the weight of the cylinder acting vertically down through the center of mass,* **N** is the normal force exerted by the incline on the cylinder, and **f** is the force of static friction acting along the incline at the point of contact.

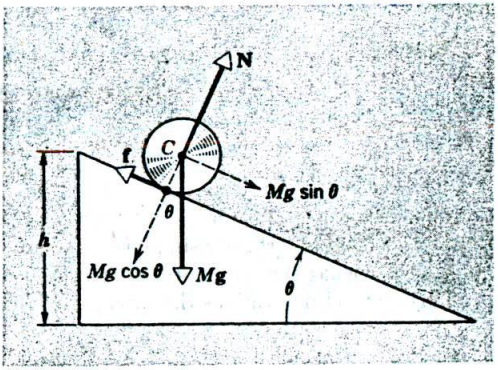


Fig. 12–17 Example 10. Dynamic solution of the motion of a cylinder rolling down an incline.

The *translational* motion of a body is obtained by assuming that all the external forces act at its center of mass. Using Newton's second law, we obtain

$$N - Mg\cos\theta = 0 \qquad \text{for motion normal to the incline,}$$

and

$$Mg\sin\theta - f = Ma \qquad \text{for motion along the incline.}$$

* In drawing the vector diagram for this problem we tacitly assume that the total weight of the body can be thought of as acting at the center of mass. We saw in Section 9–2 that this is justified for analyzing the translational motion. However, later in the problem we use this result in analyzing the rotational motion as well. We shall justify this procedure in Section 14–3, where it is shown that the *weight* of a body can be considered to act at its center of mass for both translational *and* rotational motion.

The *rotational* motion about the center of mass follows from

$$\tau = I_{cm}\alpha.$$

Neither **N** nor $M\mathbf{g}$ can cause rotation about C because their lines of action pass through C, and they have zero moment arms. The force of friction has a moment arm R about C, so that

$$fR = I_{cm}\alpha.$$

But

$$I_{cm} = \tfrac{1}{2}MR^2 \qquad \text{and} \qquad \alpha = \frac{a}{R}$$

so that

$$f = I_{cm}\alpha/R = Ma/2.$$

Substituting this into the second translational equation, we find

$$a = \tfrac{2}{3}g \sin\theta.$$

That is, the acceleration of the center of mass for the rolling cylinder ($\frac{2}{3}g \sin\theta$) is less than the acceleration of the center of mass for the cylinder sliding down the incline ($g \sin\theta$).

This result holds at any instant, regardless of the position of the cylinder along the incline. The center of mass moves with constant linear acceleration. To obtain the speed of the center of mass, starting from rest, we use the relation

$$v^2 = 2as,$$

so that

$$v^2 = 2(\tfrac{2}{3}g \sin\theta)s = \tfrac{4}{3}g\frac{h}{s}s = \tfrac{4}{3}gh$$

or

$$v = \sqrt{\tfrac{4}{3}gh}.$$

This result is the same as that obtained before by the energy method. The energy method is certainly simpler and more direct. However, if we are interested in knowing what the forces are, such as **N** and **f**, we must use a dynamical method.

This method determines the minimum force of static friction needed for rolling:

$$f = Ma/2 = (M/2)(\tfrac{2}{3}g \sin\theta) = \tfrac{1}{3}Mg \sin\theta.$$

What would happen if the force of static friction between the surfaces were less than this value?

Example 11. A sphere and a cylinder, having the same mass and radius, start from rest and roll down the same incline. Which body gets to the bottom first?

For a sphere I_{cm} equals $\frac{2}{5}MR^2$. Using the dynamical method we obtain

$$Mg \sin\theta - f = Ma, \qquad \text{translation of cm,}$$

$$fR = I_{cm}\alpha = (\tfrac{2}{5}MR^2)(a/R), \qquad \text{rotation about cm,}$$

or

$$f = \tfrac{2}{5}Ma \qquad \text{and} \qquad a = \tfrac{5}{7}g \sin\theta, \qquad \text{sphere.}$$

For the cylinder (Example 10)

$$a = \tfrac{2}{3}g \sin \theta, \qquad \text{cylinder.}$$

Hence the accelcration of the center of mass of the sphere is at all times greater than the acceleration of the center of mass of the cylinder. Since both bodies start from rest at the same instant, the sphere will reach the bottom first.

Which body has the greater rotational energy at the bottom? Which body has the greater translational energy at the bottom?

The student should note carefully that neither the mass nor the radius of the rolling object enters the previous results. How then would we expect the behavior of cylinders of different mass and radii to compare? How would we expect the behavior of spheres of different mass and radii to compare? How would the behavior of a cylinder and sphere having different masses and radii compare? ◀

QUESTIONS

1. What are the dimensions of angular momentum? Can you find any significance in the fact that they are the same as those of energy multiplied by time?

2. Is the vector product of two vectors necessarily an axial vector?*

3. Can the mass of a body be considered as concentrated at its center of mass for purposes of computing its rotational inertia?

4. About what axis would a uniform cube have its minimum rotational inertia?

5. If two circular disks of the same weight and thickness are made from metals having different densities, which disk, if either, will have the larger rotational inertia about its central axis?

6. The rotational inertia of a body of rather complicated shape is to be determined. The shape makes a mathematical calculation from $\int r^2\, dm$ exceedingly difficult. Suggest ways in which the rotational inertia could be measured *experimentally*.

7. Five solids are shown in cross section (Fig. 12–18). The cross sections have equal heights and maximum widths. The axes of rotation are perpendicular to the sections through the points shown. The solids have equal masses. Which one has the largest rotational inertia about a perpendicular axis through the center of mass? Which the smallest?

* See Supplementary Topic II.

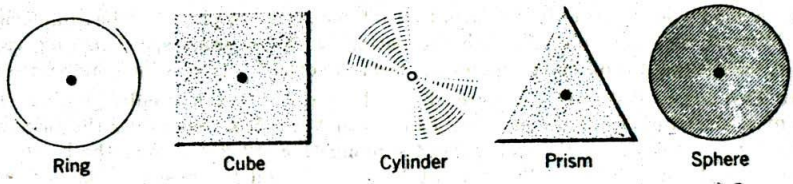


Fig. 12–18

8. In Fig. 12–19*a* a meter stick, half of which is wood—the other half steel—is pivoted at the wooden end at O and a force is applied to the steel end at a. In Fig. 12–19*b* the stick is pivoted at the steel end at O' and the same force is applied at the wooden end at a'. Does one get the same angular acceleration in each case? Explain.

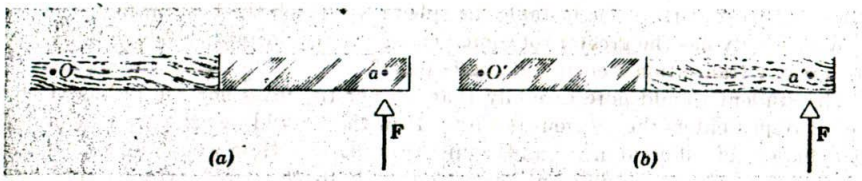


Fig. 12–19

9. A person can distinguish between a raw egg and a hard-boiled one by spinning each one on the table. Explain how.

10. Do the expressions for a and T in Example 5 give reasonable results for the special cases in which $g = 0$, $M = 0$, $M \to \infty$, $m = 0$, and $m \to \infty$?

11. The total momentum of a system of particles does not depend on the motions of the particles relative to the center of mass of the system. Can a similar statement be made about the total kinetic energy of a system of particles?

12. A cylindrical drum, pushed along by a board from an initial position shown in Fig. 12–20, rolls forward on the ground a distance $l/2$, equal to half the length of the board. There is no slipping at any contact. Where is the board then? How far has the man walked?

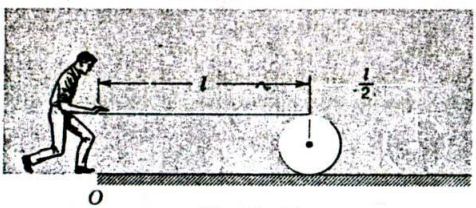


Fig. 12–20

13. A solid wooden sphere rolls down two different inclined planes of the same height but different inclines. Will it reach the bottom with the same speed in each case? Will it take longer to roll down one incline than the other? If so, which one and why?

14. Two heavy disks are connected by a short rod of much smaller radius. The system is placed on a narrow inclined plane so that the disks hang over the sides and the system rolls down on the rod without slipping (Fig. 12–21). Near the bottom of the incline the disks touch the horizontal table top and the system takes off with greatly increased translational speed. Explain carefully.

15. When a logger cuts down a tree he makes a cut on the side facing the direction in which he wants it to fall. Explain why. Would it be safe to stand directly behind the tree on the opposite side of the fall?

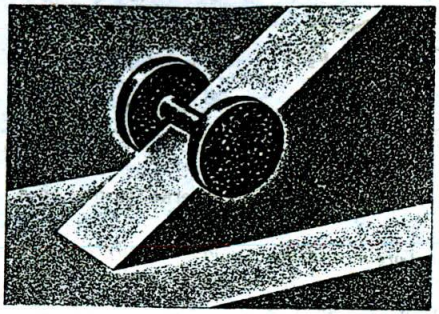


Fig. 12–21

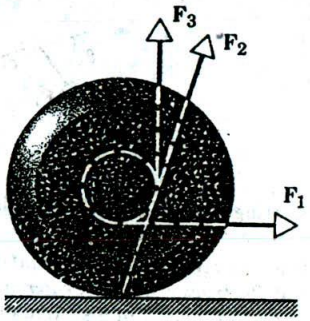


Fig. 12–22

16. A Yo-yo is resting on a horizontal table and is free to roll (Fig. 12–22). If the string is pulled by a horizontal force such as $\mathbf{F}_1$, which way will the Yo-yo roll? What happens when the force $\mathbf{F}_2$ is applied (its line of action passes through the point of contact of the Yo-yo and table)? If the string is pulled vertically with the force $\mathbf{F}_3$, what happens?

17. State Newton's three laws of motion in words suitable for rotating bodies.

PROBLEMS

1. (*a*) Given that $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ and $\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z$ find the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. (*b*) Show that if $\mathbf{r}$ and $\mathbf{F}$ lie in a given plane then $\boldsymbol{\tau}$ has no component in that plane.

2. If we are given r, p, and θ, we can calculate the angular momentum of a particle from Eq. 12–4*a*. Sometimes, however, we are given the components (x,y,z) of $\mathbf{r}$ and (p_x,p_y,p_z) of $\mathbf{p}$ instead. (*a*) Show that the components of $\mathbf{l}$ along the x-, y-, and z-axes are then given by

$$l_x = yp_z - zp_y,$$

$$l_y = zp_x - xp_z,$$

$$l_z = xp_y - yp_x.$$

(*b*) Show that if the particle moves only in the x–y plane, the resultant angular momentum vector has only a z-component.

3. (*a*) In Example 1, express $\mathbf{F}$ and $\mathbf{r}$ in terms of unit vectors and compute $\boldsymbol{\tau}$. Do the same in Example 3. (*b*) In Example 1, express $\mathbf{p}$ and $\mathbf{r}$ in terms of unit vectors and compute $\mathbf{l}$.

4. Show that the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion.

5. Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of this system of particles is the same about *any* point taken as origin.

6. In Fig. 12–23, show the line of action and the moment arm of each force about the origin O. Imagine these forces to be acting on a rigid body pivoted at O, all vectors shown being in the plane of the figure, and find the magnitude and the direction of the resultant torque on the body.

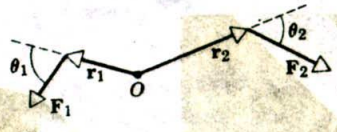


Fig. 12–23

7. Starting from Newton's third law, prove that the resultant internal torque on a system of particles is zero.

8. *Relation between the Resultant External Torque and the Angular Momentum of a System of Particles about the Center of Mass of the System.* Let $\mathbf{r}_{cm}$ be the position vector of the center of mass C of a system of particles with respect to the origin O of an inertial reference frame, and let $\mathbf{r}_i'$ be the position vector of the ith particle, of mass m_i, with respect to the center of mass C. Hence $\mathbf{r}_i = \mathbf{r}_{cm} + \mathbf{r}_i'$ (see Fig. 12–24). Now define the total angular momentum of the system of particles relative to the center of mass C to be $\mathbf{L}' = \sum_i \mathbf{r}_i' \times \mathbf{p}_i'$, where $\mathbf{p}_i' = m_i\, d\mathbf{r}_i'/dt$.

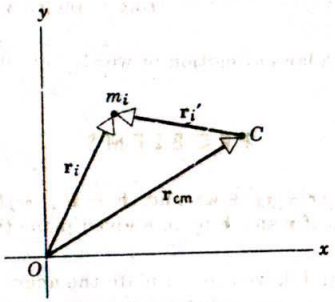


Fig. 12–24

(*a*) Show that $\mathbf{p}_i' = m_i\, d\mathbf{r}_i/dt - m_i\, d\mathbf{r}_{cm}/dt = \mathbf{p}_i - m_i\mathbf{v}_{cm}$. (*b*) Show next that $d\mathbf{L}'/dt = \sum_i \mathbf{r}_i' \times d\mathbf{p}_i'/dt$. (*c*) Combine the results of (*a*) and (*b*) and, using the definition of center of mass and Newton's third law, show that $\boldsymbol{\tau}'_{ext} = d\mathbf{L}'/dt$, where $\boldsymbol{\tau}'_{ext}$ is the sum of all the external torques acting on the system about its center of mass.

9. (*a*) Show that the sum of the rotational inertias of a plane laminar body about any two perpendicular axes in the plane of the body is equal to the rotational inertia of the body about an axis through their point of intersection perpendicular to the plane. (*b*) Apply this to a circular disk to find its rotational inertia about a diameter as axis.

10. (*a*) Prove that the rotational inertia of a thin rod of length l about an axis through its center perpendicular to its length is $I = \frac{1}{12}Ml^2$. (See Table 12–1.) (*b*) Use the parallel-axis theorem to show that $I = \frac{1}{3}Ml^2$ when the axis of rotation is through one end perpendicular to the length of the rod.

11. Show that the rotational inertia of a rectangular plate of sides a and b about an axis perpendicular to the plate through its center is $\frac{1}{12}M(a^2 + b^2)$.

12. (*a*) Show that a solid cylinder of mass M and radius R is equivalent to a thin hoop of mass M and radius $R/\sqrt{2}$, for rotation about a central axis. (*b*) The radial distance from a given axis at which the mass of a body could be concentrated without

altering the rotational inertia of the body about that axis is called the *radius of gyration*. Let k represent radius of gyration and show that

$$k = \sqrt{I/M}.$$

This gives the radius of the "equivalent hoop" in the general case.

13. The oxygen molecule has a total mass of 5.30×10^{-26} kg and a rotational inertia of 1.94×10^{-46} kg-m^2 about an axis through the center perpendicular to the line joining atoms. Suppose that such a molecule in a gas has a mean speed of 500 meters/sec and that its rotational kinetic energy is two-thirds of its translational kinetic energy. Find its average angular velocity.

14. A hoop of radius 10 ft weighs 320 lb. It rolls along a horizontal floor so that its center of mass has a speed of 0.50 ft/sec. How much work has to be done to stop it?

15. Assume the earth to be a sphere of uniform density. (*a*) What is its rotational kinetic energy? Take the radius of the earth to be 6.4×10^3 km and the mass of the earth to be 6.0×10^{24} kg. (*b*) Suppose this energy could be harnessed for man's use. For how long could the earth supply 1 kw of power to each of the 3.5×10^9 persons on earth?

16. A thin rod of length l and mass m is suspended freely at its end. It is pulled aside and swung about a horizontal axis, passing through its lowest position with an angular speed ω. How high does its center of mass rise above its lowest position? Neglect friction and air resistance.

17. An automobile engine develops 100 hp when rotating at a speed of 1800 rev/min. What torque does it deliver?

18. A wheel of mass M and radius of gyration k (see Problem 12) spins on a fixed horizontal axle passing through its hub. The hub rubs the axle, of radius a, at only one point, the coefficient of friction being μ. The wheel is given an initial angular velocity ω_0. Assume uniform deceleration and find the elapsed time and the number of revolutions before the wheel comes to a stop.

19. A uniform steel rod of length 1.20 meters and mass 6.40 kg has attached to each end a small ball of mass 1.06 kg. The rod is constrained to rotate in a horizontal plane about a vertical axis through its midpoint. At a certain instant it is observed to be making 39.0 rev/sec. Because of axle friction it comes to rest 32.0 sec later. Compute, assuming a *constant* frictional torque, (*a*) the angular acceleration, (*b*) the retarding torque exerted by axle friction, (*c*) the total work done by the axle friction, and (*d*) the number of revolutions executed during the 32.0 sec. (*e*) Suppose, however, that the frictional torque is known *not* to be constant. Which, if any, of the quantities (*a*), (*b*), (*c*), or (*d*) can still be computed without requiring any additional information? If such exists, give its value.

20. In an Atwood's machine (Fig. 5-9) one block has a mass of 500 gm and the other a mass of 460 gm. The pulley, which is mounted in horizontal frictionless bearings, has a radius of 5.0 cm. When released from rest the heavier block is observed to fall 75 cm in 5.0 sec. What is the rotational inertia of the pulley?

21. A 6.0-lb block is put on a plane inclined 30° to the horizontal and is attached by a cord parallel to the plane over a pulley at the top to a hanging block weighing 18 lb. The pulley weighs 2.0 lb and has a radius of 0.33 ft. The coefficient of kinetic friction between block and plane is 0.10. Find the acceleration of the hanging block and the tension in the cord on each side of the pulley. Assume the pulley to be a uniform disk.

22. Take the center of mass of a car to be 2.5 ft above the road and its width between wheels to be 4.5 ft. (*a*) If the car races around an unbanked curve having a radius of 100 ft without skidding, what is the largest speed possible without its overturning? (*b*) What minimum value would the coefficient of friction need to be at this speed?

23. A box 6 ft high by 4 ft wide by 3 ft deep containing a refrigerator sits in the back of a truck vertically. The weight of the refrigerator plus that of the box is 300 lb, and this is presumed to be uniformly distributed throughout the volume of the box. The box is tipped over by an acceleration of the truck. What was the minimum value that this acceleration must have had?

24. Show that a cylinder will slip on an inclined plane of inclination angle θ if the coefficient of static friction between plane and cylinder is less than $\frac{1}{3}\tan\theta$.

25. A sphere rolls up an inclined plane of inclination angle 30°. At the bottom of the incline the center of mass of the sphere has a translational speed of 16 ft/sec. (*a*) How far does the sphere travel up the plane? (*b*) How long does it take to return to the bottom?

26. A body of radius R and mass m is rolling horizontally without slipping with speed v. It then rolls up a hill to a maximum height h. If $h = 3v^2/4g$, (*a*) what is the body's rotational inertia? (*b*) What might the body be?

27. A small sphere rolls without slipping on the inside of a large hemisphere whose axis of symmetry is vertical. It starts at the top from rest. (*a*) What is its kinetic energy at the bottom? What fraction is rotational? What translational? (*b*) What normal force does the small sphere exert on the hemisphere at the bottom? Take the radius of the small sphere to be r, that of the hemisphere to be R, and let m be the mass of the sphere.

28. A homogeneous sphere starts from rest at the upper end of the track shown in Fig. 12–25 and *rolls* without slipping until it rolls off the right-hand end. If $H = 204$ ft and $h = 64$ ft and the track is horizontal at the right-hand end, determine the distance to the right of point A at which the ball strikes the horizontal base line.

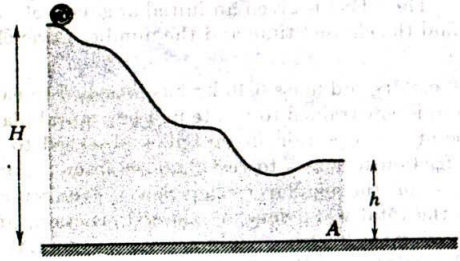


Fig. 12–25

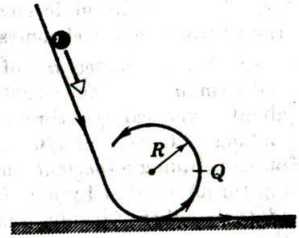


Fig. 12–26

29. A small solid marble of mass m and radius r *rolls* without slipping along the loop-the-loop track shown in Fig. 12–26. If it starts from rest at a height of $6R$ above the bottom (R = radius of the circular part of the track), what is the force (horizontal and vertical components) acting on it at the point Q?

30. A Yo-yo of mass M has a shaft of radius r about which a string is wound. A child lets the Yo-yo unwind as he holds the loose end of the string in a fixed position. The Yo-yo accelerates down, reaches the bottom, and climbs back up, the string winding around the shaft in the opposite sense. Find the tension in the string during the descent and the ascent, assuming r to be small enough to consider the string being vertical at all times. Let I represent the rotational inertia of the Yo-yo about its central axis.

31. A uniform disk, of mass M and radius R, lies on one side initially at rest on a frictionless horizontal surface. A constant force **F** is then applied tangentially at its

perimeter by means of a string wrapped around its edge. Describe the subsequent (rotational and translational) motion of the disk.

32. A cylinder of length 1.0 ft and radius 1.0 in. weighs 6.0 lb. Two cords are wrapped around the cylinder, one near each end, and the cord ends are attached to hooks on the ceiling. The cylinder is held horizontally with the two cords exactly vertical and is then released (Fig. 12–27). Find the tension in the cords as they unwind, and determine the linear acceleration of the cylinder as it falls.

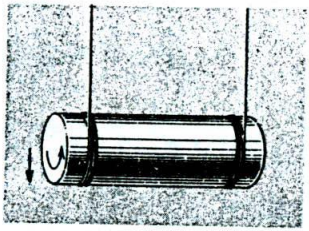


Fig. 12–27

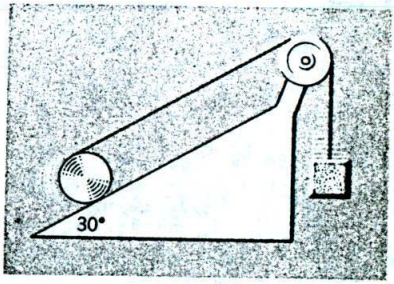


Fig. 12–28

33. A string is wrapped around a cylinder of mass M, radius R. The string is pulled vertically upward to prevent the center of mass from falling as the cylinder unwinds the string. (*a*) What is the tension in the string? (*b*) How much work has been done on the cylinder once it has reached an angular speed ω? (*c*) What is the length of string unwound in this time?

34. A solid cylinder of weight 50 lb and radius 3 in. has a light thin tape wound around it. The tape passes o er a light, smooth fixed pulley to a 10-lb body (Fig. 12–28). Find the tension in the tape and the linear acceleration of the cylinder up the incline, assuming no slipping.

35. A bowling ball is thrown down the alley in such a way that it slides with a speed v_0 initially without rolling. Prove that it will roll without any sliding when its speed falls to $\frac{5}{7}v_0$. The transition from pure sliding to pure rolling is gradual, so that both sliding *and* rolling take place during this interval. (*Hint:* Sliding ceases when the forward speed of the lowest point of the sphere is zero.)

36. A billiard ball is struck by a cue as in Fig. 12–29. The line of action of the applied impulse is horizontal and passes through the center of the ball. The initial velocity $\mathbf{v}_0$ of



Fig. 12–29

the ball, its radius R, its mass M, and the coefficient of friction μ between the ball and the table are all known. How far will the ball move before it ceases to slip on the table?

37. A 10-ft-long ladder rests against a wall and makes an angle of 60° with the horizontal floor. If it starts to slip, where is the instantaneous axis of rotation?

38. A tall chimney cracks near its base and falls over. (*a*) Express the radial and tangential linear acceleration of the top of the chimney as a function of the angle θ made by the chimney with the vertical. (*b*) Can the resultant linear acceleration exceed g? (*c*) The chimney cracks up during the fall. Explain how this can happen.

39. A meter stick is held vertically with one end on the floor and is then allowed to fall. Find the speed of the other end when it hits the floor, assuming that the end on the floor does not slip.







Rotational Dynamics II and the Conservation of Angular Momentum

CHAPTER 13

13–1 Introduction

In Chapter 12 we discussed the dynamics of the rotational motion of a rigid body about an axis that was fixed in an inertial reference frame. We saw that the scalar relation $\tau = I\alpha$ (Eq. 12–17), in which only torque components along the axis of rotation were considered, were sufficient to solve dynamical problems in this special case.

In this chapter we shall first consider the rotation of a rigid body about an axis that is *not* fixed in an inertial reference frame. To solve dynamical problems in this more general case we shall use the general (vector) relation for rotational motion, namely,

$$\boldsymbol{\tau} = d\mathbf{L}/dt \qquad (12\text{–}9)$$

in which we have dropped the subscript on $\boldsymbol{\tau}_{\text{ext}}$ for convenience.

Later we shall consider once more the rotation of particles and rigid bodies about fixed axes. This time, however, we specifically examine the action of torques which have components at right angles to the axis. Our point of departure here will not be Eq. 12–17 ($\tau = I\alpha$) but again the more general Eq. 12–9 ($\boldsymbol{\tau} = d\mathbf{L}/dt$).

Finally we shall consider systems on which no external torques act and shall introduce the important principle of *conservation of angular momentum.*

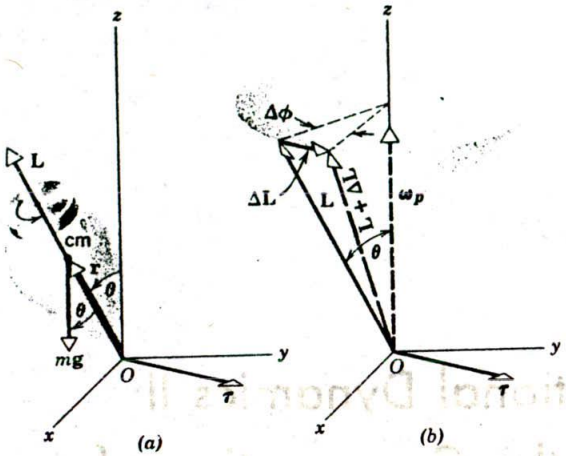


Fig. 13–1 (*a*) A precessing top, showing the angular momentum **L**, the weight $m\mathbf{g}$ and the vector **r** which locates the center of mass. (*b*) The cone swept out by the precessing axis of the top. The angular velocity of precession is shown pointing vertically upward.

13–2 The Top

Figure 13–1*a* shows a top spinning about its axis of symmetry, the point of the top being fixed at the origin O of an inertial reference frame. We know from experience that the axis of such a rapidly spinning top will move around the vertical axis, sweeping out a cone. This motion is called *precession*. Let us see if we can predict this motion from the principles of classical mechanics and, in particular, if we can calculate ω_p, the angular speed of the precessional motion.

At the instant shown in Fig. 13–1*a* the top has an angular velocity $\boldsymbol{\omega}$ about its own axis. It also has an angular momentum **L** about this same axis,* the axis making an angle θ with the vertical.

Two forces act on the top, an upward force on the pivot at O and the pull of gravity, or weight, which acts downward at the center of mass. The upward force passes through O and thus can exert no torque about that point because its moment arm is zero. The weight $m\mathbf{g}$, however, exerts a torque about O given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times m\mathbf{g},$$

where **r** locates the center of mass with respect to the pivot. This equation

* The vector $\boldsymbol{\omega}$ always points along the (fixed) axis of rotation of a spinning body but, in general, the vector **L** does not (see Section 13–3). For bodies with symmetry about the rotational axis, however, both $\boldsymbol{\omega}$ and **L** point along this axis, assuming that the axis is fixed. We can assume that $\boldsymbol{\omega}$ and **L** are coaxial for the spinning top of Fig. 13–1*a* if $\omega \gg \omega_p$, that is, if the precession rate is relatively slow so that the axis, although not fixed, changes direction only slowly.

requires that $\boldsymbol{\tau}$ be perpendicular to the plane formed by $\mathbf{r}$ and $m\mathbf{g}$; application of the right-hand rule shows that its direction is as shown in Fig. 13–1*a*. Note that $\boldsymbol{\tau}$, as well as $\mathbf{L}$ and $\mathbf{r}$, rotates about the axis at angular speed ω_p as the top precesses.

When a torque acts on a rigid body it changes the angular momentum of that body according to the fundamental relation (Eq. 12–9)

$$\boldsymbol{\tau} = d\mathbf{L}/dt. \qquad \text{(Eq. 12–9)}$$

Being a vector, $\mathbf{L}$ can change in magnitude, in direction, or in both. Equation 12–9 shows that the *change* in $\mathbf{L}$ (that is, $d\mathbf{L}$) must point in the direction of $\boldsymbol{\tau}$. Figure 13–1*a* shows us that $\boldsymbol{\tau}$ is *at right angles* to $\mathbf{L}$; thus the change in $\mathbf{L}$ brought about by the action of the torque must also be at right angles to $\mathbf{L}$.

To examine the matter quantitatively let us observe the top for a time Δt. During this interval a change in $\mathbf{L}$ of

$$\Delta\mathbf{L} = \boldsymbol{\tau}\,\Delta t$$

is predicted by Eq. 12–9 (if Δt is small enough). This change $\Delta\mathbf{L}$, which, like $\boldsymbol{\tau}$, is at right angles to $\mathbf{L}$, is displayed in Fig. 13–1*b* where we see the cone swept out by the precessing axis of the top; the top itself is omitted here for clarity.

The angular momentum of the top at the end of the time interval Δt is the vector sum of $\mathbf{L}$ and $\Delta\mathbf{L}$. Since $\Delta\mathbf{L}$ is perpendicular to $\mathbf{L}$ and is assumed to be very small in magnitude compared to it, the new angular momentum vector has the same *magnitude* as the old one but a different *direction*. Hence the head of the angular momentum vector swings around in a horizontal circle as time goes on (Fig. 13–1*b*). Since this vector always lies along the axis of rotation of the top, we have accounted qualitatively for the precession of the top.

The angular speed of precession ω_p follows from Fig. 13–1*b* in which

$$\omega_p = \Delta\phi/\Delta t.$$

But, since $\Delta L \ll L$,

$$\Delta\phi \cong \Delta L/L \sin\theta = \tau\,\Delta t/L \sin\theta$$

or

$$\omega_p = \Delta\phi/\Delta t = \tau/L \sin\theta. \qquad (13\text{–}1)$$

Since (Fig. 13–1*a*)

$$\tau = rmg \sin(180° - \theta) = rmg \sin\theta$$

we have finally

$$\omega_p = mgr/L. \qquad (13\text{–}2a)$$

Notice that the precessional angular velocity is independent of θ and varies inversely as the magnitude of the angular momentum. If the angular momentum is large, the precessional angular velocity will be small.

We can express Eq. 13–2*a* in vector form. We start by rewriting Eq. 13–1 as

$$\tau = \omega_p L \sin\theta.$$

Now ω_p is a vector pointing vertically upward in Fig. 13–1*b*, and θ in that figure is the angle between ω_p and **L**. We recognize the right side of the above equation as the magnitude of the vector product $\omega_p \times \mathbf{L}$ and we see that this equation gives the magnitude of τ in the vector relation

$$\boldsymbol{\tau} = \boldsymbol{\omega}_p \times \mathbf{L}. \qquad (13\text{–}2b)$$

This is the general vector expression relating the precessional angular velocity to τ and to **L**; the student should show that Eq. 13–2*a* may be derived readily from it. Application of right-hand rule to Fig. 13–1*b* shows that the order of factors on the right side of Eq. 13–2*b* is correct, that is, $\omega_p \times \mathbf{L}$ gives the correct direction as well as the correct magnitude for τ.

▶ **Example 1.** A student holds a rim-loaded bicycle wheel, rotating at a relatively high angular speed ω, with its shaft horizontal as in Fig. 13–2*a*. His physics instructor now asks him to turn the shaft rapidly (for a time Δt) so that the shaft points at a small angle $\Delta\theta$ above the horizontal as in Fig. 13–2*b*. He also asks the student to keep the shaft in a vertical plane at all times. What torques must the student exert on the shaft if he is to follow these instructions?

The student will be well aware from the strain in his wrists that he must exert a torque on the shaft simply to hold it in a horizontal position. This torque, which is needed to counteract the turning effect of the force of gravity that acts at the center of mass, is directed along a horizontal axis and emerges perpendicularly out of the plane of Fig. 13–2. The student must supply this torque whether or not the wheel is rotating.

If now the student turns the shaft of the spinning wheel upward, he will find that the wheel will swerve around to his right, perhaps rather violently, so that he will have failed to keep the shaft in a vertical plane. If he is to keep the shaft in this plane while he is tilting it upward, he must exert a torque on the shaft (about an almost vertical axis) tending to turn it to his left to counteract this effect. Let us see why this is so.

In tilting the shaft one changes its angular momentum **L**, in time Δt, by an amount $\Delta\mathbf{L}$, as Fig. 13–2*b* shows. During this interval, then, the student must exert an average torque on the wheel given from Eq. 12–9 as

$$\bar{\tau} = \Delta L/\Delta t;$$

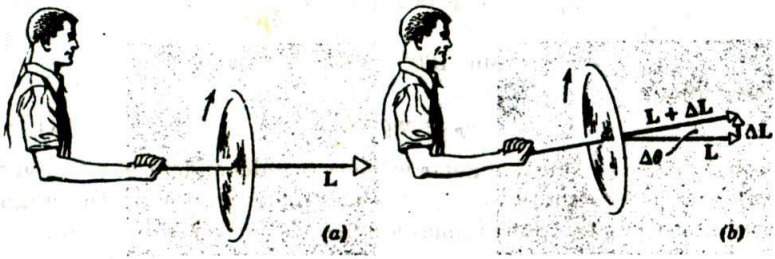


Fig. 13–2 A student holds a heavy, rim-loaded, rapidly spinning bicycle wheel by the shaft and tilts the shaft upward from the horizontal through a small angle.

the magnitude of $\bar{\tau}$ is given by

$$\bar{\tau} = \Delta L/\Delta t = L \sin \Delta\theta/\Delta t.$$

This average torque $\bar{\tau}$ has the same direction as $\Delta\mathbf{L}$, that is, it is approximately vertically upwards if the angle $\Delta\theta$ in Fig. 13–2*b* is not too large. We can see that such a torque would tend to turn the shaft to the left if the wheel was not rotating. This torque *must* be supplied by the student as he is tilting the shaft of the spinning wheel upward; if he fails to do so, the shaft will not remain in a vertical plane.

The student should experiment with such a spinning wheel, working out the relationships between the vectors $\mathbf{L}$, $\Delta\mathbf{L}$, and $\bar{\tau}$. If one is not available he can experiment with a toy gyroscope, although this fails to give the kinesthetic appreciation of $\tau = d\mathbf{L}/dt$ that is provided by a rim-loaded, rapidly spinning wheel.

There is an analogy between the experiment of Fig. 13–2 and another experiment in which the student is asked to swing a heavy weight (attached to a stout cord) around in a horizontal circle at constant speed. In this latter experiment the student, during a time Δt, must change the direction of the *linear momentum* $\mathbf{P}$ of the weight, leaving its magnitude unchanged. To do so, he must supply a *force* that points at right angles to $\mathbf{P}$ (in the direction of $\Delta\mathbf{P}$), that is, radially inward. In the experiment of Fig. 13–2 the student must, during a time Δt, change the direction of the *angular momentum* $\mathbf{L}$ of the wheel, leaving its magnitude unchanged. To do so he must supply a *torque* that points at right angles to $\mathbf{L}$ (in the direction of $\Delta\mathbf{L}$), that is, vertically upward.*

13–3 Angular Momentum and Angular Velocity

In this section it is our purpose to examine the relationship between the angular momentum and the angular velocity for particles and rigid bodies rotating about an axis fixed in an inertial reference frame.

First we consider a single particle of mass m moving with speed v in a circle about the z-axis of an inertial reference frame as in Fig. 13–3. Its angular velocity $\boldsymbol{\omega}$ lies on the z-axis and points upward. Its angular momentum $\mathbf{l}$ with respect to the origin O of the reference frame is given by Eq. 12–3, or

$$\mathbf{l} = \mathbf{r} \times \mathbf{p},$$

where $\mathbf{r}$ and $\mathbf{p}$ ($= m\mathbf{v}$) are shown in the figure. The vector $\mathbf{l}$ is perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{p}$, which means that $\mathbf{l}$ is *not* parallel to $\boldsymbol{\omega}$. Note that $\mathbf{l}$ has a (vector) component $\mathbf{l}_z$ which is parallel to $\boldsymbol{\omega}$, but it has another (vector) component $\mathbf{l}_\perp$ which is at right angles to $\boldsymbol{\omega}$. Note, too, that if we choose our origin to lie in the plane of the circulating particle, then $\mathbf{l}$ *is* parallel to $\boldsymbol{\omega}$; often, however, we do not find it convenient to make this choice.

The perhaps unexpected result that $\mathbf{l}$ and $\boldsymbol{\omega}$ are not parallel in this simple case may cause the student some concern. However, this result is quite in accord with the general relationship $\boldsymbol{\tau} = d\mathbf{l}/dt$ for a torque acting on a single particle. The vector $\mathbf{l}$ is changing with time as the particle motion

* This analogy is explored by A. E. Benfield, *American Journal of Physics*, September 1958. See also Problem 3.

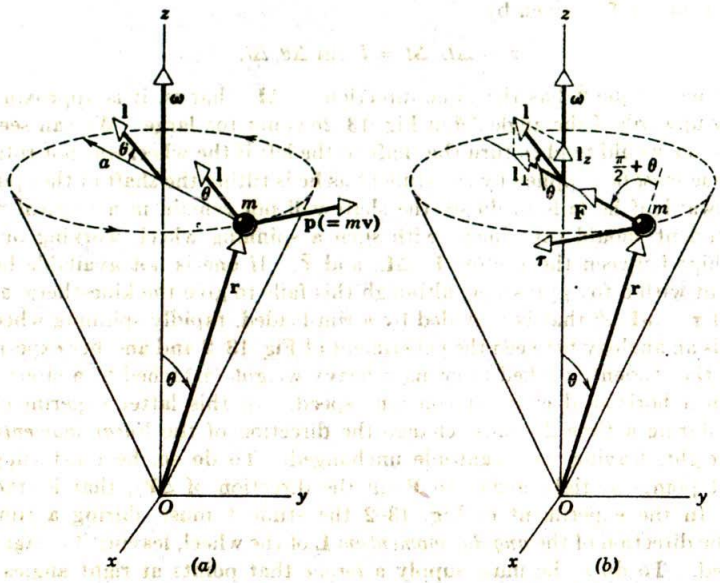


Fig. 13–3 (*a*) A particle of mass m rotating with speed v in a circle of radius a about the z-axis of an inertial reference frame. The angular momentum about O, $\mathbf{l}$ ($= \mathbf{r} \times \mathbf{p}$), is shown; for convenience, this vector is also shown translated to the center of the circle. (*b*) The same configuration, showing $\mathbf{l}$ and its components and also the centripetal force $\mathbf{F}$ and the torque $\boldsymbol{\tau}$ about O.

proceeds, the change being entirely in direction and not in magnitude, just as it was for the precessing top in the preceding section. Since the right side of the preceding relationship ($= d\mathbf{l}/dt$) has a nonzero value, the left side ($= \boldsymbol{\tau}$) must also have a nonzero value; that is, a torque must act on the particle with respect to origin O.

There is indeed such a torque. For if the particle moves in a circle, a centripetal force $\mathbf{F}$ must act on it, as in Fig. 13–3*b*. We may imagine that $\mathbf{F}$ is provided by the tension in a light cord that ties the rotating particle to the z-axis. The torque about O is provided by $\mathbf{F}$ and is given by Eq. 12–1.

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

The torque $\boldsymbol{\tau}$ is tangent to the circle (perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{F}$) and in the direction shown in Fig. 13–3*b*, as the student may verify from the right-hand rule.

▶ **Example 2.** Show that the moving particle of Fig. 13–3 satisfies the relation $\boldsymbol{\tau} = d\mathbf{l}/dt$ quantitatively.

The proof is along the same lines as that of Section 13–2 for the spinning top because, from a vector point of view, the two problems are identical. In each case we have the precession of an angular momentum vector ($\mathbf{L}$ for the top and $\mathbf{l}$ for the particle of Fig. 13–3) about a vertical axis, at a rate which we called ω_p

for the top and which we call ω for the particle. In each case we have a torque which is always at right angles to the plane formed by $\mathbf{L}$ (or $\mathbf{l}$) and $\boldsymbol{\omega}_p$ (or $\boldsymbol{\omega}$).

Thus, since the two problems are formally identical, it suffices to inquire whether the rotating particle of Fig. 13–3 obeys the vector equation for precession ($\boldsymbol{\tau} = \boldsymbol{\omega}_p \times \mathbf{L}$; Eq. 13–2*b*). This equation was derived for the precessing top directly from—and is directly equivalent to—the relation $\boldsymbol{\tau} = d\mathbf{L}/dt$ (Eq. 12–9). We can write the relation $\boldsymbol{\tau} = \boldsymbol{\omega}_p \times \mathbf{L}$ in terms of magnitudes as

$$\tau = \omega l \sin (90° - \theta) = \omega l \cos \theta, \tag{13–3}$$

in which we have substituted ω for ω_p and l for L and have noted in Fig. 13–3*a* that the angle between ω and l is $90° - \theta$. For τ and l, again using the notation of Fig. 13–3*c*, we can write

$$\tau = Fr \sin (90° + \theta) = [m\omega^2(r \sin \theta)](r)(\cos \theta)$$

and

$$l = rp \sin 90° = r(mv) = (r)(m)[\omega(r \sin \theta)],$$

in which $r \sin \theta$ is the radius a of the circle in which the particle moves, $(90° + \theta)$ is the angle between $\mathbf{r}$ and $\mathbf{F}$, and 90° is the angle between $\mathbf{r}$ and $\mathbf{p}$. Substituting these two expressions into Eq. 13–3 yields

$$m\omega^2 r^2 \sin \theta \cos \theta = \omega(m\omega r^2 \sin \theta) \cos \theta,$$

which is an identity. In terms of *magnitudes* we have proved our point. The student should refer to Fig. 13–3 and make certain that the *direction* of $\boldsymbol{\tau}$ is that of $d\mathbf{l}/dt$ (Eq. 12–7), or alternatively of $\boldsymbol{\omega} \times \mathbf{l}$ (Eq. 13–2*b*). ◀

Let us now investigate the relationship between $\mathbf{l}_z$ and $\boldsymbol{\omega}$ for the particle of Fig. 13–3. From Example 2 we have

$$l = mr^2\omega \sin \theta.$$

From Fig. 13–3*b* we see that

$$l_z = l \sin \theta = m\omega r^2 \sin^2 \theta.$$

Now $r \sin \theta = a$, the radius of the circle in which the particle moves. This leads to

$$l_z = ma^2\omega, \tag{13–4}$$

in which ma^2 is the rotational inertia I of the particle with respect to the z-axis. Thus

$$l_z = I\omega, \tag{13–5}$$

which is to be compared with Eq. 12–18 ($L = I\omega$) for the rotation of a rigid body about a fixed axis. Note that the vector relation $\mathbf{l} = I\boldsymbol{\omega}$ is *not* correct in this case because $\mathbf{l}$ and $\boldsymbol{\omega}$ do not point in the same direction. However, $\mathbf{l}_z$ and $\boldsymbol{\omega}$ *do*, so that we could write Eq. 13–5 in vector form as $\mathbf{l}_z = I\boldsymbol{\omega}$.

Now let us add another particle of mass m to the system of Fig. 13–3. In particular, let us add this particle in the same orbit, moving with the same speed, but always at a diametrically opposite point, on the other side of the axis of rotation. The angular momentum $\mathbf{l}_2$ with respect to O for this second particle will have the same magnitude as that of $\mathbf{l}_1$ for the first

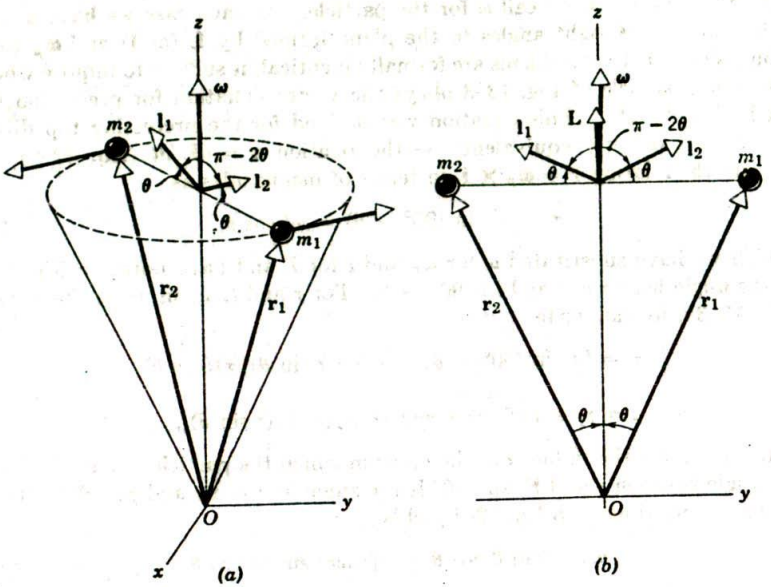


Fig. 13–4 (*a*) Two particles of mass m rotating as in Fig. 13–3 but maintaining diametrically opposite positions. (*b*) A cross section through the two particles, showing that the total angular momentum $\mathbf{L}$ $(= \mathbf{l}_1 + \mathbf{l}_2)$ for the two-particle system points along the axis of rotation, in the same direction as $\boldsymbol{\omega}$.

particle and it will make the same angle $(90° - \theta)$ with the z-axis, but it will have a different orientation around that axis. As Fig. 13–4*a* shows, $\mathbf{l}_2$ will lie in a plane formed by $\boldsymbol{\omega}$ and by $\mathbf{l}_1$ but will be on the opposite side of the z-axis from $\mathbf{l}_1$. The vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ include an angle of $180° - 2\theta$.

The total angular momentum $\mathbf{L}$ of the system of two particles is the vector sum of the angular momenta of the separate particles, that is, $\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2$. The resultant vector $\mathbf{L}$, as Fig. 13–4*b* shows, points along the z-axis (in the direction of $\boldsymbol{\omega}$) and is constant in magnitude. Note that this statement is true no matter where the origin O is located along the axis of rotation.

The fact that $\mathbf{L}$ = a constant (in both magnitude and direction), for this two-particle system, means that $d\mathbf{L}/dt = 0$, which in turn (Eq. 12–9) means that $\boldsymbol{\tau} = 0$ for this system. The student should convince himself (Fig. 13-3*b* will be helpful) that this is the case, the torques for the two particles about O being equal in magnitude but oppositely directed so that the torque acting on the two-particle system is zero.

The fact that $\boldsymbol{\omega}$ and $\mathbf{L}$ point in the same direction in this problem but did not for the case of a single particle can be traced to the fact that, in the two-particle system, the particles have the same mass and are in diametrically opposite positions at the same distance from the rotation axis.

We can now extend our system to a rigid body, made up of many par-

ticles. If the body is symmetric about the axis of rotation, by which we mean that for every mass element in the body there must be an identical mass element diametrically opposite the first element and at the same distance from the axis of rotation, then the body can be regarded as made up of sets of particle pairs of the kind we have been discussing. Since $\mathbf{L}$ and $\boldsymbol{\omega}$ are parallel for all such pairs, they are also parallel for rigid bodies that possess this kind of symmetry. Note that, in Table 12–1, all the systems except *f* and *j* meet this criterion.

For such symmetrical rigid bodies $\mathbf{L}$ and $\boldsymbol{\omega}$ are parallel and we can write Eq. 12–18 ($L = I\omega$) in vector form as

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{13–6}$$

The student must not forget, however, that, if $\mathbf{L}$ stands for the *total* angular momentum, then Eq. 13–6 applies *only* to bodies that have symmetry* about the (fixed) rotational axis. If $\mathbf{L}$ stands for the vector component of angular momentum along the rotational axis (that is, for $\mathbf{L}_z$), then Eq. 13–6 is equivalent to Eq. 12–8 and holds for *any* rigid body, symmetrical or not, that is rotating about a fixed axis.

▶ **Example 3.** Solve the problem of Example 5, Chapter 12, by direct application of Eq. 12–9 ($\boldsymbol{\tau} = d\mathbf{L}/dt$).

The system of Fig. 12–12, consisting of the wheel M and the mass m is acted on by two external forces, the downward pull of gravity $m\mathbf{g}$ acting on mass m and the upward force exerted by the bearings of the shaft of the cylinder, which we take as our origin. The tension in the cord is an internal force and does not act from the outside on the system (wheel + weight). Only the first of these external forces exerts a torque about the origin and its magnitude is $(mg)R$.

The angular momentum of the system about the origin at any instant is

$$L = I\omega + (mv)R,$$

in which $I\omega$ is the angular momentum of the (symmetrical) disk and $(mv)R$ is the angular momentum (= linear momentum × moment arm) of the falling mass about the origin. Both these contributions to L point in the same direction, namely, perpendicularly out of the plane of Fig. 12–12.

Applying $\boldsymbol{\tau} = d\mathbf{L}/dt$ (in scalar form) yields

$$\begin{aligned}(mg)R &= \frac{d}{dt}(I\omega + mvR)\\ &= I(d\omega/dt) + mR(dv/dt)\\ &= I\alpha + mRa.\end{aligned}$$

* We have oversimplified the symmetry requirement. Every rigid body, no matter how irregular its shape, has three perpendicular axes through its center of mass, about each of which $\mathbf{L}$ and $\boldsymbol{\omega}$ have the same direction, being related by $\mathbf{L} = I\boldsymbol{\omega}$. These axes are called the *principal axes*. The axis of a figure of revolution is always a principal axis, as are axes at right angles to it through the center of mass. In general, however, $\mathbf{L}$ and $\boldsymbol{\omega}$ point in different directions for axes that are not principal axes. See Arnold Sommerfeld, *Mechanics*, Chapter IV, Academic Press, New York, (1964 paperback edition).

Since $a = \alpha R$ and $I = \frac{1}{2}MR^2$, this reduces to

$$mgR = (\tfrac{1}{2}MR^2)(a/R) + mRa$$

or

$$a = \frac{2mg}{M + 2m}.$$

Example 4. A simple example of an unsymmetrical rotating rigid body is a dumbbell-type rod whose bar makes an angle θ with the fixed axis of rotation passing through its center of mass. The rod rotates at a constant angular velocity ω about this axis, the vector $\boldsymbol{\omega}$ thus pointing along this axis, as shown in Fig. 13–5. Experience tells us that such a system is "unbalanced" or "lop-sided," and if it were not securely fastened to the vertical shaft near C it would break away from the shaft at high angular velocities. It would tend to move until the angle θ becomes 90°, in which limiting position the system would then be symmetrical about the shaft.

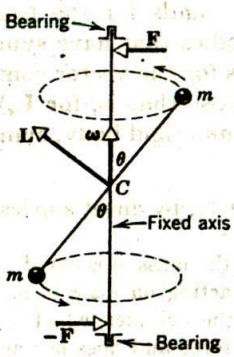


Fig. 13–5

(*a*) Show qualitatively that in the unsymmetrical case shown in Fig. 13–5 **L** and $\boldsymbol{\omega}$ are not parallel.

Each particle of mass m has an angular momentum with respect to C given by $\mathbf{r} \times \mathbf{p}$ for that particle. At the instant shown the upper particle is moving into the page at right angles to it, and the lower particle is moving out of the page at right angles to it. The momentum vectors of the two masses are therefore equal but opposite, and so are their position vectors with respect to C. Hence, by application of the right-hand rule in $\mathbf{r} \times \mathbf{p}$, we find that **l** is the same for each particle and that their sum, the total angular momentum vector **L** of the dumbbell, is as shown in the figure, at right angles to the bar in the plane of the page. Hence **L** and $\boldsymbol{\omega}$ are *not* parallel at this instant. It is clear that as the dumbbell itself rotates, the angular momentum vector, while constant in magnitude, rotates around the fixed axis of rotation.

(*b*) The fact that **L** and $\boldsymbol{\omega}$ do not point in the same direction is perfectly consistent with the fundamental relation $\boldsymbol{\tau} = d\mathbf{L}/dt$. We have seen twice before (see Section 13–2 and Example 1) that an angular momentum vector of constant magnitude that rotates around a fixed axis must have associated with it a torque $\boldsymbol{\tau}$ that is at right angles to the plane formed by **L** and $\boldsymbol{\omega}$. At the instant shown in Fig. 13–5 this plane is the plane of the figure. Is there such a torque in this problem and if so, where does it come from?

There is indeed such a torque and it arises from the unbalanced sideways forces exerted by the bearings on the shaft and transmitted by the shaft to the dumbbell bar. At the instant shown in the figure the upper end of the dumbbell would tend to move outward to the right. The shaft would be pulled to the right against the upper bearing, which in turn exerts a force **F** on the shaft that points to the left. Similarly, the lower end of the dumbbell tends to move outwards to the left. The shaft would be pulled to the left against the lower bearing, which in turn exerts a force $-\mathbf{F}$ on the shaft that points to the right. The torque $\boldsymbol{\tau}$ about C as a result of these forces points perpendicularly out of the page, at right angles

to the plane formed by $\mathbf{L}$ and $\boldsymbol{\omega}$, and in the right direction to account for the rotary motion of $\mathbf{L}$ (the student should check this).

The forces $\mathbf{F}$ and $-\mathbf{F}$ lie in the plane of Fig. 13–5 at the instant shown. As the dumbbell rotates, these forces, and therefore the torque $\boldsymbol{\tau}$, rotate with it, so that $\boldsymbol{\tau}$ always remains at right angles to the plane formed by $\boldsymbol{\omega}$ and $\mathbf{L}$ (compare with Fig. 13–1). The rotating forces $\mathbf{F}$ and $-\mathbf{F}$ cause a "wobble" in the upper and lower bearings. The bearings and their supports must be made strong enough to provide these forces. For a symmetrical rotating body there is no bearing wobble and the shaft rotates smoothly.

Bearing wobble and internal strains can cause serious practical problems when objects, such as turbine rotors, are made to rotate at high speeds. Although designed to be symmetrical, such rotors, because of small errors of blade placement, etc., may be slightly unsymmetrical. They may be restored to symmetry by the addition or removal of metal at appropriate places; this is done by spinning the wheel in a special device such that bearing wobble can be measured quantitatively and the appropriate corrective measure computed and indicated automatically. We are all familiar with lead weights placed at strategic points on automobile tire rims to reduce wobble at high speeds due to unbalance. ◀

13–4 Conservation of Angular Momentum

In Chapter 12, we found that the time rate of change of the total angular momentum of a system of particles about a point fixed in an inertial reference frame (or about the center of mass) is equal to the sum of the *external* torques acting on the system, that is,

$$\boldsymbol{\tau}_{ext} = d\mathbf{L}/dt. \tag{12–9}$$

Suppose now that $\boldsymbol{\tau}_{ext} = 0$; then $d\mathbf{L}/dt = 0$ so that $\mathbf{L}$ = a constant.

When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant. This is the *principle of the conservation of angular momentum.*

For a system of n particles, the total angular momentum $\mathbf{L}$ about some point is

$$\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2 + \cdots + \mathbf{l}_n.$$

When the resultant external torque on the system is zero, we have

$$\mathbf{L} = \text{a constant} = \mathbf{L}_0, \tag{13–7}$$

where $\mathbf{L}_0$ is the constant total angular momentum vector. The angular momenta of the individual particles may change, but their vector sum $\mathbf{L}_0$ remains constant in the absence of a net external torque.

Angular momentum is a vector quantity so that Eq. 13–7 is equivalent to three scalar equations, one for each coordinate direction through the reference point. The conservation of angular momentum therefore supplies us with three conditions on the motion of a system to which it applies.

For a system consisting of a rigid body rotating about an axis (the z-axis, say) that is fixed in an inertial reference frame, we have

$$\mathbf{L}_z = I\boldsymbol{\omega}, \tag{13–6}$$

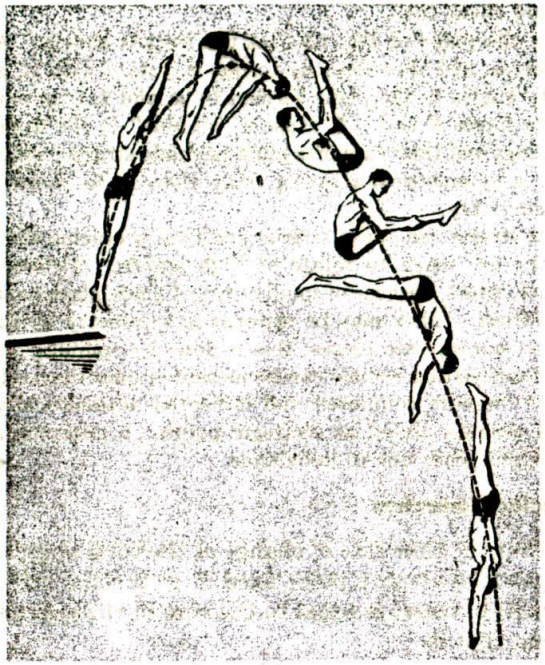


Fig. 13–6 A diver leaves the diving board with arms and legs outstretched and with some initial angular velocity. Since no torques are exerted on him about his center of mass, $L\ (= I\omega)$ is constant while he is in the air. When he pulls his arms and legs in, since I decreases, ω increases. When he again extends his limbs, his angular velocity drops back to its initial value. Notice the parabolic motion of his center of mass, common to all two-dimensional motion under the influence of gravity.

where $\mathbf{L}_z$ is the component of the angular momentum along the rotation axis and I is the rotational inertia for this same axis. It is possible for the rotational inertia I of a rotating body to change by rearrangement of its parts. If no net external torque acts, then $\mathbf{L}_z$ must remain constant and, if I does change, there must be a compensating change in ω. The principle of conservation of angular momentum in this case is expressed as

$$I\omega = I_0\omega_0 = \text{a constant.} \tag{13–8}$$

Equation 13–8 holds not only for rotation about a fixed axis but also about an axis through the center of mass of the system that moves so that it always remains parallel to itself (see p. 268).

Acrobats, divers, ballet dancers, ice skaters, and others often use this principle. Because I depends on the square of the distance of the parts of the body from the axis of rotation, a large variation is possible by extending or pulling in the limbs. Consider the diver in Fig. 13–6. Let us assume that as he leaves the diving board he has a certain angular speed ω_0 about a horizontal axis through the center of mass, such that he would rotate through half a turn before he strikes water. If he wishes to make a one and one-half turn somersault instead, in the same time, he must triple his angular speed. Now there are no external forces acting on him except gravity, and gravity exerts no torque about his center of mass. His angular momentum therefore remains constant, and $I_0\omega_0 = I\omega$. Since

$\omega = 3\omega_0$, the diver must change his rotational inertia about the horizontal axis through the center of mass from the initial value I_0 to a value I, such that I equals $\frac{1}{3}I_0$. This he does by pulling in his arms and legs toward the center of his body. The greater his initial angular speed and the more he can reduce his rotational inertia, the greater the number of revolutions he can make in a given time.

We should notice that the rotational kinetic energy of the diver is not constant. In fact, in our example, since

$$I\omega = I_0\omega_0$$

and

$$I < I_0,$$

it follows that

$$\tfrac{1}{2}I\omega^2 > \tfrac{1}{2}I_0\omega_0^2,$$

and the diver's rotational kinetic energy *increases*. This increase in energy is supplied by the diver, who does work when he pulls the parts of his body together.

In a similar way the ice skater and ballet dancer can increase or decrease the angular speed of a spin about a vertical axis. A cat manages to land on its feet after a fall by using the same principles, the tail serving as a useful, but unessential, extra appendage.

▶ **Example 5.** A small object of mass m is attached to a light string which passes through a hollow tube. The tube is held by one hand and the string by the other. The object is set into rotation in a circle of radius r_1 with a speed v_1. The string is then pulled down, shortening the radius of the path to r_2 (Fig. 13–7). Find the new linear speed v_2 and the new angular speed ω_2 of the object in terms of the initial values v_1 and ω_1 and the two radii.

The downward pull on the string is transmitted as a radial force on the object. Such a force exerts a zero torque on the object about the center of rotation. Since no torque acts on the object about its axis of rotation, its angular momentum in that direction is constant. Hence

initial angular momentum = final angular momentum,

$$mv_1r_1 = mv_2r_2,$$

and

$$v_2 = v_1\left(\frac{r_1}{r_2}\right).$$

Since $r_1 > r_2$, the object speeds up on being pulled in.

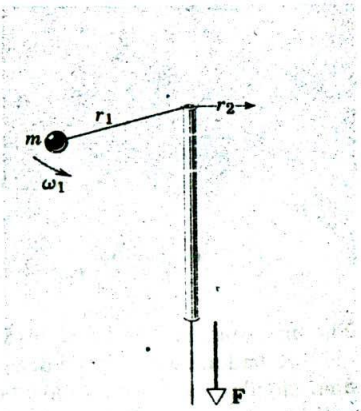


Fig. 13–7 Example 5. A mass at the end of a cord moves in a circle of radius r_1 with angular speed ω_1. The cord passes down through a tube. F supplies the centripetal force.

In terms of angular speed, since v_1 equals $\omega_1 r_1$ and v_2 equals $\omega_2 r_2$,

$$mr_1^2\omega_1 = mr_2^2\omega_2$$

and

$$\omega_2 = \left(\frac{r_1}{r_2}\right)^2 \omega_1,$$

so that there is an even greater increase in angular speed over the initial value (see Problem 23). What effect does the force of gravity (the object's weight) have on this analysis?

Example 6. A student sits on a stool that is free to rotate about a vertical axis. He holds his arms extended horizontally with an 8.0-lb weight in each hand. The instructor sets him rotating with an angular speed of 0.50 rev/sec. Assume that friction is negligible and exerts no torque about the vertical axis of rotation. Assume that the rotational inertia of the student remains constant at 4.0 slug-ft^2 as he pulls his hands to his sides and that the change in rotational inertia is due only to pulling the weights in. Take the original distance of the weights from the axis of rotation to be 3.0 ft and their final distance 0.50 ft. Find the final angular speed of the student.

The only external force is gravity acting through the center of mass, and that exerts no torque about the axis of rotation. Hence the angular momentum is conserved about this axis and

$$\text{initial angular momentum} = \text{final angular momentum},$$

$$I_0\omega_0 = I\omega.$$

We have

$$I = I_{\text{student}} + I_{\text{weights}},$$

$$I_0 = 4.0 + 2\left(\frac{8.0}{32}\right)(3.0)^2 = 8.5 \text{ slug-ft}^2,$$

$$I = 4.0 + 2\left(\frac{8.0}{32}\right)\left(\frac{1}{2}\right)^2 = 4.1 \text{ slug-ft}^2,$$

$$\omega_0 = 0.50 \text{ rev/sec} = \pi \text{ radians/sec}.$$

Therefore

$$\omega = \frac{I_0}{I}\omega_0 = \frac{8.5}{4.1}\pi \text{ radians/sec} = 2.1\pi \text{ radians/sec} \cong 1.0 \text{ rev/sec}.$$

The final angular speed is approximately doubled.

If we had allowed for the decrease in I caused by the arms being pulled in, the final angular speed would have been much greater.

What change would friction make? Is kinetic energy conserved as the student pulls in his arms and then puts them out again, assuming there is no friction? Explain.

Example 7. A classroom demonstration that illustrates the vector nature of the law of conservation of angular momentum is worth considering.

A student stands on a platform that can rotate only about a vertical axis. In his hand he holds the axle of a rim-loaded bicycle wheel with its axis vertical. The wheel is spinning about this vertical axis with an angular speed ω_0, but the

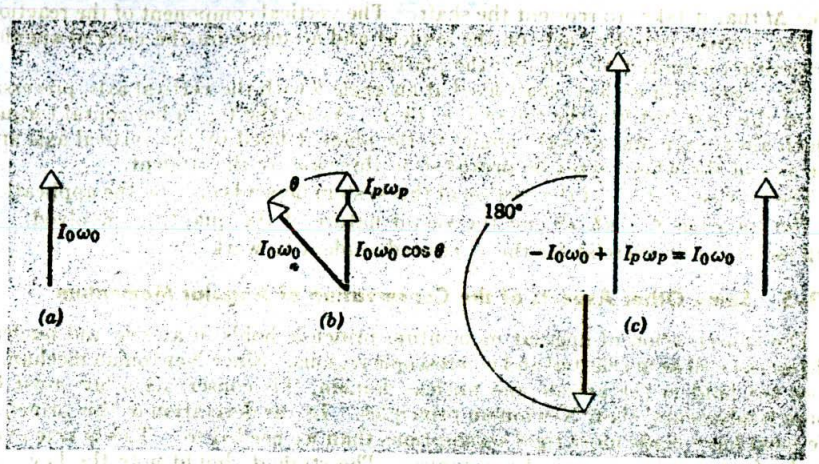


Fig. 13–8 Example 7. (*a*) The initial angular momentum of the system is shown. In (*b*), the wheel has been tilted an angle θ. Since no external torque in the vertical direction has been exerted on the system, the angular momentum in that direction must be conserved. The deficit, $(1 - \cos\theta)I_0\omega_0$, is made up by rotation of the student and platform. In (*c*), the wheel has been tilted 180°. The deficit is now $2I_0\omega_0$, which is now, as before, made up by the student and platform.

student and platform are at rest. The student tries to change the direction of rotation of the wheel. What happens?

Let us choose as the system the student plus platform plus wheel. The initial total angular momentum of this system is $I_0\omega_0$, arising from the spinning wheel, I_0 being the rotational inertia of the wheel about its axis and ω_0 pointing vertically upward. Figure 13–8*a* shows the initial condition.

The student next turns the axis of the wheel through an angle θ from the vertical (to do this he must supply a torque; see Example 1. This torque, however, is *internal* to the system as we have defined it). Since there is no *external* component of torque on the system about the vertical axis, the vertical component of angular momentum of the system must be conserved. The wheel, however, is now spinning about an axis making an angle θ with the vertical so that it contributes a vertical component of angular momentum of only $I_0\omega_0 \cos\theta$ to the system. Hence the student and platform must supply the additional angular momentum about the vertical axes, and they begin to rotate about a vertical axis. This extra vertical angular momentum $I_p\omega_p$ when added to $I_0\omega_0 \cos\theta$ must equal the initial vertical angular momentum of the system $I_0\omega_0$. That is,

$$I_p\omega_p = I_0\omega_0(1 - \cos\theta).$$

This is shown in Fig. 13–8*b*. I_p is the rotational inertia of student and platform with respect to the vertical axis, and ω_p is their angular speed about this axis.

If the student turns the wheel through an angle $\theta = 180°$, the student and platform acquire a vertical angular momentum of $2I_0\omega_0$. The total vertical angular momentum of the system is still being conserved at the initial value $I_0\omega_0$, as shown in Fig. 13–8*c*.

Consider now the angular momentum of the wheel alone. As the student turns the axis of the wheel through an angle θ he exerts a torque on it which lasts for the

time Δt that it takes to reorient the shaft. The vertical component of the reaction to this "torque-impulse" acts on the student and accounts for the vertical angular momentum acquired by him and the platform.

The wheel, held with its shaft fixed at an angle θ with the vertical axis, precesses about this axis just like the top of Fig. 13–1. As for the top, a horizontal torque, which always remains at right angles to the plane defined by the vertical axis and the axis of the wheel, must be provided, in this case by the student.

The precise analysis of the motion of this system depends only on the application of the equation $\boldsymbol{\tau} = d\mathbf{L}/dt$ and the vector nature of the quantities involved. It will be left as an exercise for the interested student to work out. ◀

13–5 Some Other Aspects of the Conservation of Angular Momentum

The conservation of angular momentum principle holds in atomic and nuclear physics as well as in celestial and macroscopic regions. Since Newtonian mechanics does not hold in the atomic and nuclear domain, this conservation law must be more fundamental than Newtonian principles. In our derivation of this principle we must have made more rigid assumptions than we needed to. This is true even in the framework of classical mechanics. The student should note the key role played by Newton's third law in our deduction of this conservation principle. This law was used to justify the assumption that the sum of the internal torques was zero. It was necessary to assert not only that the action and reaction forces were equal and opposite (the "weak" form of the third law) but also that these forces were directed along the line joining the two particles (the "strong" form of the third law). The strong form is known to be violated in some electromagnetic interactions. However, the assumption that the sum of the internal torques in a system of particles is zero can be proven on the basis of a much less stringent requirement than that the third law should hold.*

The law of conservation of angular momentum, as we have formulated it, holds for a system of bodies whenever the bodies can be treated as particles, that is, whenever effects due to the rotation of the individual bodies can be neglected. When the individual bodies have rotation, the conservation of angular momentum principle is still valid, providing we include the angular momentum associated with this rotation. However, the bodies then are no longer simple particles whose motion can be described by particle dynamics.

In atomic and nuclear physics we find that the "elementary particles" such as electrons, protons, mesons, and neutrons have angular momentum associated with an intrinsic spinning motion, as well as with orbital motion about some external point. When we use the law of conservation of total angular momentum we must include this *spin* angular momentum in the total. A fundamental aspect of atomic, molecular, and nuclear systems is that their angular momenta can take on only definite discrete values, rather than a continuum of values. Angular momentum is said to be *quantized*. Hence, angular momentum plays a central role in the description of the behavior of such systems (see Problems 4 and 6). These ideas will be developed in later chapters.

If we were to regard the sun, planets, and satellites as particles having no intrinsic spinning motion, the angular momentum of the solar system would turn out not to be constant. But these bodies do have intrinsic rotations; in fact, tidal forces convert some of the intrinsic spinning angular momentum into orbital angular momentum of the planets and satellites. When we use the law of conservation of the angular momentum, we must include this spin angular momentum in the total. The conservation of angular momentum plays a key role in the evaluation of theories of the origin of the solar system, the contraction of giant

* See E. Gerjuoy, *American Journal of Physics*, Vol. 17, 477 (1949).

stars, and other problems in astronomy.* Some astronomical applications will be considered in Chapter 16.

The basis for this rather simple way of analyzing the total angular momentum of atomic or astronomical systems is a theorem (see Problem 10) that the *total* angular momentum **L** of any system with respect to the origin of an inertial reference frame may be computed by adding the angular momentum with respect to its center of mass (*spin* angular momentum) to the angular momentum arising from the motion of the center of mass with respect to the origin (*orbital* angular momentum).

The conservation laws of total energy and of linear momentum and angular momentum are fundamental to physics, being valid in all modern physical theories. We shall have occasion to use them many times in later chapters.

13–6 Rotational Dynamics—A Review

The subject of the rotary motions of particles and rigid bodies is reasonably complicated, so much so in fact that a completely general treatment is beyond our scope here. It seems advisable to collect in one place all equations dealing with rotational dynamics and to comment on the conditions under which they can be used. We do this in Table 13–1.

Table 13–1

SUMMARY OF EQUATIONS FOR ROTARY MOTION

Eq. No.	Equation	Remarks
		I. Defining Equations
12–1	$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$	Torque on a particle about a point O, due to a resultant force $\mathbf{F}$
	$\boldsymbol{\tau}_{ext} = \Sigma\boldsymbol{\tau}_i = \Sigma(\mathbf{r}_i \times \mathbf{F}_i)$	Resultant external torque on a system of particles about a point O
12–3	$\mathbf{l} = \mathbf{r} \times \mathbf{p}$	Angular momentum of a particle about a point O
	$\mathbf{L} = \Sigma\mathbf{l}_i = \Sigma(\mathbf{r}_i \times \mathbf{p}_i)$	Resultant angular momentum of a system of particles about a point O
		II. General Relations
12–7	$\boldsymbol{\tau} = d\mathbf{l}/dt$	The law of motion for a single particle acted on by a torque. It is the rotational analog of $\mathbf{F} = d\mathbf{p}/dt$ (Eq. 9–12). Equation 12–7 holds only if $\boldsymbol{\tau}$ and $\mathbf{l}$ are measured with respect to any point O fixed in an inertial reference frame
12–9	$\boldsymbol{\tau}_{ext} = d\mathbf{L}/dt$	The law of motion for a system of particles acted on by a resultant external torque $\boldsymbol{\tau}_{ext}$. It is the rotational analog of $\mathbf{F} = d\mathbf{P}/dt$ (Eq. 9–7). Equation 12–9 holds only if $\boldsymbol{\tau}_{ext}$ and $\mathbf{L}$ are measured with respect to (*a*) any point O fixed in an inertial reference frame *or* (*b*) the center of mass of the system
		III. Special Case of a Rigid Body Rotating about an Axis Fixed in an Inertial Reference Frame (see footnote, p. 268)
12–17	$\tau = I\alpha$	$\boldsymbol{\alpha}$ is constrained to lie along the axis; I must also refer to this axis and τ must be the scalar component of $\boldsymbol{\tau}_{ext}$ directed along this same axis. It is the rotational analog of $F = Ma$ for rectilinear motion
12–18	$L = I\omega$	$\boldsymbol{\omega}$ is constrained to lie along the axis; I must also refer to this axis and L must be the scalar component along this axis of the total angular momentum. If the rotation axis has special symmetry (that is, if it is a principal axis; see footnote, p. 303), then $\mathbf{L}$ and $\boldsymbol{\omega}$ are both axially directed. It is the rotational analog of $P = Mv$ for rectilinear motion

QUESTIONS

1. (*a*) In Example 1, why would merely turning the shaft up send the wheel to the student's right? (*b*) If the student is anchored to the floor of a large spaceship that is drifting in a region free from gravity, in what way, if any, would this affect his performance of the experiment?

2. If the top of Fig. 13–1 were not spinning, it would tip over. If its spin angular momentum is large compared to the change caused by the applied torque, the top precesses. What happens in between when the top spins slowly?

3. A famous physicist (R. W. Wood), who was fond of practical jokes, mounted a rapidly spinning flywheel in a suitcase which he gave to a porter with instructions to follow him. What happens when the porter is led quickly around a corner? Explain in terms of $\tau = d\mathbf{L}/dt$.

4. A single-engine airplane must be "trimmed" to fly level. (Trimming consists of raising one aileron and lowering the opposite one.) Why is this necessary? Is this necessary on a bi-engine plane under normal circumstances?

5. The propeller of an aircraft rotates clockwise as seen from the rear. When the pilot pulls upward out of a steep dive, he finds it necessary to apply left rudder at the bottom of the dive if he is to maintain his heading. Explain.

6. Describe, in terms of $\tau = d\mathbf{L}/dt$, the rotational dynamics of the wheels on a fast train going around a curve.

7. You are walking along a narrow railroad track and you start to lose your balance. If you start falling to the right, which way do you turn your body to regain balance? Explain.

8. In order to get a billiard ball to roll without sliding from the start the cue must hit the ball not at the center (that is, a height above the table equal to the ball's radius R) but exactly at a height $\frac{7}{5}R$ *above* the center. Explain.†

9. A cylinder rotates with angular speed ω about an axis through one end, as in Fig. 13–9. Choose an appropriate origin and show qualitatively the vectors $\mathbf{L}$ and ω. Are these vectors parallel? Do symmetry considerations enter here?

10. Explain, in terms of angular momentum and rotational inertia, exactly how one "pumps up" a swing.

11. Consider the motion of a football tumbling through the air. Is the angular momentum with respect to the center of mass of the football conserved in flight? Does the magnitude or the direction of the angular velocity change with respect either to axes fixed in space or in the body?

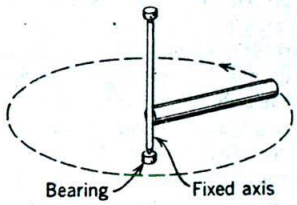


Fig. 13–9

12. In Chapter 1 the melting of the polar icecaps was cited as a possible cause of the variation in the earth's time of rotation. Explain.

13. Many great rivers flow toward the equator. What effect does the sediment they carry to the sea have on the rotation of the earth?

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* See, for example, *Frontiers of Astronomy*, by Fred Hoyle, Harper and Brothers, 1955.

† See Arnold Sommerfeld, *Mechanics, Volume I of Lectures on Theoretical Physics*, Academic Press, New York (1964 paperback edition), pp. 158–161, for a supplement on the mechanics of billiards.

14. A man turns on a rotating table with an angular speed ω. He is holding two equal masses at arm's length. Without moving his arms, he drops the two masses. What change, if any, is there in his angular speed? Is the angular momentum conserved? Explain.

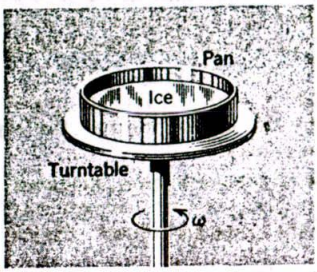


Fig. 13–10

15. In Example 5, if the string is released suddenly back to where the object can travel in a circle of radius r_1, will the object return to its original speed? What happens if one repeatedly pulls down on and suddenly releases the string? Explain the behavior in terms of work-energy and torque-angular momentum considerations.

16. A circular turntable rotates at constant angular velocity about a vertical axis. There is no friction and no driving torque. A circular pan rests on the turntable and rotates with it; see Fig. 13–10. The bottom of the pan is covered with a layer of ice of uniform thickness, which is, of course, also rotating with the pan. The ice melts but none of the water escapes from the pan. Is the angular velocity now greater than, the same as, or less than the original velocity? Give reasons for your answer.

PROBLEMS

1. A top is spinning 30 rev/sec about an axis making an angle of 30° with the vertical. Its mass is 0.50 kg and its rotational inertia is 5.0×10^{-4} kg-m. The center of mass is 4.0 cm from the pivot point. If the spin is clockwise as seen from above, what is the magnitude and direction of the angular velocity of precession?

2. With center and spokes of negligible mass, a certain bicycle wheel has a thin rim of radius 1.14 ft and weight 8.36 lb; it can turn on its axle with negligible friction. A man holds the wheel above his head with the axis vertical while he stands on a turntable free to rotate without friction; the wheel rotates clockwise, as seen from above, with an angular speed 57.7 radians/sec, and the turntable is initially at rest. The rotational inertia of wheel-plus-man-plus-turntable about the common axis of rotation is 1.54 slug ft^2. (*a*) The man's hand suddenly stops the rotation of the wheel (relative to the turntable). Determine the resulting angular velocity (magnitude *and* direction) of the turntable. (*b*) The experiment is repeated with noticeable friction introduced into the axle of the wheel, which, starting from the same initial angular speed 57.7 radians/sec, *gradually* comes to rest (relative to the turntable) while the man holds the wheel as described above. (The turntable is still free to rotate without friction.) Describe what happens to the system, giving as much quantitative information as the data permit.

3. Start from Eq. 11–20*b*, $\mathbf{a}_R = \boldsymbol{\omega} \times \mathbf{v}$, for a particle in circular motion and show that the force required for uniform circular motion is $\mathbf{F} = \boldsymbol{\omega} \times \mathbf{p}$. Compare this to Eq. 13–2*b*, $\boldsymbol{\tau} = \boldsymbol{\omega}_p \times \mathbf{L}$, and explain how the precessing spinning top can be regarded as a rotational analog to uniform circular motion.

4. (*a*) Assume that the electron moves in a circular orbit about the proton in a hydrogen atom. If the centripetal force on the electron is supplied by an electrical force $e^2/4\pi\epsilon_0 r^2$, where e is the magnitude of the charge of an electron and of a proton, r is the orbit radius, and ϵ_0 is a constant, show that the radius of the orbit is

$$r = \frac{e^2}{4\pi\epsilon_0 m v^2},$$

where m is the mass of the electron and v is its speed.

(*b*) Assume now that the angular momentum of the electron about the nucleus can only have values that are integral multiples n of $h/2\pi$, where h is a constant called Planck's constant. Show that the only electronic orbits possible are those with a radius

$$r = \frac{nh}{2\pi mv}.$$

(*c*) Combine these results to eliminate v and show that the only orbits consistent with both requirements have radii

$$r = \frac{n^2\epsilon_0 h^2}{\pi me^2}.$$

Hence the allowed radii are proportional to the square of the integers $n = 1, 2, 3$, etc. When $n = 1$, r is smallest and has the value 0.528×10^{-10} meter.

5. Show that $\mathbf{L} = I\boldsymbol{\omega}$ for the two-particle system of Fig. 13–4.

6. In 1913, Niels Bohr postulated that any mechanical rotating system with rotational inertia I can have an angular momentum whose values can take on only integral multiples of a particular number $\hbar = 1.054 \times 10^{-34}$ joule-sec. (This number is $1/2\pi$ times Planck's constant h.) In other words,

$$L = I\omega = n\hbar,$$

where n is any positive integer or zero. We say that L is quantized, since it is no longer allowed to have any value whatsoever. (*a*) Show that this postulate restricts the kinetic energy the rotating system can have to a set of discrete values, that is, that the energy is quantized. (*b*) Consider the so-called *rigid rotator*, consisting of a mass m constrained to rotate in a circle of radius R. With what angular speeds could the mass rotate if the postulate were correct? What values of kinetic energy may it assume? (*c*) Draw an energy-level diagram of some sort indicating how the spacing between the energy levels varies as n increases. It might look something like Fig. 13–11. Certain low-energy diatomic molecules behave like a rigid rotator.

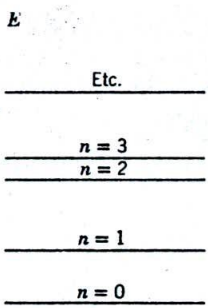


Fig. 13–11

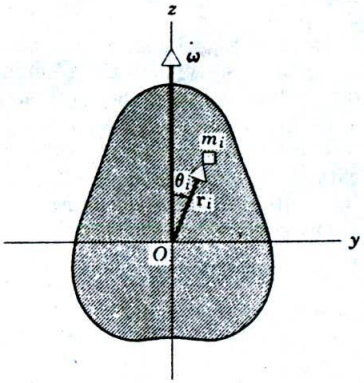


Fig. 13–12

7. Figure 13–12 shows a symmetrical rigid body rotating about a fixed axis. The origin of coordinates is fixed for convenience at the center of mass. Prove, by summing over the contributions made to the angular momentum by all of the mass elements m_i into which the body is divided, that $\mathbf{L} = I\boldsymbol{\omega}$, where $\mathbf{L}$ is the *total* angular momentum.

8. A cylinder rolls down an inclined plane of angle θ. Show, by direct application of Eq. 12-9 ($\tau = dL/dt$), that the acceleration of its center of mass is $\frac{2}{3}g \sin \theta$. Compare this method with that used in Example 10 of Chapter 12.

9. A thin rectangular sheet, of length a and width b, rotates about one of its diagonals with constant angular speed ω, the axis being fixed in an inertial reference frame. Find the direction and the magnitude of the angular momentum $\mathbf{L}$ with respect to an origin at the center of mass.

10. *Relation between the Total Angular Momentum of a System of Particles and the Orbital and Spin Angular Momenta.* The total angular momentum of a system of particles relative to the origin O of an inertial reference frame is given by $\mathbf{L} = \Sigma \mathbf{r}_i \times \mathbf{p}_i$, where $\mathbf{r}_i$ and $\mathbf{p}_i$ are measured with respect to O.

(*a*) Use the relations $\mathbf{r}_i = \mathbf{r}_{cm} + \mathbf{r}_i'$ and $\mathbf{p}_i = m_i\mathbf{v}_{cm} + \mathbf{p}_i'$ of Problem 8, Chapter 12, to express $\mathbf{L}$ in terms of the positions $\mathbf{r}_i'$ and momenta $\mathbf{p}_i'$ relative to the center of mass C. (*b*) Use the definition of center of mass and the definition of angular momentum $\mathbf{L}'$ with respect to the center of mass (Problem 8, Chapter 12) to obtain $\mathbf{L} = \mathbf{L}' + \mathbf{r}_{cm} \times M\mathbf{v}_{cm}$. (*c*) Show how this result can be interpreted as regarding the total angular momentum to be the sum of spin angular momentum (angular momentum relative to the center of mass) and orbital angular momentum (angular momentum of the motion of the center of mass C with respect to O if all the system's mass were concentrated at C).

11. The moon revolves about the earth so we always see the same face of the moon. (*a*) How are the spin and orbital parts of the angular momentum of the moon with respect to the earth related? (*b*) By how much would its spin angular momentum have to change if we were to be able to see all the moon's surface during the course of a month?

12. Using data in the appendix, find (*a*) the angular momentum of the earth's spin about its own axis, (*b*) the angular momentum of the earth's orbital motion about the sun.

13. In a playground there is a small merry-go-round of radius 4.0 ft and mass 12.0 slugs. The radius of gyration (see Problem 12, Chapter 12) is 3.0 ft. A child of mass 3.0 slugs runs at a speed of 10 ft/sec tangent to the rim of the merry-go-round when it is at rest and then jumps on. Neglect friction and find the angular velocity of the merry-go-round and child.

14. The time integral of a torque is called the *angular impulse*. Starting from $\tau = d\mathbf{L}/dt$, show that the resultant angular impulse equals the change in angular momentum. This is the rotational analog of the linear impulse-momentum theorem.

15. A meter stick lies on a frictionless horizontal table. It has a mass M and is free to move in any way on the table. A hockey puck m, moving as shown in Fig. 13–13, with speed v collides elastically with the stick. (*a*) What quantities are conserved in the collision? (*b*) What must be the mass m of the puck so that it remains at rest immediately after the collision?

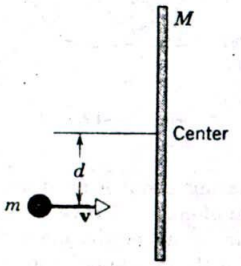


Fig. 13–13

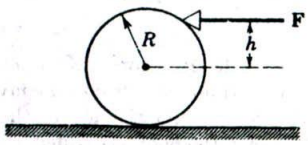


Fig. 13–14

16. A stick has a mass 0.30 slug and a length 4.0 ft. It is initially at rest on a frictionless horizontal plane and is struck perpendicularly by an impulsive force of impulse 3.0 lb-sec at a distance $l = 1.5$ ft from the center. Determine the subsequent motion.

17. A billiard ball, initially at rest, is given a sharp impulse by a cue. The cue is held horizontally a distance h above the centerline as in Fig. 13–14. The ball leaves the cue with a speed v_0 and, because of its "forward english," eventually acquires a final speed of $\frac{9}{7}v_0$. Show that

$$h = \tfrac{4}{5}R,$$

where R is the radius of the ball.

18. In the preceding problem, imagine **F** to be applied below the centerline. Show that it is impossible, with this "reverse english," to reduce the forward speed to zero unless $h = R$. Show similarly that it is impossible to give the ball a backward velocity unless **F** has a downward (vertical) component.

19. A wheel is rotating with an angular speed of 500 rev/min on a shaft whose rotational inertia is negligible. A second identical wheel, initially at rest, is suddenly coupled to the same shaft. What is the angular speed of the resultant combination of the shaft and the two wheels?

20. Two cylinders having radii R_1 and R_2 and rotational inertia I_1 and I_2, respectively, are supported by fixed axes perpendicular to the plane of Fig. 13–15. The large cylinder is initially rotating with angular velocity ω_0. The small cylinder is moved to the right until it touches the large cylinder and is caused to rotate by the frictional force between the two. Eventually, slipping ceases, and the two cylinders rotate at constant rates in opposite directions. Find the final angular velocity ω_2 of the small cylinder in terms of I_1, I_2, R_1, R_2, and ω_0. Is total angular momentum conserved in this case?

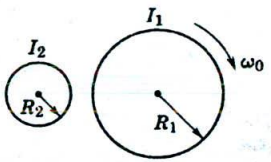


Fig. 13–15

21. The axis of the cylinder in Fig. 13–16 is fixed. The cylinder is initially at rest. The block of mass M is initially moving to the right without friction with speed v_1. It passes over the cylinder to the dotted position. When it first makes contact with the cylinder, it slips on the cylinder, but the friction is large enough so that slipping ceases

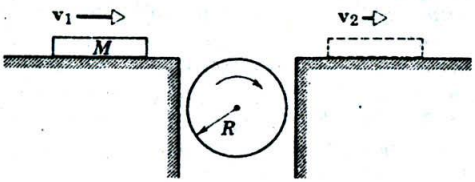


Fig. 13–16

before M loses contact with the cylinder. The cylinder has a radius R and a rotational inertia I. Find the final speed v_2 in terms of v_1, M, I, and R. This can be done most easily by using the relation between impulse and change in momentum.

22. A man stands on a frictionless rotating platform which is rotating with a speed of 1.0 rev/sec; his arms are outstretched and he holds a weight in each hand. With his hands in this position the total rotational inertia of the man, the weights and the platform is 6.0 kg-m². If by drawing in the weights the man decreases the rotational inertia to 2.0 kg-m², (*a*) what is the resulting angular speed of the platform? (*b*) By how much is the kinetic energy increased?

23. In Example 5 compare the kinetic energies of the object in two different orbits. Use the work-energy theorem to explain the difference quantitatively.

24. A cockroach, mass m, runs counterclockwise around the rim of a lazy-susan (a circular dish mounted on a vertical axle) of radius R and rotational inertia I with frictionless bearings. The cockroach's speed (relative to the earth) is v, whereas the lazy-susan turns clockwise with angular speed ω_0. The cockroach finds a bread crumb on the rim and of course, stops. What is the angular speed of the lazy-susan after the cockroach stops? Is energy conserved?

25. A particle is projected horizontally along the interior of a smooth hemispherical bowl of radius r which is kept at rest (Fig. 13–17). We wish to find the initial speed v_0 required for the particle to just reach the top of the bowl. Find v_0 as a function of θ_0, the initial angular position of the particle. (*Hint:* Use conservation principles.)

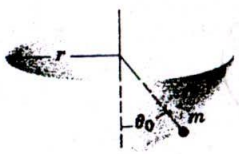


Fig. 13–17

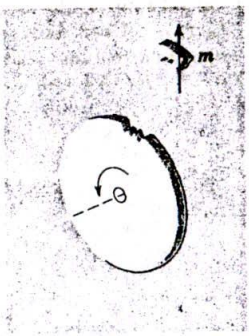


Fig. 13–18

26. A uniform flat disk of mass M and radius R rotates about a horizontal axis through its center with angular speed ω_0. (*a*) What is its kinetic energy? Its angular momentum? (*b*) A chip of mass m breaks off the edge of the disk at an instant such that the chip rises *vertically* above the point at which it broke off (Fig. 13–18). How high above the point does it rise before starting to fall? (*c*) What is the final angular speed of the broken disk? The final angular momentum and energy?

27. Two skaters, each of mass 50 kg, approach each other along parallel paths separated by 3.0 meters. They have equal and opposite velocities of 10 meters/sec. The first skater carries a long light pole, 3.0 meters long, and the second skater grabs the end of it as he passes. (Assume frictionless ice.) (*a*) Describe quantitatively the motion of the skaters after they are connected by the pole. (*b*) By pulling on the pole, the skaters reduce their distance apart to 1.0 meter. What is their motion then? (*c*) Compare the kinetic energy of the system in parts (*a*) and (*b*). Where does the change come from?

28. On a large horizontal frictionless circular track, radius R, lie two small masses m and M, free to slide on the track. Between the two masses is squeezed a spring which, however, is *not fastened* to m and M. The two masses are held together by a string

(*a*) If the string breaks, the compressed spring (assumed massless) shoots off the two masses in opposite directions; the spring itself is left behind. The balls collide when they again meet on the track (Fig. 13–19). Where does this collision take place? (You might find it convenient to express the answer in terms of the angle m or M travels through.) (*b*) If the potential energy initially stored in the spring was U_0, what is the time it takes after the string breaks for the collision to take place? (*c*) Assuming the collision to be perfectly elastic and head-on, where will the balls again collide after the first collision?

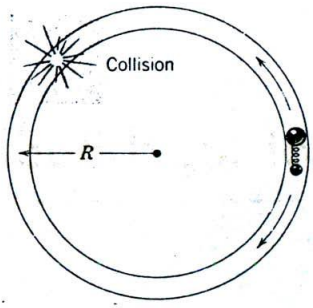


Fig. 13–19

Equilibrium of Rigid Bodies

CHAPTER 14

14–1 Rigid Bodies

The towers supporting a suspension bridge must be strong enough so that they do not collapse under the weight of the bridge and its traffic load; the landing gear of an aircraft must not collapse if the pilot makes a poor landing; the tines of a fork must not bend when we cut a tough steak. In all such problems the engineer is concerned that these presumed rigid structures do indeed remain rigid under the forces, and the associated torques, that act on them.

In such problems the engineer must ask two questions. (1) What forces and torques act on the presumed rigid body? (2) Considering its design and the materials used, will the body remain rigid under the action of these forces and torques? In this chapter we are concerned only with the first of these questions; the engineering student will deal at length with the second question in later courses.

14–2 The Equilibrium of a Rigid Body

We note that the presumed rigid bodies of the preceding section (that is, the bridge towers, the landing gear, and the fork) are in *mechanical equilibrium*. A rigid body is in mechanical equilibrium if, as viewed from an inertial reference frame, (1) the linear acceleration $\mathbf{a}_{cm}$ of its center of mass is zero and (2) its angular acceleration $\boldsymbol{\alpha}$ about any axis fixed in this reference frame is zero.

This definition does not require the body to be at rest with respect to the observer but only to be unaccelerated. Its center of mass, for example, may be moving with constant velocity $\mathbf{v}_{cm}$ and the body may be rotating

about a fixed axis with constant angular velocity ω. If the body is actually at rest (so that $v_{cm} = 0$ and $\omega = 0$), we often speak of *static equilibrium*. However, as we shall see, the restrictions imposed on the forces and torques are the same whether or not the equilibrium is static. Furthermore, we can transform any case of (nonstatic) equilibrium to one of static equilibrium by choosing an appropriate new reference frame.

The translational motion of a rigid body of mass M is governed by Eq. 9-10, or

$$\mathbf{F}_{ext} = M\mathbf{a}_{cm},$$

in which $\mathbf{F}_{ext}$ is the vector sum of all the external forces acting on the body. Because $\mathbf{a}_{cm}$ must be zero for equilibrium, the first condition of equilibrium (static or otherwise) is: *The vector sum of all the external forces acting on a body in equilibrium must be zero.*

We can write condition (1) as

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots = 0, \tag{14–1}$$

in which we have dropped the subscript on $\mathbf{F}_{ext}$ for convenience. This vector equation leads to three scalar equations.

$$\begin{aligned} F_x &= F_{1x} + F_{2x} + \cdots = 0, \\ F_y &= F_{1y} + F_{2y} + \cdots = 0, \\ F_z &= F_{1z} + F_{2z} + \cdots = 0, \end{aligned} \tag{14–2}$$

which state that the sum of the components of the forces along each of any three mutually perpendicular directions is zero.

The second requirement for equilibrium is that $\alpha = 0$ for any axis. Since the angular acceleration of a rigid body is associated with torque—recall that $\tau = I\alpha$ for a fixed axis—we can state this second condition of equilibrium (static or otherwise) as: *The vector sum of all the external torques acting on a body in equilibrium must be zero.*

We can write condition (2) as

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \cdots = 0. \tag{14–3}$$

This vector equation leads to three scalar equations

$$\begin{aligned} \tau_x &= \tau_{1x} + \tau_{2x} + \cdots = 0, \\ \tau_y &= \tau_{1y} + \tau_{2y} + \cdots = 0, \\ \tau_z &= \tau_{1z} + \tau_{2z} + \cdots = 0, \end{aligned} \tag{14–4}$$

which state that, at equilibrium, the sum of the components of the torques acting on a body, along each of any three mutually perpendicular directions, is zero.

The resultant torque $\boldsymbol{\tau}$ in Eq. 14–3, which must be zero for mechanical equilibrium, is defined with respect to a particular origin O. The quantities τ_x, τ_y, and τ_z in Eq. 14–4 are the scalar components of $\boldsymbol{\tau}$ and refer to

any set of three mutually perpendicular axes whose origin is at O, no matter how these axes are oriented in space. This follows because, if a vector is zero, its scalar components must be zero no matter how we orient the axes of the reference frame. The student may wonder whether the choice of an origin is essential. The answer—as we shall show below—is that it is not, because (for a body in translational equilibrium), if $\boldsymbol{\tau} = 0$ for any single origin O it is also zero for any other origin in the reference frame. The substance of this paragraph then is that condition 2 is satisfied for a body in translational equilibrium if we can show either that (*a*) $\boldsymbol{\tau} = 0$ with respect to *any* one point (Eq. 14–3) or that (*b*) the torque components along *any* three mutually perpendicular axes are zero (Eq. 14–4).

Let us now assume that we have a rigid body in translational equilibrium, so that $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots = 0$ (Eq. 14–1). We now wish to show that the torque about *any* point (such as P in Fig. 14–1) will be zero if the torque about one particular point (such as O in Fig. 14–1) is zero. The figure shows three of the n forces, $\mathbf{F}_1$, $\mathbf{F}_2 \cdots \mathbf{F}_n$, applied at various points on a rigid body and pointing in various directions. The points of application with respect to O are identified by displacement vectors, of which $\mathbf{r}_1$ is an example. The arbitrary point P is identified by displacement vector $\mathbf{r}_P$; the vector $\mathbf{r}_1 - \mathbf{r}_P$ locates the point of application of $\mathbf{F}_1$ with respect to point P.

We can write for the resultant torque about O (see Eq. 12–1)

$$\boldsymbol{\tau}_O = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \cdots + \mathbf{r}_n \times \mathbf{F}_n$$

and for the torque about P,

$$\boldsymbol{\tau}_P = (\mathbf{r}_1 - \mathbf{r}_P) \times \mathbf{F}_1 + (\mathbf{r}_2 - \mathbf{r}_P) \times \mathbf{F}_2 + \cdots + (\mathbf{r}_n - \mathbf{r}_P) \times \mathbf{F}_n.$$

We can expand the latter equation as

$$\boldsymbol{\tau}_P = [\mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \cdots + \mathbf{r}_n \times \mathbf{F}_n] - [\mathbf{r}_P \times (\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n)].$$

Now if, as we have assumed, the first condition of equilibrium is satisfied, then $\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = 0$ and the second term above in the square brackets

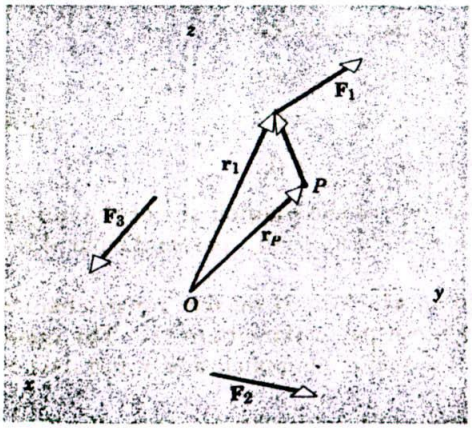


Fig. 14–1 Three of the n forces, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ . . . $\mathbf{F}_n$, that act on a rigid body, not shown. In the text we show that if $\boldsymbol{\tau} = 0$ for point O it also vanishes for any point such as P, assuming that the body is in translational equilibrium.

vanishes. The first term in the square brackets is simply τ_O (see above) so that, under these conditions

$$\tau_P = \tau_O.$$

Thus, for a body in translational equilibrium, if $\tau_O = 0$ then $\tau_P = 0$, where P is an arbitrary point.

Hence we have *six independent conditions* on our forces for a body to be in equilibrium. These conditions are the six algebraic relations of Eqs. 14–2 and 14–4. These six conditions are a condition on each of the six degrees of freedom of a rigid body, three translational and three rotational.

Often we deal with problems in which all the forces lie in a plane. Then we have only three conditions on our forces: The sum of their components must be zero for each of any two perpendicular directions in the plane, and the sum of their torques about any one axis perpendicular to the plane must be zero. These conditions correspond to the three degrees of freedom for motion in a plane, two of translation and one of rotation.

We shall limit ourselves henceforth mostly to planar problems in order to simplify the calculations. This does not impose any fundamental restriction on the general principles. Also, as a matter of convenience, we shall consider only the case of static equilibrium, in which bodies are actually at rest.

14–3 Center of Gravity

One of the forces encountered in rigid-body motions is the force of gravity. Actually, for an extended body, this is not just one force but the resultant of a great many forces. Each particle in the body is acted on by a gravitational force. If the body of mass M is imagined to be divided into a large number of particles, say n, the gravitational force exerted by the earth on the ith particle of mass m_i is $m_i\mathbf{g}$. This force is directed down toward the earth. If the acceleration due to gravity $\mathbf{g}$ is the same everywhere in a region, we say that a uniform gravitational field exists there; that is, $\mathbf{g}$ has the same magnitude and direction everywhere in that region. For a rigid body in a uniform gravitational field, $\mathbf{g}$ must be the same for each particle in the body and the weight forces on the particles must be parallel to one another. If we assume that the earth's gravitational field is uniform, we can show that all the individual weight forces acting on a body can be replaced by a single force $M\mathbf{g}$ acting down at the center of mass of the body. This is equivalent to showing that the individual weight forces, acting downward, can be counteracted in their accelerating effects by a single force $\mathbf{F}$ $(= -M\mathbf{g})$ acting upward, *provided this force* $\mathbf{F}$ *is applied at the center of mass of the body.*

Figure 14–2 shows two typical particles or mass elements m_1 and m_2, selected from the n such elements into which the rigid body has been divided. An upward force $\mathbf{F}$ $(= -M\mathbf{g})$ is applied at a certain point O. It remains to show that the body is in mechanical equilibrium *if* (and only if) point O is the center of mass. Condition 1 for equilibrium (Eq. 14–1) has already been satisfied by our choice of the magnitude and direction of

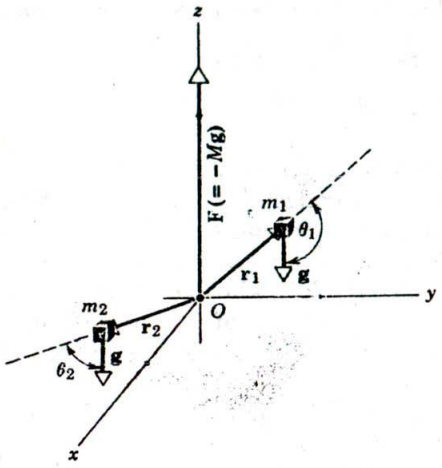


Fig. 14–2 An irregular body is divided into n mass elements of which two typical elements m_1 and m_2 are shown. In the text we prove that the body can be held in translational and rotational equilibrium by a single force $\mathbf{F}\,(=-M\mathbf{g})$ directed upward and applied at the center of mass of the body.

F. That is,

$$\mathbf{F} + m_1\mathbf{g} + m_2\mathbf{g} + \cdots + m_n\mathbf{g} = 0,$$

or

$$\mathbf{F} = -(m_1 + m_2 + \cdots + m_n)\mathbf{g} = -M\mathbf{g},$$

which corresponds to our assumption.

It remains to prove that $\boldsymbol{\tau} = 0$ for any single point in the body, such as O. This is the second condition for equilibrium. By choosing O as our origin we insure that the torque of **F** about this point is zero, because the moment arm of **F** is zero for this point. The torque about O due to the gravitational pull on the mass elements is

$$\boldsymbol{\tau} = \mathbf{r}_1 \times m_1\mathbf{g} + \mathbf{r}_2 \times m_2\mathbf{g} + \cdots + \mathbf{r}_n \times m_n\mathbf{g}$$

which (because m_1, m_2, etc., are scalars) we can write as

$$\boldsymbol{\tau} = m_1\mathbf{r}_1 \times \mathbf{g} + m_2\mathbf{r}_2 \times \mathbf{g} + \cdots + m_n\mathbf{r}_n \times \mathbf{g}.$$

Since **g** is the same in each term, we can factor it out, obtaining

$$\boldsymbol{\tau} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n) \times \mathbf{g}$$

$$= \left(\sum_1^n m_i\mathbf{r}_i\right) \times \mathbf{g},$$

in which the sum is taken over all the mass elements that make up the body.

Now *if* point O is the center of mass of the body, the sum above is zero. This follows from the definition of the center of mass (see Eq. 9–5*b* and the discussion following it). We conclude then that *if* (and only if) point O is the center of mass, then $\boldsymbol{\tau} = 0$ and the second condition for mechanical equilibrium is satisfied.

Thus the gravitational forces acting on the individual mass elements that make up a rigid body are equivalent in their translational and rotational effects to a single force equal to Mg, the total weight of the body, acting at the center of mass. We can obtain the same result if the body is continuous and divided into an infinite number of particles. The student should be able to do this by the methods of integral calculus (see Section 9–1). The point of application of the equivalent resultant gravitational force is often called the *center of gravity.*

The coincidence of the center of gravity and the center of mass came about because of the assumption that the earth's gravitational field was uniform. Actually this assumption is not strictly true, for the magnitude of **g** changes with distance from the center of the earth and furthermore the direction of **g** is radially in toward the center of the earth from any point (Chapter 16). To see the effect this has, let us consider a uniform stick many miles long inclined to the vertical in the earth's gravitational field, as in Fig. 14–3. The center of gravity of a body is the point at which the equivalent resultant gravitational force on it acts. This point must be the same as the point at which a single oppositely directed force is applied for the body to be kept in equilibrium. If the field were uniform, a single upward force of magnitude Mg at the center of mass would keep the stick in translational and rotational equilibrium. But the field is not uniform, and the value of g at m_1 is less than the value of g at m_n. The point at which a single force must be applied to keep the body in equilibrium is therefore at a point P some distance below the center of mass. Furthermore, if the orientation of the body is changed, the position of the point P, required for application of an equilibrium force, changes. Hence center of gravity really has little usefulness in such a case. Not only does it not coincide with the center of mass, but its position changes with respect to the body as the body is moved.

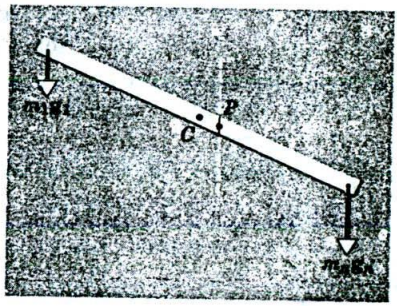


Fig. 14–3 The center of mass C and center of gravity P in reality do not coincide, since the earth's gravitational field is not uniform.

Because almost all problems in mechanics involve objects having dimensions small compared to the distances over which **g** changes appreciably, we can assume that **g** is uniform over the body. The center of mass and the center of gravity can then be taken as the same point. In fact, we can use this coincidence to determine experimentally the center of mass in irregularly shaped objects. For example, let us locate the center of mass of a thin plate of irregular shape, as in Fig. 14–4. We suspend the body by a cord from some point A on its edge. When the body is at rest, the center of gravity must lie directly under the point of support somewhere on the line Aa, for only then can the torque resulting from the cord and the weight add to zero. We next suspend the body from another point B on its edge. Again, the center of gravity must lie somewhere on Bb. The only

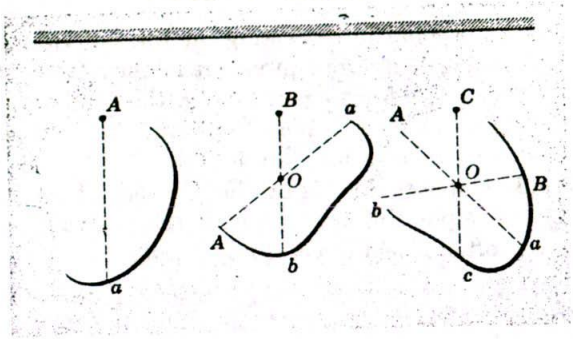


Fig. 14–4 Since the center of mass O always hangs directly below the point of suspension, hanging a plate from two different points determines O.

point common to the lines Aa and Bb is O, the point of intersection, so that this point must be the center of gravity. If now we suspend the body from any other point on its edge, as C, the vertical line Cc will pass through O. Since we have assumed a uniform field, the center of gravity coincides with the center of mass, which is therefore located at O.

14–4 Examples of Equilibrium

In applying the conditions for equilibrium (zero resultant force and zero resultant torque about any axis), we can clarify and simplify the procedure in many ways.

First, we draw an imaginary boundary around the system under consideration. This assures that we see clearly just what body or system of bodies it is to which we are applying the laws of equilibrium. This process is called isolating the system.

Second, we draw vectors representing the magnitude, direction, and point of application of all *external* forces. An external force is one that acts from outside the boundary which was drawn earlier. Examples of external forces often encountered are gravitational forces and forces transmitted by strings, wires, rods, and beams which cross the boundary. A question sometimes arises about the direction of a force. In this case make an imaginary cut through the member transmitting the force at the point where it crosses the boundary. If the ends of this cut tend to pull apart, the force acts outward. If you are in doubt, choose the direction arbitrarily. A negative value for a force in the solution means that the force acts in the direction opposite to that assumed. Note that only external forces acting on the system need be considered; all internal forces cancel one another in pairs.

Third, we choose a convenient reference frame along whose axes we resolve the external forces before applying the first condition of equilibrium (Eq. 14–2). The object here is to simplify the calculations. The preferable reference frame is usually obvious.

Fourth, we choose a convenient reference frame along whose axes we resolve the external torques before applying the second condition of equilibrium (Eq. 14–4). The object again is to simplify calculations and we may use different reference frames in applying the two conditions for static equilibrium if this proves to be convenient. Suppose that an axis passes through the point at which two forces concur and is at right angles to the plane formed by these forces; these forces will automatically have no torque component along (or about) this axis. The torque components resulting from all external forces must be zero about any axis for equilibrium. Internal torques will cancel in pairs and need not be considered.

▶ **Example 1.** (*a*) A uniform steel meter bar rests on two scales at its ends (Fig. 14–5). The bar weighs 4.0 lb. Find the readings on the scales.

Our system is the bar. The forces acting on the bar are **W**, the gravitational force acting down at the center of gravity, and $\mathbf{F}_1$ and $\mathbf{F}_2$, the forces exerted upward on the bar at its ends by the scales. These are shown in Fig. 14–5*a*. By Newton's third law, the force exerted by a scale on the bar is equal and opposite to that exerted by the bar on the scale. Therefore, to obtain the readings on the scales, we must determine the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$.

For translational equilibrium (Eq. 14–1) the condition is

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{W} = 0.$$

All forces act vertically, so that if we choose the y-axis to be vertical, no other axes need be considered. Then we get the scalar equation

$$F_1 + F_2 - 4.0 \text{ lb} = 0.$$

For rotational equilibrium, the component of the resultant torque on the bar must be zero about *any* axis. We have seen that it is enough to show that the torque components are zero for any set of three mutually perpendicular axes. These components are certainly zero for any two perpendicular axes that lie in the plane of Fig. 14–5*a* (Why?). It remains to require that the resultant torque is zero about any one axis at right angles to the plane of the figure. Let us choose an axis through the center of gravity. Then, taking clockwise rotation as positive and counterclockwise rotation as negative, the condition for rotational equilibrium

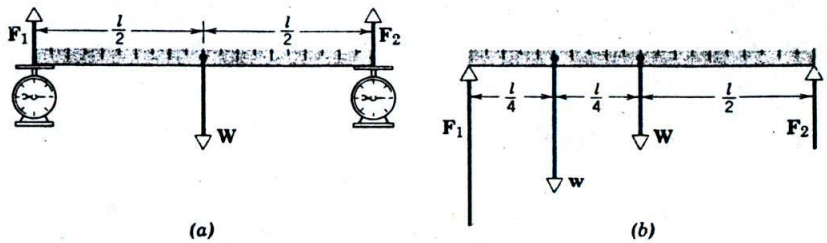


Fig. 14–5 (*a*) Example 1*a*. A uniform steel bar rests on two spring scales. (*b*) Example 1*b*. A weight is suspended a quarter of the way from one end.

(Eq. 14–4) is

$$F_1\left(\frac{l}{2}\right) - F_2\left(\frac{l}{2}\right) + W(0) = 0,$$

or

$$F_1 - F_2 = 0.$$

Combining the two equations, we obtain

$$F_1 + F_2 = 2F_1 = 2F_2 = 4.0 \text{ lb},$$

$$F_1 = F_2 = 2.0 \text{ lb}.$$

Each scale reads 2.0 lb, as we might have expected.

If we had chosen an axis through one end of the bar, we would have obtained the same result. For example, taking torques about an axis through the right end, we obtain

$$F_1(l) - W\left(\frac{l}{2}\right) + F_2(0) = 0,$$

or

$$F_1 = \frac{W}{2} = \frac{4.0 \text{ lb}}{2} = 2.0 \text{ lb}.$$

Combining this with $F_1 + F_2 = 4.0$ lb, we obtain $F_2 = 2.0$ lb, as before.

(*b*) Suppose that a 6.0-lb block is placed at the 25-cm mark on the meter bar. What do the scales read now?

The external forces acting on the bar are shown in Fig. 14–5*b*, where w is the force exerted on the bar by the block. The first condition for equilibrium is

$$F_1 + F_2 - W - w = 0.$$

With $W = 4.0$ lb and $w = 6.0$ lb, we obtain

$$F_1 + F_2 = 10 \text{ lb}.$$

If we take an axis through the left end of the bar, the second condition for equilibrium is

$$w\left(\frac{l}{4}\right) + W\left(\frac{l}{2}\right) - F_2(l) = 0.$$

With $W = 4.0$ lb and $w = 6.0$ lb, we obtain

$$F_2 = 3.5 \text{ lb}.$$

Putting this result into the first equation, we obtain

$$F_1 + 3.5 \text{ lb} = 10 \text{ lb},$$

$$F_1 = 6.5 \text{ lb}.$$

The left-hand scale reads 6.5 lb and the right-hand scale reads 3.5 lb at equilibrium.

Why do we obtain only two conditions on the forces in this problem rather than the three conditions expected for problems in which all forces lie in the same plane?

Example 2. (*a*) A 60-ft ladder weighing 100 lb rests against a wall at a point 48 ft above the ground. The center of gravity of the ladder is one-third the way up. A 160-lb man climbs halfway up the ladder. Assuming that the wall is frictionless, find the forces exerted by the system on the ground and the wall.

The forces acting on the ladder are shown in Fig. 14–6. **W** is the weight of the man standing on the ladder, and **w** is the weight of the ladder itself. A force $\mathbf{F}_1$ is exerted by the ground on the ladder. $\mathbf{F}_{1v}$ is the vertical component and $\mathbf{F}_{1h}$ is the horizontal component of this force (due to friction). The wall, being frictionless, can exert only a force normal to its surface, called $\mathbf{F}_2$. We are given the following data:

$$W = 160 \text{ lb}, \qquad w = 100 \text{ lb},$$
$$a = 48 \text{ ft}, \qquad c = 60 \text{ ft}.$$

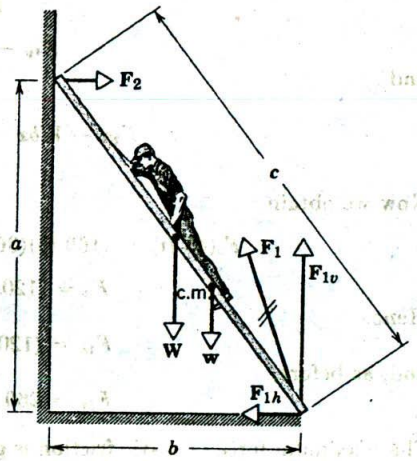


Fig. 14–6 Example 2.

From the geometry we conclude that $b = 36$ ft. The line of action of **W** intersects the ground at a distance $b/2$ from the wall and the line of action of **w** intersects the ground at a distance $2b/3$ from the wall.

We choose the x-axis to be along the ground and the y-axis along the wall. Then, the conditions on the forces for translational equilibrium (Eq. 14–2) are

$$F_2 - F_{1h} = 0,$$
$$F_{1v} - W - w = 0.$$

For rotational equilibrium (Eq. 14–4) choose an axis through the point of contact with the ground and obtain

$$F_2(a) - W\left(\frac{b}{2}\right) - w\left(\frac{b}{3}\right) = 0.$$

Using the data given, we obtain

$$F_2(48 \text{ ft}) - (160 \text{ lb})(18 \text{ ft}) - (100 \text{ lb})(12 \text{ ft}) = 0,$$
$$F_2 = 85 \text{ lb},$$
$$F_{1h} = F_2 = 85 \text{ lb},$$
$$F_{1v} = 160 \text{ lb} + 100 \text{ lb} = 260 \text{ lb}.$$

By Newton's third law the forces exerted by the ground and the wall on the ladder are equal but opposite to the forces exerted by the ladder on the ground and the wall, respectively. Therefore, the normal force on the wall is 85 lb, and the force on the ground has components of 260 lb down and 85 lb to the right.

(*b*) If the coefficient of static friction between the ground and the ladder is $\mu_s = 0.40$, how high up the ladder can the man go before it starts to slip?

Let x be the fraction of the total length of the ladder the man can climb before slipping begins. Then our equilibrium conditions are

$$F_2 - F_{1h} = 0,$$

$$F_{1v} - W - w = 0,$$

and

$$F_2 a - Wbx - w\left(\frac{b}{3}\right) = 0.$$

Now we obtain

$$F_2(48 \text{ ft}) = (160 \text{ lb})(36 \text{ ft})x + (100 \text{ lb})(12 \text{ ft}),$$

$$F_2 = (120x + 25) \text{ lb}.$$

Hence

$$F_{1h} = (120x + 25) \text{ lb},$$

and, as before,

$$F_{1v} = 260 \text{ lb}.$$

The maximum force of static friction is given by

$$F_{1h} = \mu_s F_{1v} = (0.40)(260 \text{ lb}) = 104 \text{ lb}.$$

Therefore,

$$F_{1h} = (120x + 25) \text{ lb} = 104 \text{ lb}$$

and

$$x = \tfrac{79}{120},$$

so that the man can climb up the ladder

$$60x \text{ ft} = 39.5 \text{ ft}$$

before slipping begins.

In this example the ladder is treated as a one-dimensional object, with only one point of contact at the wall and ground. The student should reflect on how this limits consideration of the less artificial case of two contact points at each end.

The reason for assuming that the wall is frictionless is discussed later. Can you guess what it is?

Example 3. A uniform beam is hinged at the wall. A wire connected to the wall a distance d above the hinge is attached to the other end of the beam. The beam makes an angle of 30° with the horizontal when a weight w is hung from a string fastened to the end of the beam. If the beam has a weight W and a length l, find the tension in the wire and the forces exerted by the hinge on the beam.

The situation is depicted in Fig. 14–7, in which all the forces acting on the beam are shown. The wire pulling on the beam makes some angle α with the horizontal so that the tension **T** in the wire has horizontal and vertical components $\mathbf{T}_h$ and $\mathbf{T}_v$, respectively, as shown. The force **F** exerted by the hinge on the beam also has horizontal and vertical components $\mathbf{F}_h$ and $\mathbf{F}_v$, respectively. **W** is the weight of the beam, acting at its center of gravity, and **w** is the tension in the string that transmits the weight of the suspended body to the beam.

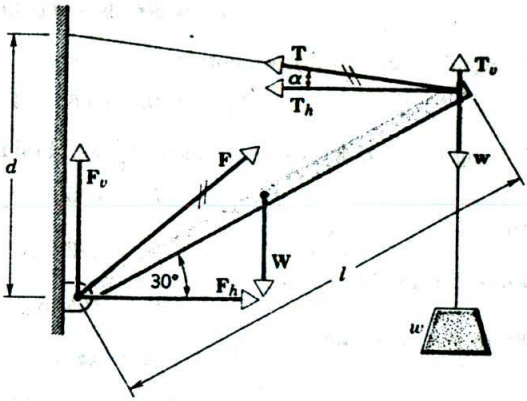


Fig. 14–7 Example 3.

Choosing our axes to be horizontal and vertical, we obtain for translational equilibrium

$$F_v + T_v - W - w = 0,$$

and

$$F_h - T_h = 0.$$

Choosing an axis through the point of intersection of **T** and **w** (Why?), we obtain for rotational equilibrium

$$F_v(l \cos 30°) - F_h(l \sin 30°) - \frac{W(l \cos 30°)}{2} = 0.$$

Our unknowns are T_h, T_v, F_h, and F_v. Let us assign the following values to the other quantities:

$$W = 300 \text{ lb}, \qquad w = 200 \text{ lb}, \qquad l = 10 \text{ ft}, \qquad d = 6.0 \text{ ft}.$$

Therefore

(1) $$F_v + T_v = 500 \text{ lb},$$

(2) $$F_h = T_h,$$

and

$$F_v(10)(0.866) = F_h(5.0) + (300)(5.0)(0.866),$$

or

(3) $$F_v = F_h(5.0/8.66) + 150 \text{ lb}.$$

Recall that we have four unknowns, namely F_v, F_h, T_v, and T_h. We need another relation between these quantities if we are to solve the problem. This relation follows from the fact that $\mathbf{T}_v$ and $\mathbf{T}_h$ must add to give a resultant vector **T** directed along the wire. The wire cannot supply or support a force transverse to its orientation. (Notice that this is not true for the beam, however.) Hence our fourth relation is

$$T_v = T_h \tan \alpha,$$

where $\tan \alpha = (d - l \sin 30°)/l \cos 30° = 1.0/8.66$, so that

(4) $$T_v = T_h/8.66.$$

Combining (1) and (4) we obtain

$$F_v = 500 \text{ lb} - T_h/8.66.$$

Combining (2) and (3), we obtain

$$F_v = T_h(5.0/8.66) + 150 \text{ lb}.$$

Solving these equations simultaneously, we obtain

$$T_h = 505 \text{ lb},$$

and

$$F_v = 442 \text{ lb}.$$

From (2) we obtain

$$F_h = 505 \text{ lb}.$$

From (1) we obtain

$$T_v = 58 \text{ lb}.$$

The tension in the wire will then be

$$T = \sqrt{T_h^2 + T_v^2} = 509 \text{ lb},$$

and the hinge will exert a horizontal force of 505 lb and a vertical force of 442 lb on the beam. ◀

In the preceding examples we have been careful to limit the number of unknown forces to the number of independent equations relating the forces. When all the forces act in a plane, we can have only three independent equations of equilibrium, one for rotational equilibrium about any axis normal to the plane and two others for translational equilibrium in the plane. However, we often have more than three unknown forces. For example, in the ladder problem of Example 2*a*, if we drop the artificial assumption of a frictionless wall, we have four unknown scalar quantities, namely, the horizontal and vertical components of the force acting on the ladder at the wall and the horizontal and vertical components of the force acting on the ladder at the ground. Since we have only three scalar equations, these forces cannot be determined. For any value assigned to one unknown force, the other three forces can be determined. But if we have no basis for assigning any particular value to an unknown force, there are an infinite number of solutions possible mathematically. We must therefore find another independent relation between the unknown forces if we hope to solve the problem uniquely.

Another simple example of such underdetermined structures is the automobile. In this case we wish to determine the forces exerted by the ground on each of the four tires when the car is at rest on a horizontal surface. If we assume that these forces are normal to the ground, we have four unknown scalar quantities. All other forces, such as the weight of the car and passengers, act normal to the ground. Therefore we have only three independent equations giving the equilibrium conditions, one for translational equilibrium in the single direction of all the forces and two for rotational equilibrium about the two axes perpendicular to each other in a

horizontal plane. Again the solution of the problem is indeterminate, mathematically. A four-legged table with all its legs in contact with the floor is a similar example.

Of course, since there is actually a unique solution to any real physical problem, we must find a physical basis for the additional independent relation between the forces that enable us to solve the problem. The difficulty is removed when we realize that structures are never perfectly rigid, as we have tacitly assumed throughout. Actually our structures will be somewhat deformed. For example, the automobile tires and the ground will be deformed, as will the ladder and wall. The laws of elasticity and the elastic properties of the structure determine the nature of the deformation and will provide the necessary additional relation between the four forces. A complete analysis therefore requires not only the laws of rigid body mechanics but also the laws of elasticity. In courses in civil and mechanical engineering, many such problems are encountered and analyzed in this way. We shall not consider the matter further here.

14–5 Stable, Unstable, and Neutral Equilibrium of Rigid Bodies in a Gravitational Field

In Chapter 8 we saw that the gravitational force is a conservative force. For conservative forces we can define a potential energy function $U(x,y,z)$, where U is related to $\mathbf{F}$ by

$$F_x = -\partial U/\partial x, \qquad F_y = -\partial U/\partial y, \qquad F_z = -\partial U/\partial z.$$

At points where $\partial U/\partial x$ is zero, a particle subject to this conservative force will be in translational equilibrium in the x-direction, for then F_x equals zero. Likewise, at points where $\partial U/\partial y$ or $\partial U/\partial z$ are zero, a particle will be in translational equilibrium in the y- and z-directions, respectively. The derivative of U at a point will be zero when U has an extreme value (maximum or minimum) at that point or when U is constant with respect to the variable coordinate.

When U is a minimum, the particle is in *stable* equilibrium; any displacement from this position will result in a restoring force tending to return the particle to the equilibrium position. Another way of stating this is to say that if a body is in stable equilibrium, work must be done on it by an external agent to change its position. This results in an increase in its potential energy.

When U is a maximum, the particle is in *unstable* equilibrium; any displacement from this position will result in a force tending to push the particle farther from the equilibrium position. In this case no work must be done on the particle by an external agent to change its position; the work done in displacing the body is supplied internally by the conservative force, resulting in a decrease in potential energy.

When U is constant, the particle is in *neutral* equilibrium. In this case a particle can be displaced slightly without experiencing either a repelling or restoring force.

Notice that a particle can be in equilibrium with respect to one coordinate without necessarily being in equilibrium with respect to another coordinate, as for example a freely falling ball. Furthermore, a particle may be in stable equilibrium with respect to one coordinate and in unstable equilibrium with respect to another coordinate, as for example a particle at a saddle point (Fig. 14–8).

All these remarks apply to particles, that is, to translational motion. Suppose now we treat a rigid body. We must consider rotational equilibrium as well as translational equilibrium. The problem of a rigid body in a gravitational field is

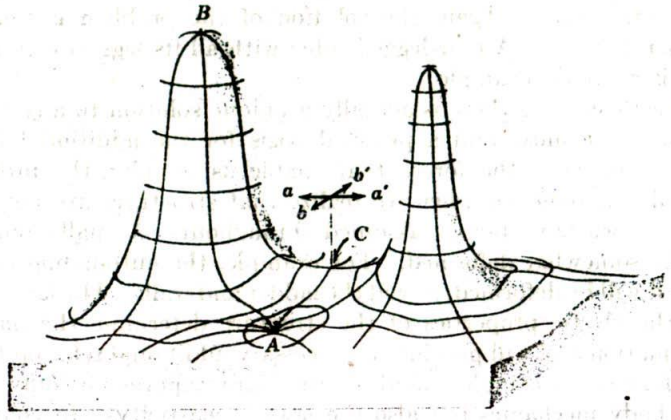


Fig. 14–8 A gravitational potential surface, which may be thought of as a real surface. A particle placed at A, B or C remains at rest; a plane tangent to any of these points is horizontal. We say that a particle here is *in equilibrium*. At A, a particle, if slightly displaced, tends to return to A. A represents a point of *stable equilibrium*. At B, a particle, if slightly displaced, tends to increase its displacement. Thus B represents a point of *unstable equilibrium*. At C, the particle, if slightly displaced in the direction aa', will tend to return to C, but if it is displaced in direction bb', it will tend to increase its displacement. C is called a *saddle point* since a saddle has somewhat this shape. Neutral equilibrium, experienced by a particle anywhere on a plane horizontal surface, is not illustrated.

particularly simple, however, because *all the gravitational forces on the particles of the rigid body can be considered to act at one point, both for translational and rotational purposes.* We can replace this entire rigid body, for purposes of equilibrium under gravitational forces, by a single particle having the equivalent mass at the center of gravity.

For example, consider a cube at rest on one side on a horizontal table. The center of gravity is shown at the center of the central cross section of the cube in Fig. 14–9*a*. Let us supply a force to the cube so as to rotate it without its slipping about an axis along an edge. Notice that the center of gravity is raised and that work is done on the cube, which increases its potential energy. If the force is removed, the cube tends to return to its original position, its increased potential energy being converted into kinetic energy as it falls back. This initial position is, therefore, one of *stable* equilibrium. In terms of a particle of equivalent mass at the center of gravity, this process is described by the dotted line which indicates the path taken by the center of gravity during this motion. The particle is seen to have a minimum potential energy in the position of stable equilibrium, as required. We can conclude that the rigid body will be in stable equilibrium if the application of any force can raise the center of gravity of the body but not lower it.

If the cube is rotated until it balances on an edge, as in Fig. 14–9*b*, then once again the cube is in equilibrium. This equilibrium position is seen to be unstable. The application of even the slightest horizontal force will cause the cube to fall away from this position with a decrease of potential energy. The particle of equivalent mass at the center of gravity follows the dotted path shown. At the position of unstable equilibrium this particle has a maximum potential energy, as required. We can conclude that the rigid body will be in unstable equilibrium if

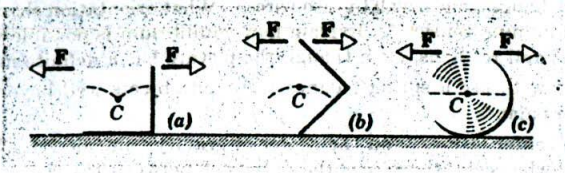


Fig. 14–9 Equilibrium of an extended body. (*a*) A cube resting on one side is in *stable equilibrium* since its center of gravity *C* is raised if the cube is tipped by a horizontal force **F**. (*b*) A cube resting on one edge is in *unstable equilibrium* since *C* falls if the cube is tipped by **F**. (*c*) A sphere or cylinder is in *neutral equilibrium* since *C* neither rises nor falls when **F** is applied. Compare these criteria for equilibrium with those given in Fig. 14–8. How are the criteria in the two figures related?

the application of any horizontal force tends to lower the center of gravity of the body.

The neutral equilibrium of a rigid body is illustrated by a sphere on a horizontal table (Fig. 14–9c). If the sphere is subjected to any horizontal force, the center of gravity is neither raised nor lowered but moves along the horizontal dotted line. The potential energy of the sphere is constant during the displacement, as is that of the particle of equivalent mass at the center of gravity. The system has no tendency to move in any direction when the applied force is removed. A rigid body will be in neutral equilibrium if the application of any horizontal force neither raises nor lowers the center of gravity of the body.

Under what circumstances would a *suspended* rigid body be in stable equilibrium? When would a *suspended* rigid body be in unstable equilibrium, and when would it be in neutral equilibrium?

QUESTIONS

1. Are Eqs. 14–1 and 14–3 both necessary and sufficient conditions for mechanical equilibrium? For static equilibrium?

2. A wheel rotating at constant angular velocity ω about a fixed axis is in mechanical equilibrium because no net external force or torque acts on it. However, the particles that make up the wheel undergo a centripetal acceleration $\mathbf{a}$ directed toward the axis. Since $\mathbf{a} \neq 0$ how can the wheel be said to be in equilibrium?

3. Give several examples of a body which is not in equilibrium, even though the resultant of all the forces acting on it is zero.

4. If a body is not in translational equilibrium, will the torque about any point be zero if the torque about some particular point is zero?

5. Which is more likely to break in use, a hammock stretched tightly between two trees or one that sags quite a bit? Prove your answer.

6. A ladder is at rest with its upper end against a wall and the lower end on the ground. Is it more likely to slip when a man stands on it at the bottom or at the top? Explain.

7. In Example 2, if the wall were rough, would the empirical laws of friction supply us with the extra condition needed to determine the extra (vertical) force exerted by the wall on the ladder?

8. In Example 3, why isn't it necessary to consider friction at the hinge?

9. A picture hangs from a wall by two wires. What orientation should the wires have to be under minimum tension? Explain how equilibrium is possible with any number of orientations and tensions, even though the picture has a definite mass.

10. Show how to use a spring balance to weigh objects well beyond the maximum reading of the balance.

11. Do the center of mass and the center of gravity coincide for a building? For a lake? Under what conditions does the difference between the center of mass and the center of gravity of a body become significant?

12. If a rigid body is thrown into the air without spinning, it does not spin during its flight, provided air resistance can be neglected. What does this simple result imply about the location of the center of gravity?

13. A uniform block, in the shape of a rectangular parallelepiped of sides in the ratio 1:2:3, lies on a horizontal surface. In which position, that is, on which of its three different faces, can it be said to be most stable, if any?

14. A virus particle in a rotating liquid-filled centrifuge tube is in uniform circular motion (that is, in *accelerated* motion) as viewed by an observer in the laboratory. An observer rotating with the centrifuge, however, would declare the particle to be *unaccelerated*. Explain how the virus particle can be in equilibrium for this second observer but not for the first.

15. In Chapter 5 we *defined* force in terms of acceleration from the relation $\mathbf{F} = m\mathbf{a}$. For a body in equilibrium, however, there are no accelerations. How, then, can we give meaning to the forces acting on such a body?

PROBLEMS

1. What force F applied horizontally at the axle of the wheel is necessary to raise the wheel over an obstacle of height h? Take r as the radius of the wheel and W as its weight (Fig. 14–10).

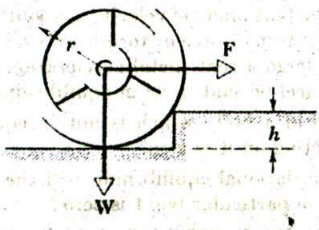


Fig. 14–10

2. A beam is carried by three men, one man at one end and the other two supporting the beam between them on a crosspiece so placed that the load is equally divided among the three men. Find where the crosspiece is placed. Neglect the mass of the crosspiece.

3. A meter stick balances on a knife edge at the 50.0-cm mark. When two nickels are stacked over the 12.0-cm mark, the loaded stick is found to balance at the 45.5-cm mark. A nickel has a mass of 5.0 gm. What is the mass of the meter stick? Try this technique sometimes and check your answer experimentally.

4. Prove that when only three forces act on a body in equilibrium, they must be coplanar, and their lines of action must meet at a point or at infinity.

5. A circular section of radius r is cut out of a uniform disk of radius R, the center of the hole being $R/2$ from the center of the original disk. Locate the center of gravity of the resulting body.

6. In Fig. 14–11 a man is trying to get his car out of the mud on the shoulder of a road. He ties one end of a rope tightly around the front bumper and the other end tightly around a telephone pole 60 ft away. He then pushes sideways on the rope at its midpoint with a force of 125 lb, displacing the center of the rope 1.0 ft from its previous position and the car almost moves. What force does the rope exert on the car? (The rope stretches somewhat under the tension.)

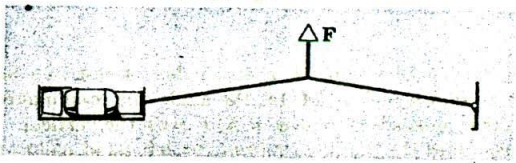


Fig. 14–11

7. A homogeneous sphere of radius r and weight W *slides* along the floor under the action of a constant *horizontal* force P applied to a string, as shown in Fig. 14–12. (*a*) Show that, if μ is the coefficient of friction between sphere and floor, the height h is given by $h = r(1 - \mu W/P)$. (*b*) Show that the sphere is not in translational equilibrium under these circumstances. Is there any point about which the sphere is in rotational equilibrium? (*c*) Can one get the sphere to be in both rotational *and* translational equilibrium by a different choice of h? . . . By a different direction for P? Explain.

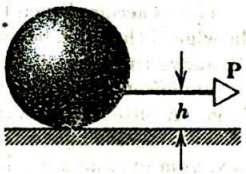


Fig. 14–12

8. A thin horizontal bar AB of negligible weight and length l is pinned to a vertical wall at A and supported at B by a thin wire BC that makes an angle θ with the hori-

zontal. A weight P can be moved anywhere along the bar as defined by the distance x from the wall (Fig. 14–13). (*a*) Find the tensile force T in the thin wire as a function of x. (*b*) Find the horizontal and vertical components of the force exerted on the bar by the pin at A.

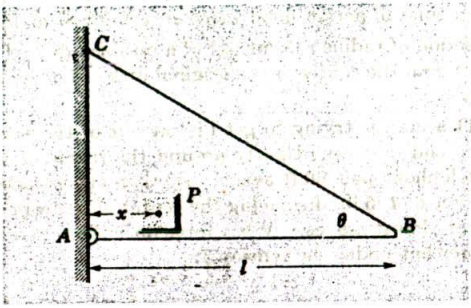


Fig. 14–13

9. A *non*uniform bar of weight W is suspended at rest in a horizontal position by two light ropes as shown in Fig. 14–14; the angle one rope makes with the vertical is $\theta = 36.9°$; the other makes the angle $\phi = 53.1°$ with the vertical. If the length l of the bar is 20.0 ft, compute the distance x from the left-hand end to the center of gravity.

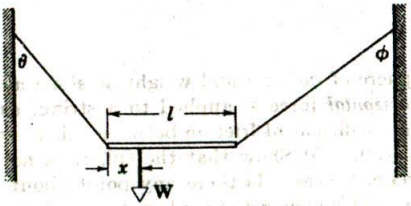


Fig. 14–14

10. A well-known problem is the following (see, for example, *Scientific American*, November 1964, p. 128): Uniform bricks are placed one upon another in such a manner as to have the maximum offset. This is accomplished by having the center of gravity of the top brick directly above the edge of the brick below it, the center of gravity of the two top bricks combined directly above the edge of the third brick from the top, etc. (*a*) Justify this criterion for maximum offset. (*b*) Show that, if the process is continued downward, one can obtain as large an offset as he wants. (Martin Gardner, in the article referred to above, states: "With 52 playing cards, the first placed so that its end is flush with a table edge, the maximum overhang is a little more than $2\frac{1}{4}$ card lengths . . . ."). (*c*) Suppose now, instead, one piles up uniform bricks so that the end of one brick is offset from the one below it by a constant fraction, $1/n$, of a brick length l. How many bricks, N, can one use in this process before the pile will fall over? Check the plausibility of your answer for $n = 1$, $n = 2$, $n = \infty$.

11. Four bricks, each of length l, are put on top of one another (see Fig. 14–15) in such a way that part of each extends beyond the one beneath. Show that the *largest*

equilibrium extensions are (*a*) top brick overhanging the one below by $l/2$, (*b*) second brick from top overhanging the one below by $l/4$, and (*c*) third brick from top overhanging the bottom one by $l/6$.



Fig. 14–15

12. A door 7.0 ft high and 3.0 ft wide weighs 60 lb. A hinge 1.0 ft from the top and another 1.0 ft from the bottom each support half the door's weight. Assume that the center of gravity is at the geometrical center of the door and determine the horizontal and vertical force components exerted by each hinge on the door.

13. An automobile weighing 3000 lb has a wheel base of 120 in. Its center of gravity is located 70 in. behind the front axle. Determine the force exerted on each of the front wheels (assumed the same) and the force exerted on each of the back wheels (assumed the same) by the level ground.

14. A balance is made up of a rigid rod free to rotate about a point not at the center of the rod. It is balanced by unequal weights placed in the pans at each end of the rod. When an unknown mass m is placed in the left-hand pan, it is balanced by a mass m_1 placed in the right-hand pan, and similarly when the mass m is placed in the right-hand pan, it is balanced by a mass m_2 in the left-hand pan. Show that

$$m = \sqrt{m_1 m_2},$$

and state any assumptions you make in doing this.

15. In the stepladder shown in Fig. 14–16, AC and CE are 8.0 ft long and hinged at C. BD is a tie rod 2.5 ft long, halfway up. A man weighing 192 lb climbs 6.0 ft along the ladder. Assuming that the floor is frictionless and neglecting the weight of the ladder, find the tension in the tie rod and the forces exerted on the ladder by the floor. (Hint: it will help to isolate parts of the ladder in applying the equilibrium conditions.)

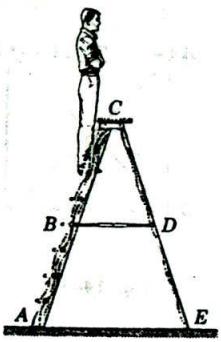


Fig. 14–16

16. A 100-lb plank, of length $l = 20$ ft, rests on the ground and on a frictionless roller at the top of a wall of height $h = 10$ ft (see Fig. 14–17). The center of gravity of the plank is at its center. The plank remains in equilibrium for any value of $\theta \geq 70°$, but slips if $\theta < 70°$. (*a*) Draw a diagram showing all forces acting on the plank. (*b*) Find the coefficient of friction between the plank and the ground.

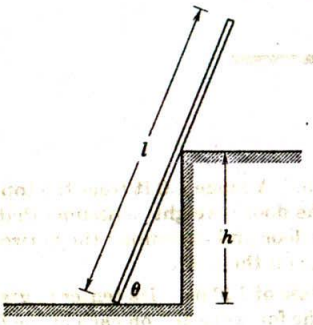


Fig. 14–17

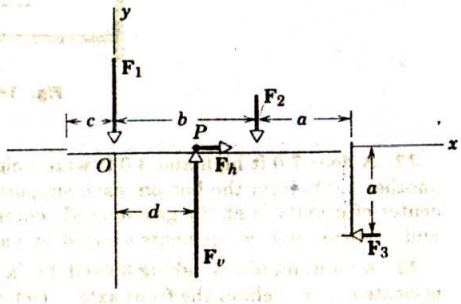


Fig. 14–18

17. Forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ act on the structure of Fig. 14–18 as shown. We wish to put the structure in equilibrium by applying a force, at a point such as P, whose vector components are $\mathbf{F}_h$ and $\mathbf{F}_v$. We are given that $a = 2.0$ ft, $b = 3.0$ ft, $c = 1.0$ ft, $F_1 = 20$ lb, $F_2 = 10$ lb, and $F_3 = 5.0$ lb. Find F_h, F_v, and d.

18. By means of a turnbuckle G, a tensile force $\mathbf{T}$ is produced in bar AB of the square frame $ABCD$, as in Fig. 14–19. Determine the forces produced in the other bars. The diagonals AC and BD pass each other freely at E. Show how symmetry considerations can lead to considerable simplification in this and similar problems.

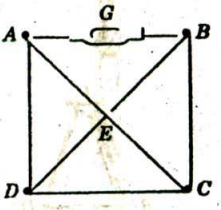


Fig. 14–19

19. A flexible chain of weight W hangs between two fixed points, A and B, at the same level, as shown in Fig. 14–20. Find (a) the vector force exerted by the chain on each end point and (b) the tension in the chain at the lowest point.



Fig. 14–20

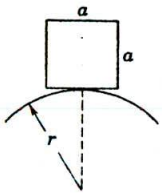


Fig. 14–21

20. A bowl having a radius of curvature r rests on a horizontal table. Show that the bowl will be in stable equilibrium about the center point at its bottom only if the center of mass of the material piled up in the bowl is not as high as r.

21. A cube of uniform density and edge a is balanced on a cylindrical surface of radius r as shown in Fig. 14–21. Show that the criterion for stable equilibrium of the cube, assuming that friction is sufficient to prevent slipping, is $r > a/2$.

Oscillations

CHAPTER 15

15–1 Oscillations

Any motion that repeats itself in equal intervals of time is called *periodic motion.* As we shall see, the displacement of a particle in periodic motion can always be expressed in terms of sines and cosines. Because the term harmonic is applied to expressions containing these functions, periodic motion is often called *harmonic motion.*

If a particle in periodic motion moves back and forth over the same path, we call the motion *oscillatory* or *vibratory.* The world is full of oscillatory motions. Some examples are the oscillations of the balance wheel of a watch, a violin string, a mass attached to a spring, atoms in molecules or in a solid lattice, and air molecules as a sound wave passes by.

Many oscillating bodies do not move back and forth between precisely fixed limits because frictional forces dissipate the energy of motion. Thus a violin string soon stops vibrating and a pendulum stops swinging. We call such motions *damped* harmonic motions. Although we cannot eliminate friction from the periodic motions of gross objects, we can often cancel out its damping effect by feeding energy into the oscillating system so as to compensate for the energy dissipated by friction. The main spring of a watch and the falling weight in a pendulum clock supply external energy in this way, so that the oscillating system, that is, the balance wheel or the pendulum, moves as if it were undamped.

Not only mechanical systems can oscillate. Radio waves, microwaves, and visible light are oscillating magnetic and electric field vectors. Thus a tuned circuit in a radio and a closed metal cavity in which microwave

energy is introduced can oscillate electromagnetically. The analogy is close, being based on the fact that *mechanical and electromagnetic oscillations are described by the same basic mathematical equations.* We will make the most of this analogy in later chapters.

The *period* T of a harmonic motion is the time required to complete one round trip of the motion, that is, one complete oscillation or *cycle*. The *frequency* of the motion ν is the number of oscillations (or cycles) per unit of time. The frequency is therefore the reciprocal of the period, or

$$\nu = 1/T. \tag{15–1}$$

The mks unit of frequency is the cycle per second, or *hertz*.* The position at which no net force acts on the oscillating particle is called its *equilibrium* position. The *displacement* (linear or angular) is the distance (linear or angular) of the oscillating particle from its equilibrium position at any instant.

Let us focus attention on a particle oscillating back and forth along a straight line between fixed limits. Its displacement **x** changes periodically in both magnitude and direction. Its velocity **v** and acceleration **a** also vary periodically in magnitude and direction and, in view of the relation $\mathbf{F} = m\mathbf{a}$, so does the force **F** acting on the particle.

Forces associated with harmonic motion are the most general kinds of forces that we have discussed so far. In the earlier chapters we dealt only with constant forces (and accelerations). Later, when we considered forces that are not constant but instead vary with time, we examined a force (and thus an acceleration) that varied in direction although its magnitude was constant (the centripetal force of Section 6–3), and a force (and thus an acceleration) which varied in magnitude although its direction was constant (the impulsive force of Section 10–1). Here, in harmonic motion, the force, and the acceleration, vary both in direction *and* magnitude.

In terms of energy, we can say that a particle undergoing harmonic motion passes back and forth through a point (its equilibrium position) at which its potential energy is a minimum. A swinging pendulum is a good example, its potential energy being a minimum at the bottom of the swing, that is, at the equilibrium position. Figure 15-1*a* shows a particle oscillating between the limits x_1 and x_2, O being the equilibrium position. Figure 15-1*b* shows the corresponding potential energy curve, which has a minimum value at that position. The force acting on the particle at any position is derivable from the potential energy function; it is given by Eq. 8–7,

$$F = -dU/dx, \tag{8–7}$$

and is displayed in Fig. 15–1*c*. The force is zero at the equilibrium position O, points to the right (that is, has a positive value) when the particle

* This frequency unit is named after Heinrich Hertz (1857–1894) whose research in electromagnetism was universally recognized as providing the experimental confirmation of the electromagnetic waves predicted by Maxwell.

is to the left of O, and points to the left (that is, has a negative value) when the particle is to the right of O. The force is *a restoring force* because it always acts to accelerate the particle in the direction of its equilibrium position. Hence in harmonic motion the position of equilibrium is always one of *stable* equilibrium.

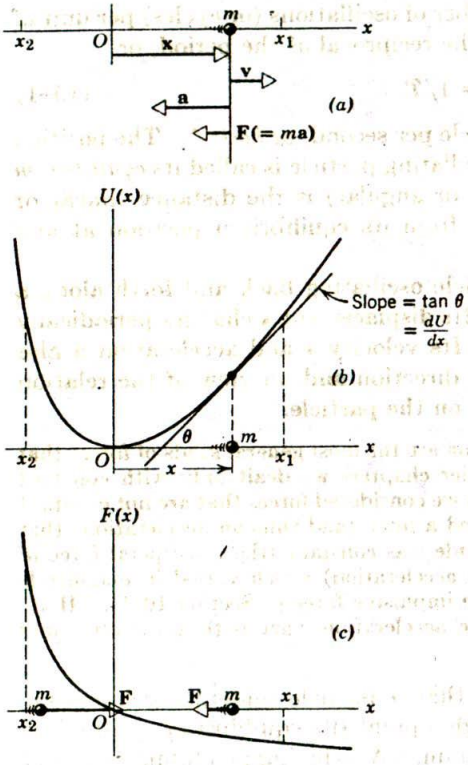


Fig. 15–1 (*a*) A particle of mass m oscillates harmonically between points x_1 and x_2, O being the equilibrium position. (*b*) The potential energy of the particle as a function of position. The force acting on the particle at position x is given by $F = -dU/dx$. (*c*) The force acting on the particle as a function of position x; note that the force is directed toward the equilibrium position.

The total mechanical energy E for an oscillating particle is the sum of its kinetic energy and its potential energy, or

$$E = K + U \tag{15–2}$$

in which E remains constant if no nonconservative forces, such as the force of friction, are acting. Figure 15–2 shows E for the motion of Fig. 15–1. Note how Eq. 15–2 is satisfied for the particle in the typical position shown. The particle cannot move outside the limits x_1 and x_2 because, in these regions, U exceeds E. This, as Eq. 15–2 shows, would require the kinetic energy to be negative, an impossibility.

For a given environment, that is, for a given function $U(x)$, an oscillating particle can have various total energies, depending on how it is set into motion initially. Thus the total energy may be E', rather than E, in

which case the limits of oscillation would be x_1' and x_2', as Fig. 15–2 shows, rather than x_1 and x_2.

15-2 The Simple Harmonic Oscillator

Let us consider an oscillating particle (Fig. 15–3*a*) moving back and forth about an equilibrium position through a potential that varies as

$$U(x) = \tfrac{1}{2}kx^2 \qquad (15\text{–}3)$$

in which k is a constant; see Fig. 15–3*b*. The force acting on the particle is given by Eq. 8–7, or

$$F(x) = -dU/dx = -d(\tfrac{1}{2}kx^2)/dx = -kx; \qquad (15\text{–}4)$$

see Fig. 15–3*c*. Such an oscillating particle is called a *simple harmonic oscillator* and its motion is called *simple harmonic motion*. In such motion, as Eq. 15–3 shows, the potential energy curve varies as the square of the displacement, and, as Eq. 15–4 shows, the force acting on the particle is proportional to the displacement but is opposite to it in direction. In simple harmonic motion the limits of oscillation are equally spaced about the equilibrium position. This is not true for the more general motion of

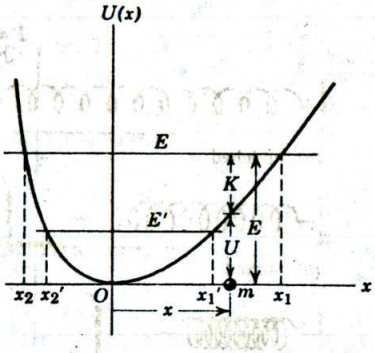


Fig. 15–2 The total mechanical energy E for the motion of Fig. 15–1 is shown. If the total mechanical energy of the oscillating particle is reduced to E', the limits of oscillation are reduced to x_1' and x_2' respectively.

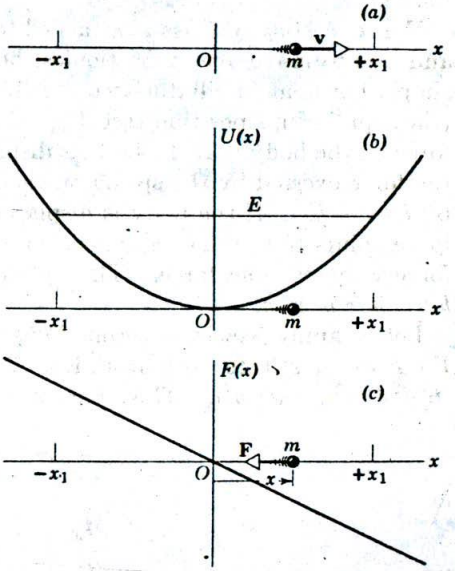


Fig. 15–3 (*a*) A particle of mass m oscillates with simple harmonic motion between points $+x_1$ and $-x_1$, O being the equilibrium position. (*b*) The potential energy and the total mechanical energy. (*c*) The force acting on the particle. The student should compare this figure carefully with Fig. 15-1, which illustrates the general case of harmonic motion.

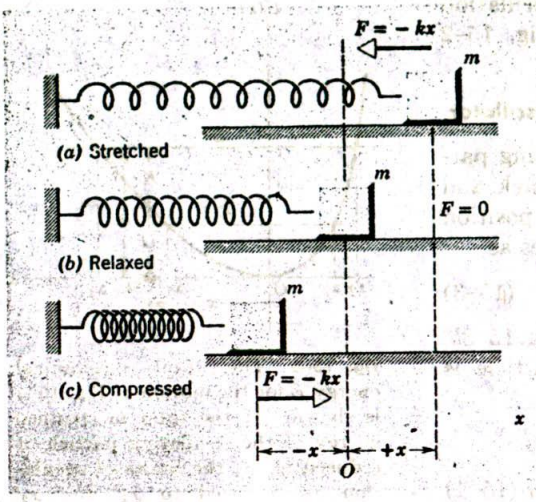


Fig. 15–4 A simple harmonic oscillator. The force exerted by the spring is shown in each case. The block slides on a frictionless horizontal table.

Fig. 15–1 which, although harmonic, is not simple harmonic. The magnitude of the maximum displacement, that is, the quantity x_1 in Fig. 15–3, always taken as positive, is called the *amplitude* of the simple harmonic motion.

The student will have recognized Eq. 15–3 [$U(x) = \frac{1}{2}kx^2$] as the expression for the potential energy of an "ideal" spring, compressed or extended by a distance x; see Section 8–4. In this same section an ideal spring was defined as one in which the force exerted by the stretched or compressed spring is given by $F(x) = -kx$ (see Eq. 15–4), k being called the *force constant.*

Hence, a body of mass m attached to an ideal spring of force constant k and free to move over a frictionless horizontal surface is an example of a simple harmonic oscillator (see Fig. 15–4). Note that there is a position (the equilibrium position; see Fig. 15–4*b*) in which the spring exerts no force on the body. If the body is displaced to the right (as in Fig. 15–4*a*), the force exerted by the spring on the body points to the left and is given by $F = -kx$. If the body is displaced to the left (as in Fig. 15–4*c*), the force points to the right and is also given by $F = -kx$. In each case the force is a *restoring* force. The motion of the oscillating mass is *simple harmonic motion.*

Let us apply Newton's second law, $F = ma$, to the motion of Fig. 15–4. For F we substitute $-kx$ (from Eq. 15–4) and for the acceleration a we put in d^2x/dt^2 ($= dv/dt$). This gives us

$$-kx = m\frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \qquad (15\text{–}5)$$

This equation involves derivatives and is, therefore, called a *differential equation.* To solve this equation means to determine how the displacement x of the particle must depend on the time t in order that the equation be satisfied. When we know how x depends on time, we know the motion of the particle; thus, Eq. 15–5 is called the *equation of motion* of a simple harmonic oscillator. We shall solve this equation and describe the motion in detail in the next section.

The simple harmonic oscillator problem is important for two reasons. First, most problems involving mechanical vibrations reduce to that of the simple harmonic oscillator at small amplitudes of vibration, or to a combination of such vibrations. This is equivalent to saying that if we consider a small enough portion of the restoring force curve of Fig. 15–1*c* (around the origin), it becomes arbitrarily close to a straight line which, as Fig. 15–3*c* shows, is characteristic of simple harmonic motion. Or, in other words, the potential energy curve of Fig. 15-1*b* for general oscillatory motion reduces to that of Fig. 15–3*b* for simple harmonic oscillation when the amplitude of vibration is made sufficiently small about the equilibrium position O.

Second, as we have indicated, equations like Eq. 15–5 turn up in many physical problems in acoustics, in optics, in mechanics, in electrical circuits, and even in atomic physics. The simple harmonic oscillator exhibits features common to many physical systems.

Equation 15–4 ($F = -kx$) is an empirical relation known as *Hooke's law.* It is a special case of a more general relation, dealing with the deformation of elastic bodies, discovered by Robert Hooke (1635–1703).* It is obeyed by springs and other elastic bodies provided the deformation is not too great. If the solid is deformed beyond a certain point, called its *elastic limit,* it will not return to its original shape when the applied force is removed (Fig. 15–5). It turns out that Hooke's law holds almost up to the elastic limit for many common materials. The range of applied forces over which Hooke's law is valid is called the "proportional region." Beyond the elastic limit, the force can no longer be specified by a potential energy function, because the force then depends on many factors including the speed of deformation and the previous history of the solid.

Notice that the restoring force and potential energy function of the simple harmonic oscillator are the same as that of a solid deformed in one dimension in the proportional region. If the deformed solid is released, it will vibrate, just as the simple harmonic oscillator does. Therefore, as long as the amplitude of the vibration is small enough, that is, as long as the deformation remains in the proportional region, mechanical vibrations behave exactly like simple harmonic oscillators. It is easy to generalize

* Robert Hooke expressed his law in the Latin words *Ut tensio sic vis* ["the (ex)tension is proportional to the force"]. Although he discovered this law in 1660, he did not publish it until sixteen years later and then only in the form of the cryptogram *ceiiinosssttuv.* Hooke may have been motivated by a desire to establish priority without revealing information to other investigators.

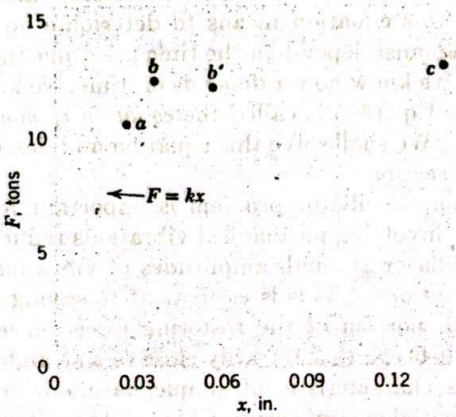


Fig. 15–5 Typical graph of applied force F versus resulting elongation of an aluminum bar under tension. The sample was a foot long and a square inch in cross section. Notice that we may write $F = kx$ only for the portion Oa, since beyond this point the slope is no longer constant but varies in a complicated way with x. At some point such as b (the *elastic limit*) the sample does not return to its original length when the applied force is removed. Between b and b' the elongation increases, even though the force is held constant; the material flows like a viscous fluid. At c, the sample can be stretched no farther; any increase in elongation results in the sample's breaking in two. The applied force is equal in magnitude to the restoring force so that no minus sign appears in the relation $F = kx$.

this discussion to show that any problem involving mechanical vibrations of small amplitude in three dimensions reduces to a combination of simple harmonic oscillators.

The vibrating string or membrane, sound vibrations, the vibrations of atoms in solids, and electrical or acoustical oscillations in a cavity can all be described in a form which is identical mathematically to a system of harmonic oscillators. The analogy enables us to solve problems in one area by using the techniques developed in other areas.

15–3 Simple Harmonic Motion

Let us now solve the equation of motion of the simple harmonic oscillator,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{15–6}$$

Recall that any system of mass m upon which a force $F = -kx$ acts will be governed by this equation. In the case of a spring, the proportionality constant k is the force constant of the spring determined by the stiffness of the spring. In other oscillating systems the proportionality constant k may be related to other physical features of the system, as we shall see later. We can use the oscillating spring as our prototype.

Equation 15–6 is a differential equation. It gives a relation between a function of the time $x(t)$ and its second time derivative d^2x/dt^2. To find

the position of the particle as a function of the time, we must find a function $x(t)$ which satisfies this relation.

We can rewrite Eq. 15–6 as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x. \tag{15–7}$$

Equation 15–7 then requires that $x(t)$ be some function whose second derivative is the negative of the function itself, except for a constant factor k/m. We know from the calculus, however, that the sine function or the cosine function has this property.* For example,

$$\frac{d}{dt}\cos t = -\sin t \quad \text{and} \quad \frac{d^2}{dt^2}\cos t = -\frac{d}{dt}\sin t = -\cos t.$$

This property is not affected if we multiply the cosine function by a constant A. We can allow for the fact that the sine function will do as well, and for the fact that Eq. 15–7 contains a constant factor, by writing as a tentative solution of Eq. 15–7,

$$x = A\cos(\omega t + \delta). \tag{15–8}$$

Here since

$$\cos(\omega t + \delta) = \cos\delta\cos\omega t - \sin\delta\sin\omega t = a\cos\omega t + b\sin\omega t,$$

the constant δ allows for any combination of sine and cosine solutions. Hence, with the (as yet) unknown constants A, ω, and δ, we have written as general a solution to Eq. 15–7 as we can. In order to determine these constants such that Eq. 15–8 is actually the solution of Eq. 15–7, we differentiate Eq. 15–8 twice with respect to the time. We have

$$\frac{dx}{dt} = -\omega A\sin(\omega t + \delta)$$

and

$$\frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \delta).$$

Putting this into Eq. 15–7, we obtain

$$-\omega^2 A\cos(\omega t + \delta) = -\frac{k}{m}A\cos(\omega t + \delta).$$

Therefore, if we choose the constant ω such that

$$\omega^2 = \frac{k}{m}, \tag{15–9}$$

then

$$x = A\cos(\omega t + \delta)$$

is in fact a solution of the equation of a simple harmonic oscillator.

---

* Harmonic motion is not only periodic but also bounded. Only the sine and cosine functions (or combinations of them) have both these properties.

The constants A and δ are still undetermined and, therefore, still completely arbitrary. This means that any choice of A and δ whatsoever will satisfy Eq. 15–7, so that a large variety of motions is possible for the oscillator. Actually, this is characteristic of a differential equation of motion, for such an equation does not describe just one single motion but a group or family of possible motions which have some features in common but differ in other ways. In this case ω is common to all the allowed motions, but A and δ may differ among them. We shall see later that A and δ are determined for a particular harmonic motion by how the motion starts.

Let us find the *physical* significance of the constant ω. If the time t in Eq. 15–8 is increased by $2\pi/\omega$, the function becomes

$$\begin{aligned} x &= A \cos [\omega(t + 2\pi/\omega) + \delta], \\ &= A \cos (\omega t + 2\pi + \delta), \\ &= A \cos (\omega t + \delta). \end{aligned}$$

That is, the function merely repeats itself after a time $2\pi/\omega$. Therefore, $2\pi/\omega$ is the *period* of the motion T. Since $\omega^2 = k/m$, we have

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}. \tag{15–10}$$

Hence, all motions given by Eq. 15–7 have the same period of oscillation, and this is determined only by the mass m of the vibrating particle and the force constant k. The *frequency* ν of the oscillator is the number of complete vibrations per unit time and is given by

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}. \tag{15–11}$$

Hence,

$$\omega = 2\pi\nu = \frac{2\pi}{T}. \tag{15–12}$$

The quantity ω is called the *angular frequency;* it differs from the frequency ν by a factor 2π. It has the dimension of reciprocal time (the same as angular speed) and its unit is the radian/sec. In Section 15–6 we shall give a geometric meaning to this angular frequency.

The constant A has a simple physical meaning. The cosine function takes on values from -1 to 1. The *displacement* x from the central equilibrium position $x = 0$, therefore, has a maximum value of A. Hence, A $(= x_{max})$ is the *amplitude* of the motion. Since A is not fixed by our differential equation, motions of various amplitudes are possible, but all have the same frequency and period. *The frequency of a simple harmonic motion is independent of the amplitude of the motion.*

The quantity $(\omega t + \delta)$ is called the *phase* of the motion. The constant δ is called the *phase constant.* Two motions may have the same amplitude

and frequency but differ in phase. If $\delta = -\pi/2$, for example,

$$x = A \cos(\omega t + \delta) = A \cos(\omega t - 90°)$$
$$= A \sin \omega t$$

so that the displacement is zero at the time $t = 0$. When $\delta = 0$, the displacement $x = A \cos \omega t$ is a maximum at the time $t = 0$. Other initial displacements correspond to other phase constants.

The amplitude A and the phase constant δ of the oscillation are determined by the initial position and speed of the particle. These two initial conditions will specify A and δ exactly.* Once the motion has started, however, the particle will continue to oscillate with a constant amplitude and phase constant at a fixed frequency, unless other forces disturb the system.

In Fig. 15–6 we plot the displacement x versus the time t for several simple harmonic motions. Three comparisons are made. In Fig. 15–6*a*,

* A phase constant may be increased by any integral multiple of 2π, or of 360°, and it will still describe the motion equally well.

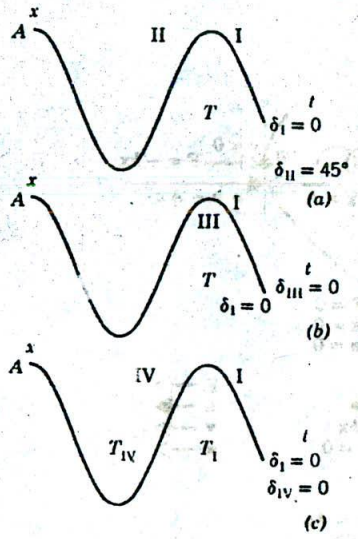


Fig. 15–6 Several solutions of the simple harmonic oscillator equation. (*a*) Both solutions have the same amplitude and period but differ in phase by 45°. (*b*) Both have the same period and phase constant but differ in amplitude by a factor of 2. (*c*) Both have the same phase constant and amplitude but differ in period by a factor of 2.

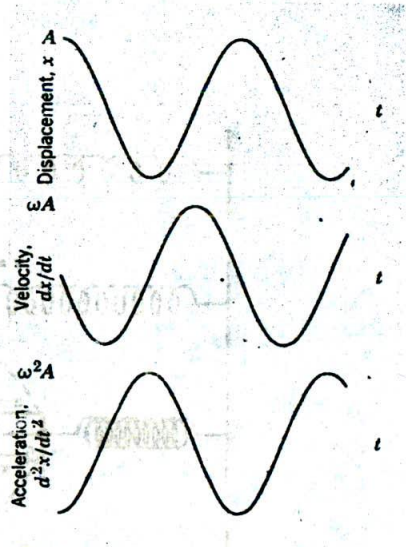


Fig. 15–7 The relations between displacement, velocity, and acceleration in simple harmonic motion. The phase constant δ is zero in this particular case since the displacement is maximum at $t = 0$; see Eq. 15–8.

I and II have the same amplitude and frequency but differ in phase by $\delta = \pi/4$ or 45°. In Fig. 15–6*b*, I and III have the same frequency and phase constant but differ in amplitude by a factor of 2. In Fig. 15–6*c*, I and IV have the same amplitude and phase constant but differ in frequency by a factor $\frac{1}{2}$ or in period by a factor 2. The student should study these curves carefully to become familiar with the terminology used in simple harmonic motion.

Another distinctive feature of simple harmonic motion is the relation between the displacement, the velocity, and the acceleration of the oscillating particle. Let us compare these quantities for curve I of Fig. 15–6, which is typical. In Fig. 15–7 we plot separately the displacement x versus the time t, the velocity $v = dx/dt$ versus the time t, and the acceleration $a = d^2x/dt^2$ versus the time t. The equations of these curves are

$$x = A \cos (\omega t + \delta),$$

$$v = \frac{dx}{dt} = -\omega A \sin (\omega t + \delta), \tag{15–13}$$

$$a = \frac{dv}{dt} = -\omega^2 A \cos (\omega t + \delta).$$

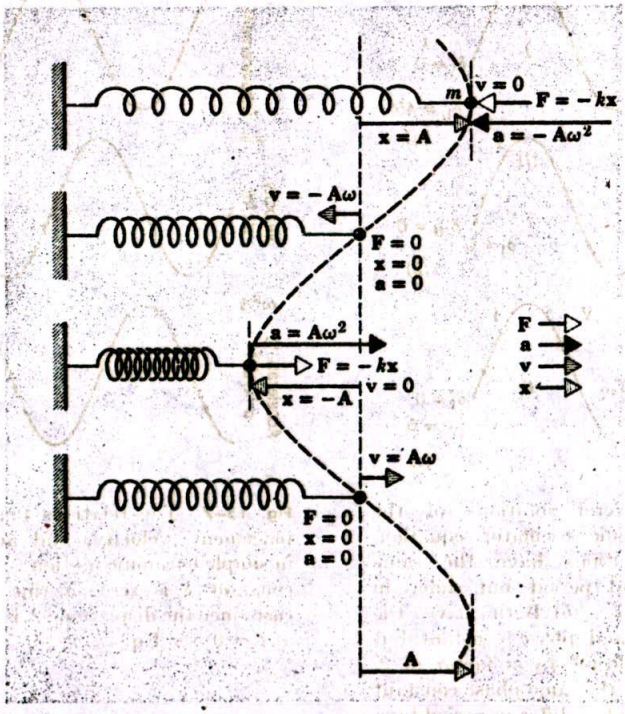


Fig. 15–8 The force acting on, and the acceleration, velocity and displacement of a mass m undergoing simple harmonic motion. Compare carefully with Fig. 15–7.

For the case plotted we have taken $\delta = 0$. The units and scale of displacement, velocity, and acceleration are omitted for simplicity of comparison. Notice that the maximum displacement is A, the maximum speed is ωA, and the maximum acceleration is $\omega^2 A$.

When the displacement is a maximum in either direction, the speed is zero because the velocity must now change its direction. The acceleration at this instant, like the restoring force, has a maximum value but is directed opposite to the displacement. When the displacement is zero, the speed of the particle is a maximum and the acceleration is zero, corresponding to a zero restoring force. The speed increases as the particle moves toward the equilibrium position and then decreases as it moves out to the maximum displacement, just as for a pendulum bob.

In Fig. 15–8 we show the instantaneous values of **x**, **v**, and **a** at four instants in the motion of a particle oscillating at the end of a spring.

15–4 Energy Considerations in Simple Harmonic Motion

Equation 15–2 tells us that for harmonic motion, including simple harmonic motion, in which no dissipative forces act the total mechanical energy E $(= K + U)$ is conserved. We can now study this in more detail for the special case of simple harmonic motion, for which the displacement is given by

$$x = A \cos (\omega t + \delta). \tag{15–8}$$

The potential energy U at any instant is given by

$$U = \tfrac{1}{2} kx^2$$

$$= \tfrac{1}{2} kA^2 \cos^2 (\omega t + \delta) \tag{15–14}$$

The potential energy has a maximum value of $\frac{1}{2}kA^2$. During the motion the potential energy varies between zero and this maximum value, as the curves in Fig. 15–9*a* and 15–9*b* show.

The kinetic energy K at any instant is $\frac{1}{2}mv^2$. Using the relations

$$v = dx/dt = -\omega A \sin (\omega t + \delta)$$

and

$$\omega^2 = k/m,$$

we obtain

$$K = \tfrac{1}{2}mv^2,$$

$$= \tfrac{1}{2}m\omega^2 A^2 \sin^2 (\omega t + \delta),$$

$$= \tfrac{1}{2}kA^2 \sin^2 (\omega t + \delta). \tag{15–15}$$

The kinetic energy, therefore, has a maximum value of $\frac{1}{2}kA^2$ or $\frac{1}{2}m(\omega A)^2$, in agreement with the maximum speed ωA noted earlier. During the motion the kinetic energy varies between zero and this maximum value, as shown by the curves in Fig. 15–9*a* and 15–9*b*.

The total mechanical energy is the sum of the kinetic and the potential energy. Using Eqs. 15–14 and 15–15 we obtain

$$E = K + U = \tfrac{1}{2}kA^2 \sin^2 (\omega t + \delta) + \tfrac{1}{2}kA^2 \cos^2 (\omega t + \delta) = \tfrac{1}{2}kA^2. \tag{15–16}$$

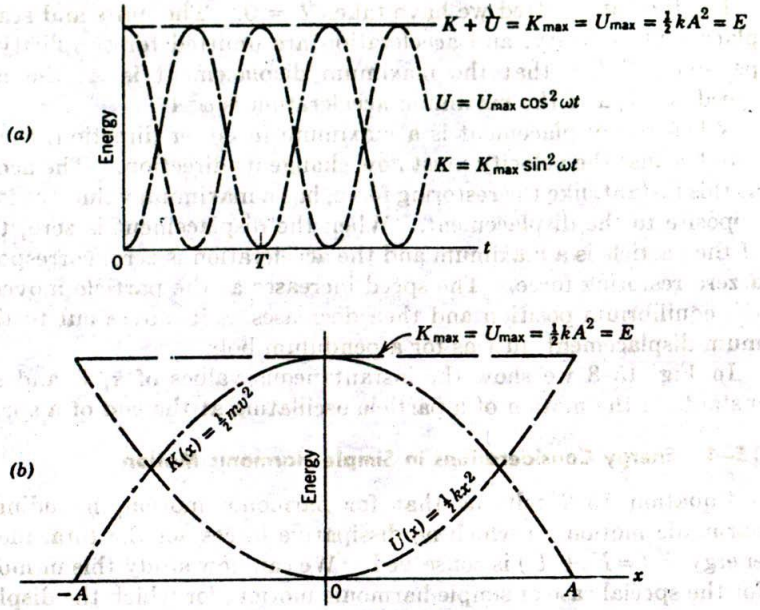


Fig. 15–9 Energies of a simple harmonic oscillator. (*a*) Potential energy (— — — —), kinetic energy (— · —), and total energy (— —) plotted as a function of time. (*b*) Potential, kinetic, and total energy plotted as a function of displacement from the equilibrium position. Compare with Fig. 8–4.

We see that the total mechanical energy is constant, as we expect, and has the value $\frac{1}{2}kA^2$. At the maximum displacement the kinetic energy is zero, but the potential energy has the value $\frac{1}{2}kA^2$. At the equilibrium position the potential energy is zero, but the kinetic energy has the value $\frac{1}{2}kA^2$. At other positions the kinetic and potential energies each contribute energy whose sum is always $\frac{1}{2}kA^2$. This constant total energy is shown in Fig. 15–9*a* and 15–9*b*. *The total energy of a particle executing simple harmonic motion is proportional to the square of the amplitude of the motion.* It is clear from Fig. 15–9*a* that the *average* kinetic energy for the motion during one period is exactly equal to the *average* potential energy and that each of these average quantities is $\frac{1}{4}kA^2$.

Equation 15-16 can be written quite generally as

$$K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2. \tag{15–17}$$

From this relation we obtain $v^2 = (k/m)(A^2 - x^2)$ or

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}. \tag{15–18}$$

This relation shows clearly that the speed is a maximum at the equilibrium position $x = 0$ and zero at the maximum displacement $x = A$. In fact,

we can start from the conservation of energy principle, Eq. 15–17 (in which $\frac{1}{2}kA^2 = E$), and by integration of Eq. 15–18 obtain the displacement as a function of time. The result is identical with Eq. 15–8 which we deduced from the differential equation of the motion, Eq. 15–6. (See Problem 18.)

The effect of dissipative forces will be discussed in Section 15–9.

▶ **Example 1.** The horizontal spring of Fig. 15–4 is found to be stretched 3.0 in. from its equilibrium position when a force of 0.75 lb acts on it. Then a 1.5-lb body is attached to the end of the spring and is pulled 4.0 in. along a horizontal frictionless table from the equilibrium position. The body is then released and executes simple harmonic motion.

(*a*) What is the force constant of the spring?

A force of 0.75 lb *on* the spring produces a displacement of 0.25 ft. Hence,

$$k = F/x = 0.75 \text{ lb}/0.25 \text{ ft} = 3.0 \text{ lb/ft}.$$

Why didn't we use $k = -F/x$ here?

(*b*) What is the force exerted by the spring on the 1.5-lb body just before it is released?

The spring is stretched 4.0 in. or $\frac{1}{3}$ ft. Hence, the force exerted *by* the spring is

$$F = -kx = -(3.0 \text{ lb/ft})(\tfrac{1}{3} \text{ ft}) = -1.0 \text{ lb}.$$

The minus sign indicates that the force is directed opposite to the displacement.

(*c*) What is the period of oscillation after release?

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1.5/32}{3.0}}\,\text{sec} = \frac{\pi}{4}\,\text{sec} = 0.79 \text{ sec}.$$

This corresponds to a frequency $\nu(= 1/T)$ of 1.3 cycles/sec and to an angular frequency $\omega(= 2\pi\nu)$ of 8.0 rad/sec.

(*d*) What is the amplitude of the motion?

The maximum displacement corresponds to zero kinetic energy and a maximum potential energy. This is the initial condition before release, so that the amplitude is the initial displacement of 4.0 in. Hence, $A = \frac{1}{3}$ ft.

(*e*) What is the maximum speed of the vibrating body?

From Eq. 15–13, $v_{max} = \omega A = (2\pi/T)A$,

$$v_{max} = \left(\frac{2\pi}{\pi/4}\,\text{sec}^{-1}\right)(\tfrac{1}{3} \text{ ft}) = 2.7 \text{ ft/sec}.$$

The maximum speed occurs at the equilibrium position, where $x = 0$. This value is achieved twice in each period, the velocity being -2.7 ft/sec when the body passes through $x = 0$ first after release and $+2.7$ ft/sec when the body passes through $x = 0$ on the return trip.

(*f*) What is the maximum acceleration of the body?

From Eq. 15–13, $a_{max} = \omega^2 A = (k/m)A$,

$$a_{max} = \left(\frac{3.0}{1.5/32}\right)\left(\frac{1}{3}\right) \text{ft/sec}^2 = 21 \text{ ft/sec}^2.$$

The maximum acceleration occurs at the ends of the path where $x = \pm A$ and $v = 0$. Hence, $a = -21$ ft/sec^2 at $x = +A$ and $a = +21$ ft/sec^2 at $x = -A$, the acceleration and displacement being oppositely directed.

(*g*) Compute the velocity, the acceleration, and the kinetic and potential energies of the body when it has moved in halfway from its initial position toward the center of motion.

At this point, $$x = \frac{A}{2} = \tfrac{1}{6} \text{ ft},$$

so that from Eq. 15–18,

$$v = -\frac{2\pi}{T}\sqrt{A^2 - x^2}$$

$$= -\frac{2\pi}{\pi/4}\sqrt{\left(\frac{1}{3}\right)^2 - \left(\frac{1}{6}\right)^2} \text{ ft/sec} = -\frac{4}{\sqrt{3}} \text{ ft/sec} = -2.3 \text{ ft/sec},$$

$$a = -\frac{k}{m}x = \frac{-3.0}{1.5/32}\left(\frac{1}{6}\right) \text{ ft/sec}^2 = -11 \text{ ft/sec}^2,$$

$$K = \tfrac{1}{2}mv^2 = \left(\frac{1}{2}\right)\left(\frac{1.5}{32}\right)\left(\frac{4}{\sqrt{3}}\right)^2 \text{ ft-lb} = \tfrac{1}{8} \text{ ft-lb},$$

$$U = \tfrac{1}{2}kx^2 = \left(\frac{1}{2}\right)(3)\left(\frac{1}{6}\right)^2 \text{ ft-lb} = \tfrac{1}{24} \text{ ft-lb}.$$

(*h*) Compute the total energy of the oscillating system.

Since the total energy is conserved, we can compute it at any stage of the motion. Using previous results, we obtain

$$E = K + U = \tfrac{1}{8} + \tfrac{1}{24} = \tfrac{1}{6} \text{ ft-lb}, \quad \text{(particle at } x = A/2)$$

$$E = U_{max} = \tfrac{1}{2}kx_{max}^2 = \left(\frac{1}{2}\right)(3)\left(\frac{1}{3}\right)^2 \text{ ft-lb} = \tfrac{1}{6} \text{ ft-lb}, \quad \text{(particle at } x = A)$$

$$E = K_{max} = \tfrac{1}{2}mv_{max}^2 = \left(\frac{1}{2}\right)\left(\frac{1.5}{32}\right)\left(\frac{8}{3}\right)^2 \text{ ft-lb} = \tfrac{1}{6} \text{ ft-lb}, \quad \text{(particle at } x = 0)$$

(*i*) What is the displacement of the body as a function of time?

In general, we have

$$x = A\cos(\omega t + \delta).$$

We have already found that $A = \frac{1}{3}$ ft. We must now determine ω and δ. We obtain

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\pi/4} = 8 \text{ radians/sec},$$

so that, with our particular units,

$$x = \tfrac{1}{3}\cos(8t + \delta).$$

At the time $t = 0$, $x = \frac{1}{3}$ ft, so that at that instant

$$x = \tfrac{1}{3}\cos\delta = \tfrac{1}{3}$$

or

$$\delta = 0 \text{ radian}.$$

Therefore, with $A = \frac{1}{3}$ ft, $\omega = 8$ radians/sec, and $\delta = 0$ radian, we obtain

$$x = \tfrac{1}{3}\cos 8t.$$

This describes the motion of the body, where x is in feet, t is in seconds, and the angle $8t$ is in radians. ◀

15–5 Applications of Simple Harmonic Motion

A few physical systems that move with simple harmonic motion are considered here. We will discuss others from time to time throughout the text.

The Simple Pendulum. A simple pendulum is an idealized body consisting of a point mass, suspended by a light inextensible cord. When pulled to one side of its equilibrium position and released, the pendulum swings in a vertical plane under the influence of gravity. The motion is periodic and oscillatory. We wish to determine the period of the motion.

Figure 15–10 shows a pendulum of length l, particle mass m, making an angle θ with the vertical. The forces acting on m are $m\mathbf{g}$, the gravitational force, and $\mathbf{T}$, the tension in the cord. Choose axes tangent to the circle of motion and along the radius. Resolve $m\mathbf{g}$ into a radial component of magnitude $mg \cos \theta$, and a tangential component of magnitude $mg \sin \theta$. The radial components of the forces supply the necessary centripetal acceleration to keep the particle moving on a circular arc. The tangential component is the restoring force acting on m tending to return it to the equilibrium position. Hence, the restoring force is

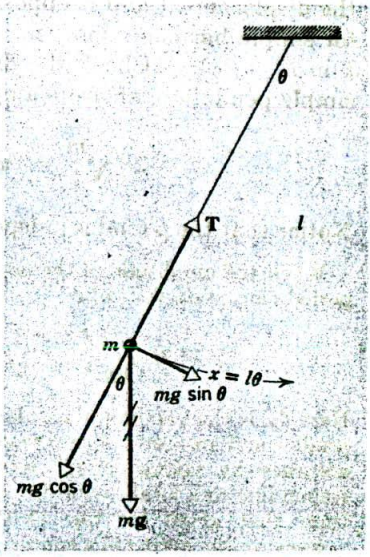


Fig. 15–10 The forces acting on a simple pendulum are the tension $\mathbf{T}$ in the string and the weight $m\mathbf{g}$ of mass m. The magnitudes of the radial and tangential components of $m\mathbf{g}$ are labeled.

$$F = -mg \sin \theta.$$

Notice that the restoring force is not proportional to the angular displacement θ but to $\sin \theta$ instead. The resulting motion is, therefore, not simple harmonic. However, *if the angle θ is small*, $\sin \theta$ is very nearly equal to θ in radians.* The displacement along the arc is $x = l\theta$, and for small angles

* For example,

		Difference, %
$\theta = 0° = 0.0000$ radian	$\sin \theta = 0.0000$	0
$\theta = 2° = 0.0349$ radian	$\sin \theta = 0.0349$	0.00
$\theta = 5° = 0.0873$ radian	$\sin \theta = 0.0872$	0.11
$\theta = 10° = 0.1745$ radian	$\sin \theta = 0.1736$	0.51
$\theta = 15° = 0.2618$ radian	$\sin \theta = 0.2588$	1.14

this is nearly straight-line motion. Hence, assuming

$$\sin\theta \cong \theta,$$

we obtain

$$F = -mg\theta = -mg\frac{x}{l} = -\frac{mg}{l}x.$$

For *small displacements*, therefore, the restoring force is proportional to the displacement and is oppositely directed. This is exactly the criterion for simple harmonic motion. The constant mg/l represents the constant k in $F = -kx$. Check the dimensions of k and mg/l. The period of a simple pendulum when its amplitude is small is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/l}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{l}{g}}. \tag{15–19}$$

Notice that the period is independent of the mass of the suspended particle.

When the amplitude of the oscillation is not small, the general equation for the period can be shown to be

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{2^2}\cdot\sin^2\frac{\theta_m}{2} + \frac{1}{2^2}\cdot\frac{3^2}{4^2}\cdot\sin^4\frac{\theta_m}{2} + \cdots\right). \tag{15–20}$$

Here θ_m is the maximum angular displacement and the succeeding terms become smaller and smaller. The period can then be computed to any desired degree of accuracy by taking enough terms in the infinite series. When $\theta_m = 15°$, corresponding to a total to-and-fro angular displacement of 30°, the true period differs from that given by Eq. 15–19 by less than 0.5%.

Because the period of a simple pendulum is practically independent of the amplitude, the pendulum is useful as a timekeeper. As damping forces reduce the amplitude of swing, the period remains very nearly unchanged. In a pendulum clock energy is supplied automatically by an escapement mechanism to compensate for frictional loss. The pendulum clock with escapement was invented by Christian Huygens (1629–1695).

The simple pendulum also provides a convenient method for measuring the value of g, the acceleration due to gravity. We need not perform a free-fall experiment here, but instead we merely measure l and T.

The Torsional Pendulum. In Fig. 15–11 we show a disk suspended by a wire attached to the center of mass of the disk. The wire is securely fixed to a solid support and to the disk. At the equilibrium position of the disk a radial line is drawn from its center to P, as shown. If the disk is rotated in a horizontal plane to the radial position Q, the wire will be twisted. The twisted wire will exert a torque on the disk tending to return it to the position P. This is a restoring torque. For small twists the restoring torque is found to be proportional to the amount of twist, or the angular displacement (Hooke's law), so that

$$\tau = -\kappa\theta. \tag{15–21}$$

Here κ is a constant that depends on the properties of the wire and is called the *torsional* constant. The minus sign shows that the torque is directed

opposite to the angular displacement θ. Equation 15–21 is the condition for *angular simple harmonic motion.*

The equation of motion for such a system is

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2},$$

so that, on using Eq. 15–21, we obtain

$$-\kappa\theta = I\frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta. \qquad (15\text{–}22)$$

Notice the similarity between Eq. 15–22 for simple angular harmonic motion and Eq. 15–7 for simple linear harmonic motion. In fact, the equations are identical mathematically. We have simply substituted angular displacement θ for linear displacement x, rotational inertia I for mass m, and torsional constant κ for force constant k. The solution of Eq. 15–22 is, therefore, a simple harmonic oscillation in the angle coordinate θ, namely

$$\theta = \theta_m \cos(\omega t + \delta). \qquad (15\text{–}23)$$

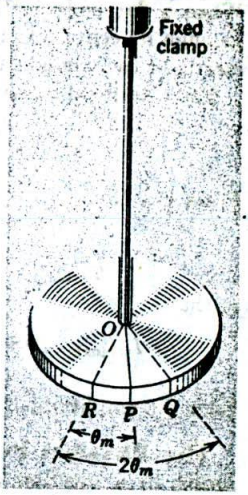


Fig. 15–11 The torsional pendulum. The line drawn from the center to P oscillates between Q and R, sweeping out an angle $2\theta_m$ where θ_m is the (angular) amplitude of the motion.

Here, θ_m is the maximum angular displacement, that is, the amplitude of the angular oscillation. In Fig. 15–11 the disk oscillates about the equilibrium position $\theta = 0$ (line OP), the total angular range being $2\theta_m$ (from OQ to OR).

The period of the oscillation by analogy with Eq. 15–10 is

$$T = 2\pi\sqrt{\frac{I}{\kappa}}. \qquad (15\text{–}24)$$

If κ is known and T is measured, the rotational inertia I about the axis of rotation of any oscillating rigid body can be determined. If I is known and T is measured, the torsional constant κ of any sample of wire can be determined.

Many laboratory instruments involve torsional oscillations, notably the galvanometer. The Cavendish balance is a torsional pendulum (Chapter 16). The balance wheel of a watch is another example of angular harmonic motion, the restoring torque here being supplied by a spiral hairspring.

▶ **Example 2.** A thin rod of mass 0.10 kg and length 0.10 meter is suspended by a wire which passes through its center and is perpendicular to its length. The wire is twisted and the rod set oscillating. The period is found to be 2.0 sec.

When a flat body in the shape of an equilateral triangle is suspended similarly through its center of mass, the period is found to be 6.0 sec. Find the rotational inertia of the triangle about this axis.

The rotational inertia of the rod is $Ml^2/12$ (see Table 12–1). Hence

$$I_{\text{rod}} = \frac{(0.10 \text{ kg})(0.10 \text{ meter})^2}{12} = 8.3 \times 10^{-5} \text{ kg-m}^2.$$

From Eq. 15–24,

$$\frac{T_{\text{rod}}}{T_{\text{triangle}}} = \left(\frac{I_{\text{rod}}}{I_{\text{triangle}}}\right)^{1/2} \quad \text{or} \quad I_{\text{triangle}} = I_{\text{rod}} \left(\frac{T_t}{T_r}\right)^2,$$

so that

$$I_{\text{triangle}} = (8.3 \times 10^{-5} \text{ kg-m}^2) \left(\frac{6.0 \text{ sec}}{2.0 \text{ sec}}\right)^2 = 7.5 \times 10^{-4} \text{ kg-m}^2.$$

Does the amplitude of the oscillation affect the period in these cases? ◀

The Physical Pendulum. Any rigid body mounted so that it can swing in a vertical plane about some axis passing through it is called a physical pendulum. This is a generalization of the simple pendulum in which a weightless cord holds a single particle. Actually all real pendulums are physical pendulums.

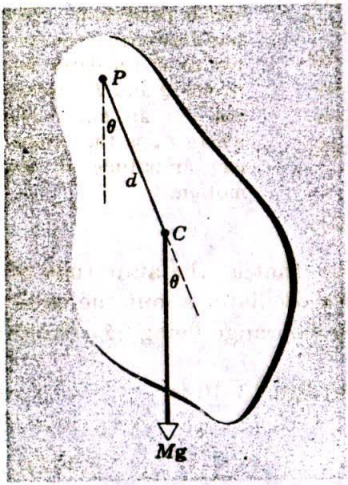


Fig. 15–12 A laminar physical pendulum, with center of mass C, is pivoted at P and displaced an angle θ from its equilibrium position (when C hangs directly below P). Its weight Mg supplies a restoring torque.

In Fig. 15–12 a body of irregular shape is pivoted about a horizontal frictionless axis through P and displaced from the equilibrium position by an angle θ. The equilibrium position is that in which the center of mass of the body, C, lies vertically below P. The distance from pivot to center of mass is d, the rotational inertia of the body about an axis through the pivot is I, and the mass of the body is M. The restoring torque for an angular displacement θ is

$$\tau = -Mgd \sin \theta$$

and is due to the tangential component of the force of gravity. Since τ is proportional to $\sin \theta$, rather than θ, the condition for simple angular harmonic motion does not, in general, hold here. For small angular displacements, however, the relation $\sin \theta \cong \theta$ is, as before, an excellent approximation, so that for *small amplitudes*,

$$\tau = -Mg\,d\,\theta$$

or

$$\tau = -\kappa\theta,$$

where

$$\kappa = Mgd.$$

But
$$\tau = I\frac{d^2\theta}{dt^2} = I\alpha,$$

so that
$$\frac{d^2\theta}{dt^2} = \frac{\tau}{I} = -\frac{\kappa}{I}\theta.$$

Hence, the period of a physical pendulum oscillating with small amplitude is

$$T = 2\pi\sqrt{\frac{I}{\kappa}} = 2\pi\sqrt{\frac{I}{Mgd}}. \tag{15–25}$$

At larger amplitudes the physical pendulum still has a harmonic motion, but not a simple harmonic one.

Notice that this treatment applies to a laminar object of any shape and that the pivot can be located anywhere. As a special case consider a point mass m suspended at the end of a weightless string of length l. Here

$$I = ml^2, \qquad M = m, \qquad d = l,$$

so that
$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{l}{g}},$$

which is the period of a simple pendulum with small amplitude. The physical pendulum is used for accurate determinations of g.

Equation 15–25 can be solved for the rotational inertia I, giving

$$I = \frac{T^2Mgd}{4\pi^2}. \tag{15–26}$$

The quantities on the right are all directly measurable. The center of mass can be determined by suspension as was shown in Fig. 14–4. Hence, the rotational inertia about an axis of rotation not through the center of mass of a body of any shape can be determined by suspending the body as a physical pendulum from that axis.

▶ **Example 3.** Find the length of a simple pendulum whose period is equal to that of a particular physical pendulum.

Equating the period of a simple pendulum to that of a physical pendulum, we obtain

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{I}{Mgd}}$$

or
$$l = \frac{I}{Md}. \tag{15–27}$$

Hence, as far as its period of oscillation is concerned, the mass of a physical pendulum may be considered to be concentrated at a point whose distance from the pivot is $l = I/Md$. This point is called the *center of oscillation* of the physical pendulum. Notice that it depends on the location of the pivot for any given body.

Example 4. A disk is pivoted at its rim (Fig. 15–13). Find its period for small oscillations and the length of the equivalent simple pendulum.

The rotational inertia of a disk about an axis through its center is $\frac{1}{2}Mr^2$, where r is the radius and M is the mass of the disk. The rotational inertia about the pivot at the rim is

$$I = \tfrac{1}{2}Mr^2 + Mr^2 = \tfrac{3}{2}Mr^2.$$

The period then, with $d = r$, is

$$T = 2\pi\sqrt{\frac{I}{Mgr}} = 2\pi\sqrt{\frac{3}{2}\frac{Mr^2}{Mgr}} = 2\pi\sqrt{\frac{3}{2}\frac{r}{g}},$$

independent of the mass of the disk.

The simple pendulum having the same period has a length

$$l = \frac{I}{Mr} = \tfrac{3}{2}r$$

or three-fourths the diameter of the disk. The center of oscillation of the disk pivoted at P is, therefore, at O, a distance $\frac{3}{2}r$ below the point of support. Is any particular mass required of the equivalent simple pendulum?

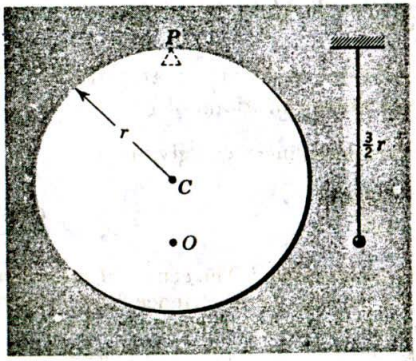


Fig. 15–13 Example 4. A physical pendulum consisting of a disk pivoted at the edge (P), along with a simple pendulum having the same period. O is the center of oscillation.

If we pivot the disk at a point midway between the rim and the center, as at O, we find that $I = \frac{3}{4}Mr^2$ and $d = \frac{1}{2}r$. The period T is

$$T = 2\pi\sqrt{\tfrac{3}{2}r/g}$$

just as before. This illustrates a general property of the center of oscillation O and the point of support P, namely, if the pendulum is pivoted about a new axis through O, its period is unchanged and P becomes the new center of oscillation.

If the disk were pivoted at the center, what would be its period of oscillation?

Example 5. The period of a disk of radius 4.00 in. executing small oscillations about a pivot at its rim is measured to be 0.784 sec. Find the value of g, the acceleration due to gravity at that location.

From $T = 2\pi\sqrt{\frac{3}{2}r/g}$, we obtain

$$g = \frac{6\pi^2 r}{T^2}.$$

With $T = 0.784$ sec and $r = \frac{1}{3}$ ft, we obtain

$$g = \frac{6\pi^2 \cdot \frac{1}{3}}{(0.784)^2}\ \text{ft/sec}^2 = 32.1\ \text{ft/sec}^2. \quad \blacktriangleleft$$

15-6 Relation between Simple Harmonic Motion and Uniform Circular Motion

Let us consider the relation between simple harmonic motion along a straight line and uniform circular motion. This relation is useful in describing many features of simple harmonic motion. It also gives a simple geometric meaning to the angular frequency ω and the phase constant δ. Uniform circular motion is also an example of a combination of simple harmonic motions, a phenomenon we deal with rather often in wave motion.

In Fig. 15–14 Q is the point moving around a circle of radius A with a constant angular speed of ω, expressed, say, in radians/sec. P is the perpendicular projection of Q on the horizontal diameter, along the x-axis. Let us call Q the *reference point* and the circle on which it moves the *reference circle.* As the reference point revolves, the projected point P moves back and forth along the horizontal diameter. The x-component of Q's

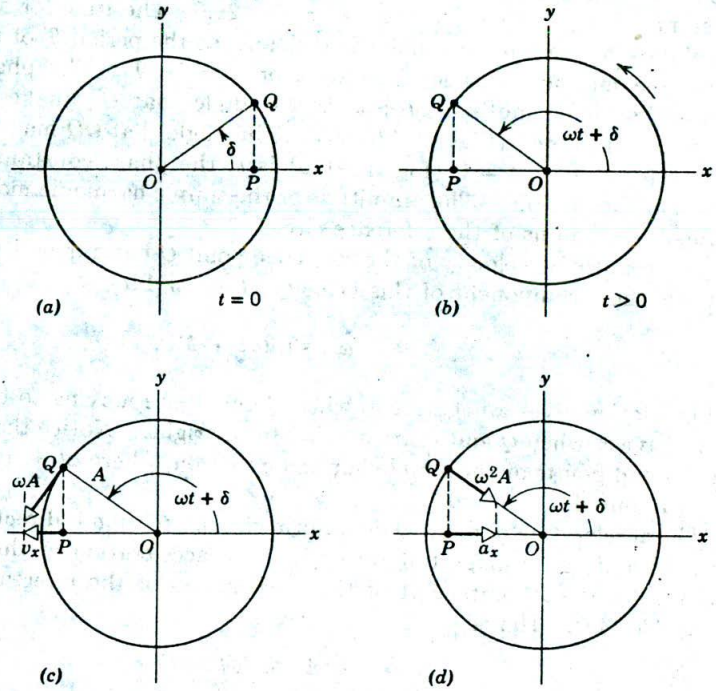


Fig. 15–14 The relation of simple harmonic motion to uniform circular motion. Q moves in uniform circular motion and P in simple harmonic motion. Q has angular speed ω, P angular frequency ω. (*a*, *b*) The x-component of Q's displacement is always equal to P's displacement. (*c*) The x-component of Q's velocity is always equal to P's velocity. (*d*) The x-component of Q's acceleration is always equal to P's acceleration.

displacement is always the same as the displacement of P; the x-component of the velocity of Q is always the same as the velocity of P; and the x-component of the acceleration of Q is always the same as the acceleration of P.

Let the angle between the radius OQ and the x-axis at the time $t = 0$ be called δ. At any later time t, the angle between OQ and the x-axis is $(\omega t + \delta)$, the point Q moving with constant angular speed. The x-coordinate of Q at any time is, therefore,

$$x = A \cos (\omega t + \delta). \tag{15–28}$$

Hence, the projected point P moves with simple harmonic motion along the x-axis. Therefore, *simple harmonic motion can be described as the projection along a diameter of uniform circular motion.*

The angular frequency ω of simple harmonic motion is the same as the angular speed of the reference point. The frequency of the simple harmonic motion is the same as the number of revolutions per unit time of the reference point. Hence, $\nu = \omega/2\pi$ or $\omega = 2\pi\nu$. The time for a complete revolution of the reference point is the same as the period T of the simple harmonic motion. Hence, $T = 2\pi/\omega$ or $\omega = 2\pi/T$. The phase of the simple harmonic motion, $\omega t + \delta$, is the angle that OQ makes with the x-axis at any time t (Fig. 15–14*b*,*c*,*d*). The angle that OQ makes with the x-axis at the time $t = 0$ (Fig. 15–14*a*) is δ, the phase constant or initial phase of the motion. The amplitude of the simple harmonic motion is the same as the radius of the reference circle.

The tangential velocity of the reference point Q has a magnitude of ωA. Hence, the x-component of this velocity (Fig. 15–14*c*) is

$$v_x = -\omega A \sin (\omega t + \delta).$$

This relation gives a negative v_x when Q and P are moving to the left and a positive v_x when Q and P are moving to the right. Notice that v_x is zero at the end points of the simple harmonic motion, where $\omega t + \delta$ is zero and π, as required.

The acceleration of point Q in uniform circular motion is directed radially inward and has a magnitude of $\omega^2 A$. The acceleration of the projected point P is the x-component of the acceleration of the reference point Q (Fig. 15–14*d*). Hence,

$$a_x = -\omega^2 A \cos (\omega t + \delta)$$

gives the acceleration of the point executing simple harmonic motion. Notice that a_x is zero at the midpoints of the simple harmonic motion, where $\omega t + \delta = \pi/2$ or $3\pi/2$, as required.

These results are all identical with those of simple harmonic motion along the x-axis; see Eqs. 15–13.

If we had taken the perpendicular projection of the reference point onto

the y-axis, instead, we would have obtained for the motion of the y-projected point

$$y = A \sin (\omega t + \delta). \tag{15–29}$$

This is again a simple harmonic motion. It differs only in phase from Eq. 15–28, for if we replace δ by $\delta - \pi/2$, then $\cos (\omega t + \delta)$ becomes $\sin (\omega t + \delta)$. It is clear that the projection of uniform circular motion along *any* diameter gives a simple harmonic motion.

Conversely, uniform circular motion can be described as a combination of two simple harmonic motions. It is that combination of two simple harmonic motions, occurring along perpendicular lines, which have the same amplitude and frequency but differ in phase by 90°. When one component is at the maximum displacement, the other component is at the equilibrium point. If we combine these components (Eqs. 15–28 and 15–29), we obtain at once the relation

$$r = \sqrt{x^2 + y^2} = A.$$

By writing the relations for v_y and a_y (the student should do this) and combining corresponding quantities, we obtain also the relations

$$v = \sqrt{v_x^2 + v_y^2} = \omega A,$$

$$a = \sqrt{a_x^2 + a_y^2} = \omega^2 A.$$

These relations correspond respectively to the magnitudes of the displacement, the velocity, and the acceleration in uniform circular motion.

It will be possible for us to analyze many complicated motions as combinations of individual simple harmonic motions. Circular motion is a particularly simple combination. In the next section we shall consider other combinations of simple harmonic motions.

Example 6. In Example 1 we considered a body executing a horizontal simple harmonic motion. The equation of that motion (units?) was

$$x = \tfrac{1}{3} \cos 8t.$$

This motion can also be represented as the projection of uniform circular motion along a horizontal diameter.

(*a*) Give the properties of the corresponding uniform circular motion.

The x-component of the circular motion is given by

$$x = A \cos (\omega t + \delta).$$

Therefore, the reference circle must have a radius $A = \frac{1}{3}$ ft, the initial phase or phase constant must be $\delta = 0$, and the angular velocity must be $\omega = 8$ radians/sec, in order to obtain the equation $x = \frac{1}{3} \cos 8t$ for the horizontal projection.

(*b*) From the motion of the reference point determine the time required for the body to come halfway in toward the center of motion from its initial position.

As the body moves halfway in, the reference point moves through an angle of

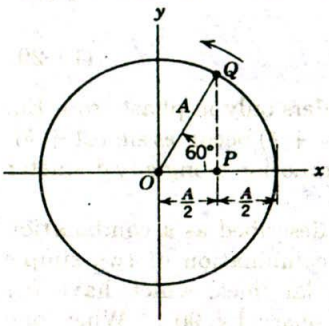


Fig. 15–15 Example 6. The particles Q and P of Fig. 15–14 are shown for $\omega t = 60°$. Since ω is known, t may be found.

$\omega t = 60°$ (Fig. 15–15). The angular velocity is constant at 8 radians/sec so that the time required to move through 60° is

$$t = \frac{60°}{\omega} = \frac{\pi/3 \text{ radians}}{8 \text{ radians/sec}} = \frac{\pi}{24} \text{ sec} = 0.13 \text{ sec}.$$

The time may also be computed directly from the equation of motion. Thus,

$$x = \tfrac{1}{3} \cos 8t \qquad \text{and} \qquad x = \frac{A}{2} = \tfrac{1}{6},$$

Hence

$$\tfrac{1}{6} = \tfrac{1}{3} \cos 8t \qquad \text{or} \qquad 8t = \cos^{-1}(\tfrac{1}{2}) = \frac{\pi}{3}.$$

Therefore,

$$t = \frac{\pi}{24} \text{ sec} = 0.13 \text{ sec}. \quad \blacktriangleleft$$

15–7 Combinations of Harmonic Motions

Often two linear simple harmonic motions *at right angles* are combined. The resulting motion is the sum of two independent oscillations. Consider first the case in which the *frequencies* of the vibrations *are the same*, such as

$$\begin{aligned} x &= A_x \cos(\omega t + \delta), \\ y &= A_y \cos(\omega t + \alpha). \end{aligned} \tag{15–30}$$

The x- and y-motions have different amplitudes and different phase constants, however.

If the phase constants are the same so that $\delta = \alpha$, the resulting motion is a straight line. This can be shown analytically, for when we eliminate t from the equations

$$x = A_x \cos(\omega t + \delta) \qquad y = A_y \cos(\omega t + \delta),$$

we obtain

$$y = (A_y/A_x)x.$$

This is the equation of a straight line, whose slope is A_y/A_x. In Fig. 15–16*a* and *b* we show the resultant motion for two cases, $A_y/A_x = 1$ and $A_y/A_x = 2$. In these cases both the x- and y-displacements reach a maximum at the same time and reach a minimum at the same time. They are in phase.

If the phase constants are different, the resulting motion will not be a straight line. For example, if the phase constants differ by $\pi/2$, the maximum x-displacement occurs when the y-displacement is zero and vice versa. When the amplitudes are equal, the resulting motion is circular; when the amplitudes are unequal, the resulting motion is elliptical. Two cases, $A_y/A_x = 1$ and $A_y/A_x = 2$, are shown in Fig. 15–16*c* and *d*, for $\delta = \alpha + \pi/2$. The cases $A_y/A_x = 1$ and $A_y/A_x = 2$, for $\delta = \alpha - \pi/4$, are shown in Fig. 15–15*e* and *f*.

All possible combinations of two simple harmonic motions at right angles having the same frequency correspond to *elliptical* paths, the circle and straight line being special cases of an ellipse. This can be shown analytically by combining Eqs. 15–30 and eliminating the time; the student can show that the resulting equation is that of an ellipse. The shape of the ellipse depends only on the ratio of the amplitudes, A_y/A_x, and the *difference* in phase between the two oscillations, $\delta - \alpha$. The actual motion can

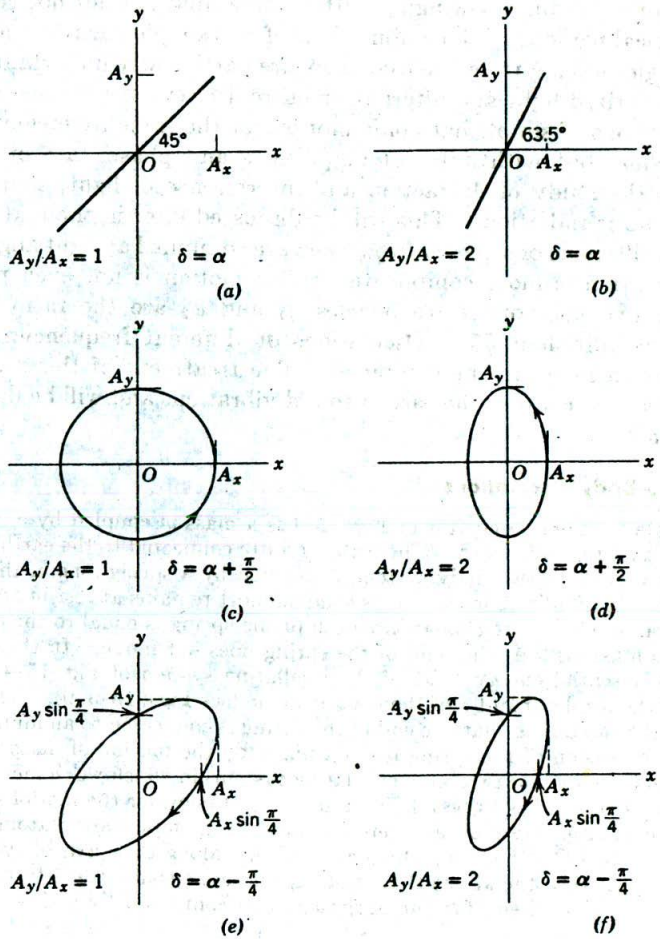


Fig. 15–16 Simple harmonic motions in two dimensions. (*a*) The amplitudes of x and y (namely A_x and A_y) are the same, as are their phase constants. (*b*) y's amplitude is twice x's, but their phase constants are the same. (*c*) Their amplitudes are equal, but x leads y in phase by 90°. (*d*) Same as (*c*) except that y's amplitude is twice x's. (*e*) Equal amplitudes, but x lags y in phase by 45°. (*f*) Same as (*e*) except that y's amplitude is twice x's.

be either clockwise or counterclockwise, depending on which component leads in phase.

A simple way to produce such patterns is by means of an oscilloscope. In this, electrons are deflected by each of two electric fields at right angles to one another. The field strengths alternate sinusoidally with the same frequency, but their phases and amplitudes can be varied. In this way the electrons can be made to trace out the various patterns discussed above on a fluorescent screen. We can also produce these patterns mechanically by means of a pendulum swinging with small amplitude but not confined to one vertical plane. Such combinations of two simple harmonic motions at right angles having the same frequency are particularly important in the study of polarized light and alternating current circuits.

Combinations of simple harmonic motions of the same frequency in the *same direction*, but with different amplitudes and phases, are of special interest in the study of diffraction and interference of light, sound, and electromagnetic radiation. This will be discussed later in the text.

If two oscillations of *different frequencies* are combined at right angles, the resulting motion is more complicated. The motion is not even periodic unless the two component frequencies ω_1 and ω_2 are the ratio of two integers (see Problem 35). Oscillations of different frequencies in the same direction may also be combined. The treatment of this motion is particularly important in the case of sound vibrations and will be discussed in Chapter 20.

15–8 Two-Body Oscillations

The simple harmonic oscillator of Fig. 15–4 is a mass m coupled by a spring of force constant k to a solid wall. The wall is rigidly connected to the earth, so that this system is really a two-body system, connected by a spring, one of the bodies being effectively of infinite mass. This solid support remains at rest in an inertial reference frame so that the change in length of the spring is equal to the displacement of the mass m; the other end of the spring does not move. In this case we defined the potential energy $U(x)$ of the oscillating system of Fig. 15–4 to be a function of the displacement x of the mass m alone (see Figs. 15–3, 9). This again is equivalent to assuming that one end of the spring is connected to an infinite mass so that the extension of the spring is determined by the motion of mass m alone.

Often in nature we find two-body oscillating systems in which we *cannot* take the mass of one of the bodies to be infinite and we must consider the motions of both bodies in an appropriate inertial reference frame. Examples are diatomic molecules such as H_2, CO, HCl, etc., which can oscillate along their axis of symmetry. The coupling between the atoms that make up these molecules is electromagnetic, but we may imagine them, for our purpose, to be connected by a tiny, massless spring.

The surprising thing about two-body oscillators is that, by redefining terms slightly and by introducing a new concept (that of *reduced mass*), we can describe the oscillations by exactly the same equations that we have already derived for the (effectively) one-body system of Fig. 15-4. Let us prove this.

Figure 15–17*a* shows two bodies m_1 and m_2 connected by a (massless) spring of force constant k; the system is free to oscillate on a frictionless horizontal surface. We locate the ends of the spring by the coordinates $x_1(t)$ and $x_2(t)$, as shown. The length of the spring at any instant is $x_1 - x_2$. If its normal, unstressed length is l,

then the *change* in length of the spring, $x(t)$, is given by

$$x = (x_1 - x_2) - l. \tag{15–31}$$

If x is positive, the spring is stretched, if $x = 0$, the spring has its normal length, and if x is negative, it is compressed.

In Fig. 15–17*a* we assume, for concreteness, that the spring is stretched, so that $x > 0$. We show also the force $\mathbf{F}$ exerted by the spring on m_2 and the force $-\mathbf{F}$ exerted on m_2. These two forces are equal and opposite, as the figure shows, and have the common magnitude $F = kx$.

If we apply Newton's second law, $F = ma$, to masses m_1 and m_2 we obtain

$$m_1 \frac{d^2x_1}{dt^2} = -kx$$

and

$$m_2 \frac{d^2x_2}{dt^2} = +kx.$$

Let us now multiply the first equation by m_2 and the second equation by m_1 and subtract. We obtain

$$m_1m_2 \frac{d^2x_1}{dt^2} - m_1m_2 \frac{d^2x_2}{dt^2} = -m_2kx - m_1kx,$$

which we can write as

$$\frac{m_1m_2}{m_1 + m_2} \frac{d^2}{dt^2}(x_1 - x_2) = -kx. \tag{15–32}$$

Let us call the quantity $m_1m_2/(m_1 + m_2)$, which has the dimensions of mass, the *reduced mass* of the system and give it the symbol μ; that is,

$$\mu = \frac{m_1m_2}{m_1 + m_2}. \tag{15–33}$$

Because l is a constant, $d^2(x_1 - x_2)/dt^2 = d^2x/dt^2$ and Eq. 15–32 now can be written as

$$\frac{d^2x}{dt^2} + \frac{k}{\mu}x = 0. \tag{15–34}$$

This is identical in form to Eq. 15–5 which we developed for the single-body oscillation of Fig. 15–4. The differences are that (1) x in Eq. 15–34 is the *relative* displacement of the two blocks from their equilibrium positions (see Eq. 15–31) rather than the displacement of a single block from its equilibrium position, and (2) μ is the *reduced mass* of the pair of blocks rather than the mass of a single block.

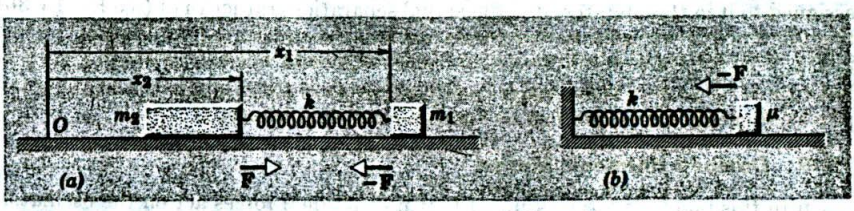


Fig. 15–17 (*a*) Two bodies of masses m_1 and m_2 connected by a (massless) spring whose unstressed length is l. (*b*) A single body of mass μ (the reduced mass) connected by an identical spring to a rigid wall.

Note from Eq. 15–33, which we can write either as

$$\mu = m_1 \frac{m_2}{m_1 + m_2} = m_2 \frac{m_1}{m_1 + m_2}$$

or as

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

that (for finite masses) μ is always *smaller* than m_1 or m_2; hence the name *reduced* mass. Equation 15–34 leads, by way of the derivation that follows Eq. 15–6, to

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{\mu}{k}} \tag{15–35}$$

for the frequency and period of oscillation of the system of Fig. 15–17*a*. It is clear that this system has the same frequency and period as a single block of mass μ, connected by a similar spring to a rigid wall, as in Fig 15–17*b*. Hence, the two-body oscillation of Fig. 15–17*a* is equivalent to the one-body oscillation of Fig. 15–17*b*. One particle moves relative to the other particle as though the other particle were fixed and the mass of the moving one were reduced to μ. The reduced mass concept is applied widely in physics, especially in quantum and solid-state physics.

We can solve Eq. 15–34, as in Section 15–3, to yield these relations:

$$x = A\cos(\omega t + \delta),$$

$$v = dx/dt = -\omega A \sin(\omega t + \delta),$$

and

$$a = dv/dt = -\omega^2 A \cos(\omega t + \delta).$$

They are identical with Eqs. 15–13 except that here x, v, and a are the *relative* displacement, velocity, and acceleration, respectively, of the two blocks. Thus

$$x = (x_1 - x_2) - l,$$

$$v = dx/dt = v_1 - v_2, \tag{15–36}$$

and

$$a = dv/dt = a_1 - a_2,$$

in which the subscripts refer to the two blocks.

The potential energy of a two-body, simple harmonic oscillator is given by $U(x) = \frac{1}{2}kx^2$ which shows clearly, because x depends on the positions of both blocks (see Eq. 15–36), that the potential energy is a characteristic of the system as a whole.

Many actual two-body oscillators, although harmonic, are not simple harmonic; their potential energy curves, like that of Fig. 8–7*a* which refers to a diatomic molecule, are not parabolic. Even such oscillators, however, behave like simple harmonic oscillators for small enough amplitudes of oscillation about the equilibrium position. Note, too, that x in Fig. 8–7*a* has a different meaning than we have assigned to it in this chapter; it is the actual separation, rather than (see Eq. 15–36) the difference between the actual separation and the equilibrium separation. Thus in Fig. 8–7*a* the stable equilibrium position corresponds, not to $x = 0$ as in Fig. 15–2, but to $x = \sqrt[6]{2a/b}$. This change is only a change in the origin of the x-axis of the potential energy curve and has no fundamental significance.

15–9 Damped Harmonic Motion

Up to this point we have assumed that no frictional forces act on the oscillator. If this assumption held strictly, a pendulum or a weight on a spring would oscillate indefinitely. Actually, the amplitude of the oscillation gradually decreases to zero

as a result of friction. The motion is said to be damped by friction and is called *damped harmonic motion*. Often the friction arises from air resistance or internal forces. The magnitude of the frictional force usually depends on the speed. In most cases of interest the frictional force is proportional to the velocity of the body but directed opposite to it. An example of a damped oscillator is shown in Fig. 15–18.

The equation of motion of the damped simple harmonic oscillator is given by the second law of motion, $F = ma$, in which F is the sum of the restoring force $-kx$ and the damping force $-b\,dx/dt$. Here b is a positive constant. We obtain

$$F = ma,$$

or
$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

or
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \qquad (15\text{–}37)$$

If b is small, the solution of this differential equation (given without proof)* is

$$x = Ae^{-bt/2m}\cos(\omega' t + \delta), \qquad (15\text{–}38)$$

where
$$\omega' = 2\pi\nu' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}. \qquad (15\text{–}39)$$

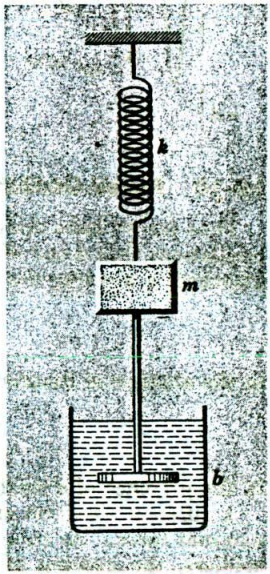


Fig. 15–18 A damped harmonic oscillator. A disk is attached to the mass and immersed in a fluid which exerts a damping force $-b\,dx/dt$. The elastic restoring force is $-kx$.

In Fig. 15–19 we plot the displacement x as a function of the time t for oscillatory motion with small damping.

We can interpret the solution as follows. First, the frequency is smaller and the period is longer when friction is present. Friction slows down the motion, as might be expected If no friction were present, b would equal zero and ω' would equal $\sqrt{k/m}$ or ω, which is the angular frequency of undamped motion. When friction is present, ω' is less than ω, as shown by Eq. 15–39. Second, the amplitude of the motion gradually decreases to zero. The time interval τ during which the amplitude drops to $1/e$ of its initial value is called the *mean lifetime* of the oscillation. The amplitude factor is $Ae^{-bt/2m}$, so that $\tau = 2m/b$. Once again, if there were no friction present, b would equal zero and the amplitude would have the constant value A as time went on; the lifetime would be infinite.

If the force of friction is great enough, b becomes so large that Eq. 15–38 is no longer a valid solution of the equation of motion.† Then the motion will not be periodic at all. The body merely returns to its equilibrium position when released from its initial displacement A.

In damped harmonic motion the energy of the oscillator is gradually dissipated by friction and falls to zero in time.

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* See, for example, H. W. Reddick and F. H. Miller, *Advanced Mathematics for Engineers*, third edition, John Wiley and Sons, 1955, pp. 76–78.

† *Ibid.*, pp. 80–83.

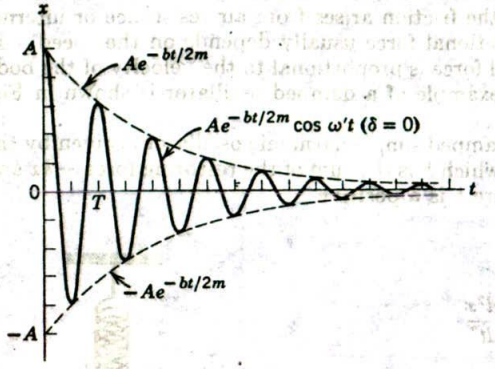


Fig. 15–19 Damped harmonic motion plotted versus time. The motion is oscillatory with everdecreasing amplitude. The amplitude (– – –) is seen to start with value A and decay exponentially to zero as $t \to \infty$.

15–10 Forced Oscillations and Resonance

Thus far we have discussed only the natural oscillations of a body, that is, the oscillations that occur when the body is displaced and then released. For a mass attached to a spring the natural frequency is

$$\omega = 2\pi\nu = \sqrt{\frac{k}{m}}$$

in the absence of friction and

$$\omega' = 2\pi\nu' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2},$$

in the presence of a small frictional force bv.

A different situation arises, however, when the body is subject to an oscillatory external force. As examples, a bridge vibrates under the influence of marching soldiers, the housing of a motor vibrates owing to periodic impulses from an irregularity in the shaft, and a tuning fork vibrates when exposed to the periodic force of a sound wave. The oscillations that result are called *forced* oscillations. These forced oscillations have the frequency of the *external force* and not the natural frequency of the body. However, the response of the body depends on the relation between the forced and the natural frequency. A succession of small impulses applied at the proper frequency can produce an oscillation of large amplitude. A child using a swing learns that by pumping at proper time intervals he can make the swing move with a large amplitude. The problem of forced oscillations is a very general one. Its solution is useful in acoustic systems, alternating current circuits, and atomic physics as well as in mechanics.

The equation of motion of a forced oscillator follows from the second law of motion. In addition to the restoring force $-kx$ and the damping force $-b\,dx/dt$, we have also the applied oscillating external force. For simplicity let this external force be given by $F_m \cos \omega'' t$. Here F_m is the maximum value of the external force and ω'' $(= 2\pi\nu'')$ is its angular frequency. We can imagine such a force applied directly to the suspended mass of Fig. 15–18, if we wish, for concreteness.

From $$F = ma,$$

we obtain $$-kx - b\frac{dx}{dt} + F_m \cos \omega'' t = m\frac{d^2x}{dt^2}$$

or $$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_m \cos \omega'' t. \qquad (15\text{–}40)$$

The solution of this equation (given without proof)* is

$$x = \frac{F_m}{G} \sin(\omega'' t - \delta), \tag{15–41}$$

where

$$G = \sqrt{m^2(\omega''^2 - \omega^2)^2 + b^2\omega''^2}, \tag{15–42}$$

and

$$\delta = \cos^{-1}\frac{b\omega''}{G}. \tag{15–43}$$

Let us consider the resulting motion in a qualitative way.

Notice (Eq. 15–41) that the system vibrates with the frequency of the driving force, ω'', rather than with its natural frequency ω, and that the motion is undamped harmonic motion.

The simplest case is that in which there is no damping, which means that $b = 0$ in Eq. 15–42. The factor G, which has the value $|m(\omega''^2 - \omega^2)|$ for $b = 0$, is large when the frequency of the driving force ω'' is very different from the natural undamped frequency of the system ω. This means that the amplitude of the resultant motion, F_m/G, is small. As the driving frequency approaches the natural frequency, that is, as $\omega'' \to \omega$, we see that $G \to 0$ and the amplitude $F_m/G \to \infty$. Actually some damping is always present so that the amplitude of oscillation, although it may become large, remains finite in practice.

For actual, damped oscillators (for which $b \neq 0$ in Eq. 15–42), there is a characteristic value of the driving frequency ω'' at which the amplitude of oscillation is a maximum. This condition is called *resonance*† and the value of ω'' at which resonance occurs is called the *resonant frequency*. The smaller the damping in a given system the closer is the resonant frequency to the natural undamped frequency ω. Frequently the damping is small enough so that the resonant frequency can be taken to equal the natural undamped frequency ω with small error. Likewise, for small damping, the natural undamped frequency ω $(= \sqrt{k/m})$ can be taken to equal the natural damped frequency ω' (see Eq. 15–39) with small error.

In Fig. 15–20 we have drawn five curves giving the amplitude of the forced vibrations as a function of the ratio of the driving frequency ω'' to the undamped natural frequency ω. Each of the five curves corresponds to a different value of the damping constant b. Curve (a) shows the amplitude when $b = 0$, that is, when there is no damping. In this case, as we have seen, the amplitude becomes infinite at $\omega'' = \omega$ because energy is being fed into the system continuously by the applied force and none of it is dissipated. In practice, some friction is always present, so the amplitude reaches a large, but finite, value. Of course, when the amplitude gets so large that Hooke's law no longer holds and the elastic limit is exceeded, the system is no longer governed by Eq. 15–40. Often the system breaks, as in the Tacoma Bridge disaster (Fig. 15–21). Curves (b) and (c) give the amplitude of forced vibration for two cases of increasing damping.

The displacement caused by a constant force F_m applied to a system with a force constant k is simply F_m/k. Notice (Fig. 15–20) that the amplitude of the forced vibrations is rather large compared to this static displacement. A column of soldiers marching in step across a bridge can set it vibrating with a destructively

* *Ibid.*, pp. 80–83.

† Resonance, defined here to occur at the frequency at which the forced oscillations have their maximum amplitude, may be defined in other ways as, for example, at the frequency at which maximum power is transferred from the driving unit to the oscillating system or at which the speed of the oscillating mass is a maximum. The definitions are not equivalent; we will discuss the matter further when we deal with forced electrical oscillations; see Problem 42.

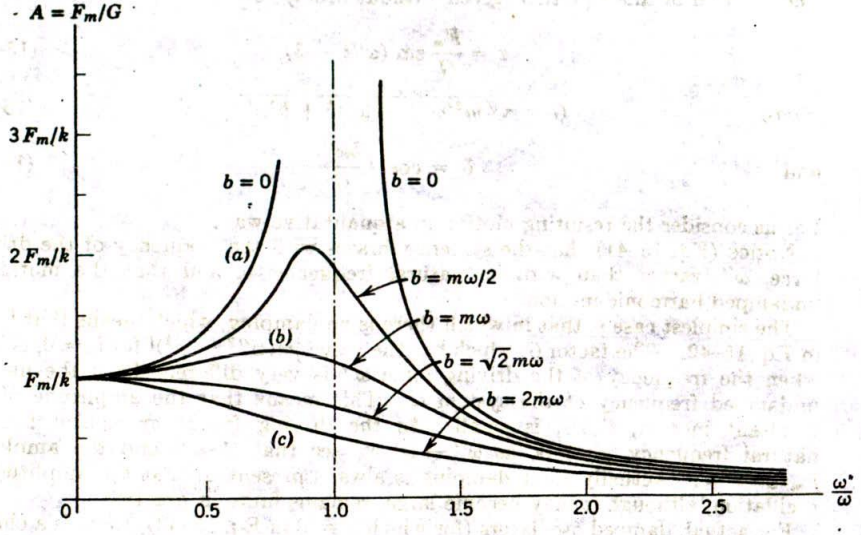


Fig. 15–20 The amplitude of a driven damped simple harmonic oscillator is plotted versus the ratio of the driving frequency ω'' to the undamped natural frequency ω. Curves for five different degrees of damping are shown; curve (*a*) shows no damping and curve (*c*) high damping. We notice that the resonant peak moves nearer and nearer the vertical line at $\omega''/\omega = 1$ as b becomes smaller and smaller.

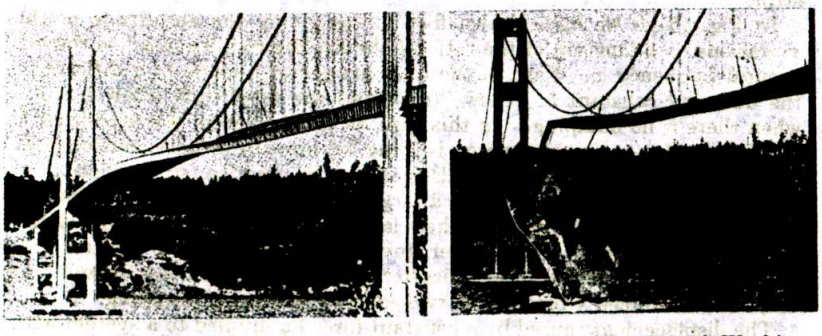


Fig. 15–21 On July 1, 1940, the Tacoma Narrows Bridge at Puget Sound, Washington, was completed and opened to traffic. Just four months later a mild gale set the bridge oscillating until the main span broke up, ripping loose from the cables and crashing into the water below. The wind produced a fluctuating resultant force in resonance with a natural frequency of the structure. This caused a steady increase in amplitude until the bridge was destroyed. Many other bridges were later redesigned to make them aerodynamically stable.

large amplitude if the frequency of their steps happens to be some natural frequency of the bridge. This is the reason why soldiers break step when crossing a bridge. Resonance considerations are very important in many electrical, acoustic, and atomic devices, as we shall see later.

QUESTIONS

1. Give some examples of motions that are approximately simple harmonic. Why are motions that are exactly simple harmonic rare?

2. A spring has a force constant k, and a mass m is suspended from it. The spring is cut in half and the same mass is suspended from one of the halves. Is the frequency of vibration the same before and after the spring is cut? How are the frequencies related?

3. An unstressed spring has a force constant k. It is stretched by a weight hung from it to an equilibrium length well within the elastic limit. Does the spring have the same force constant k for displacements from this new equilibrium position?

4. Any real spring has mass. If this mass is taken into account, explain qualitatively how this will change our expressions for the period of oscillation of a spring-and-mass system (see Problem 40).

5. Suppose we have a block of unknown mass and a spring of unknown force constant. Show how we can predict the period of oscillation of this block-spring system simply by measuring the extension of the spring produced by attaching the block to it.

6. Can one have an oscillator which even for smaller amplitudes is not simple harmonic? That is, can one have a nonlinear restoring force in an oscillator even at arbitrarily small amplitudes?

7. Could we ever construct a simple pendulum?

8. Could standards of mass, length, and time be based on properties of a pendulum? Explain.

9. Show that as the amplitude θ_m in Eq. 15–20 approaches 180° the period approaches infinity. Is this reasonable?

10. Predict by qualitative arguments whether a pendulum oscillating with large amplitude will have a period longer or shorter than the period for oscillations with small amplitude. (Consider extreme cases.)

11. How is the period of a pendulum affected when its point of suspension is (*a*) moved horizontally with acceleration a; (*b*) moved vertically upward with acceleration a; (*c*) moved vertically downward with acceleration $a < g$. Which case, if any, applies to a pendulum mounted on a cart rolling down an inclined plane?

12. Why was an axis through the center of mass excluded in using Eq. 15–26 to determine I? Does this equation apply to such an axis? How can you determine I for such an axis using physical pendulum methods?

13. A hollow sphere is filled with water through a small hole in it. It is hung by a long thread and, as the water slowly flows out of the hole at the bottom, one finds that the period of oscillation first increases and then decreases. Explain.

14. Two pendula, each consisting of a disk attached to a light bar, are identical except for the coupling between disk and bar. In one the bar is rigidly mounted to the disk; in the other ball-bearings are used so that the disk would be free to spin about the end of the bar, for example. Both pendula are hung, pulled aside to the same height, and released. Which has the greater period? Explain.

15. How can a pendulum be used so as to trace out a sinusoidal curve?

16. What component simple harmonic motions would give a figure 8 as the resultant motion?

17. Is there a connection between the F vs. x relation at the molecular level and the macroscopic relation between F and x in a spring?

18. Why are damping devices often used on machinery? Give an example.

19. Give some examples of common phenomena in which resonance plays an important role.

20. The lunar ocean tide is much more important than the solar ocean tide (see Question 13 of Chapter 16, for example). The opposite is true for tides in the earth's atmosphere, however. Explain this, using resonance ideas, given the fact that the atmosphere has a natural period of oscillation of nearly 12 hours.

PROBLEMS

1. A 4.0-kg block extends a spring 16 cm from its unstretched position. The block is removed and a 0.50-kg body is hung from the same spring. If the spring is then stretched and released, what is its period of motion?

2. A 2.0-kg mass hangs from a spring. A 300-gm body hung below the mass stretches the spring 2.0 cm farther. If the 300-gm body is removed and the mass set into oscillation, find the period of motion.

3. A small body of mass 0.10 kg is undergoing simple harmonic motion of amplitude 1.0 meter and period 0.20 sec. (*a*) What is the maximum value of the force acting on it? (*b*) If the oscillations are produced by a spring, what is the force constant of the spring?

4. A body oscillates with simple harmonic motion according to the equation

$$x = 6.0 \cos\left(3\pi t + \frac{\pi}{3}\right) \text{ meters.}$$

What is (*a*) the displacement, (*b*) the velocity, and (*c*) the acceleration at the time $t = 2$ sec. Find also (*d*) the phase constant, (*e*) the frequency ν, and (*f*) the period of the motion.

5. A particle executes linear harmonic motion about the point $x = 0$; at $t = 0$ it has displacement $x = 0.37$ cm and zero velocity. If the frequency of the motion is 0.25/sec, determine (*a*) the period, (*b*) the angular frequency, (*c*) the amplitude, (*d*) the displacement at time t (arbitrary), (*e*) the velocity at time t (arbitrary), (*f*) the maximum speed, (*g*) the maximum acceleration, (*h*) the displacement at $t = 3.0$ sec, and (*i*) the velocity at $t = 3.0$ sec.

6. Two particles execute simple harmonic motion of the same amplitude and frequency along the same straight line. They pass one another when going in opposite directions each time their displacement is half their amplitude. What is the phase difference between them?

7. A block is on a horizontal surface which is moving horizontally with a simple harmonic motion of frequency two oscillations per second. The coefficient of static friction between block and plane is 0.50. How great can the amplitude be if the block does not slip along the surface?

8. A block is on a piston which is moving vertically with a simple harmonic motion of period 1.0 sec. (*a*) At what amplitude of motion will the block and piston separate? (*b*) If the piston has an amplitude of 5.0 cm, what is the maximum frequency for which the block and piston will be in contact continuously?

9. A uniform spring whose unstressed length is l has a force constant k. The spring is cut into two pieces of unstressed lengths l_1 and l_2, where $l_1 = nl_2$ and n is an integer. What are the corresponding force constants k_1 and k_2 in terms of n and k? Check your result for $n = 1$ and $n = \infty$.

10. Two springs are joined and connected to a mass m as shown in Fig. 15–22. The surfaces are frictionless. If the springs separately have force constants k_1 and k_2, show

that the frequency of oscillation of m is

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{(k_1 + k_2)m}}.$$

(The electrical analog of this system is a parallel connection of two capacitors.)

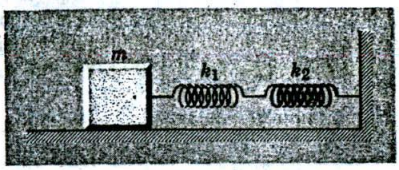


Fig. 15–22

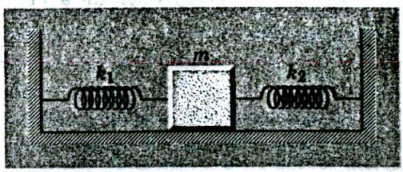


Fig. 15–23

11. The springs are now attached to m and to fixed supports as shown in Fig. 15–23. Show that the frequency of oscillation in this case is

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}}.$$

(The electrical analog of this system is a series combination of two capacitors.)

12. The vibration frequencies of atoms in solids at normal temperatures are of the order 10^{13}/sec. Imagine the atoms to be connected to one another by springs. Suppose that a single silver atom vibrates with this frequency and that all the other atoms are at rest. Then compute the force constant of a single spring. One mole of silver has a mass of 108 gm and contains 6.02×10^{23} atoms.

13. The end of one of the prongs of a tuning fork which executes simple harmonic motion of frequency 1000 per second has an amplitude of 0.40 mm. Neglect damping and find (*a*) the maximum acceleration and maximum speed of the end of the prong, and (*b*) the speed and acceleration of the end of the prong when it has a displacement 0.20 mm.

14. A spring of force constant 19.6 nt/meter hangs vertically. A body of mass 0.20 kg is attached to its free end and then released. Assume that the spring was unstretched before the body was released and find how far below the initial position the body descends. Find also the frequency and amplitude of the resulting simple harmonic motion.

15. An 8.0-lb block is suspended from a spring with a force constant of 3.0 lb/in. A bullet weighing 0.10 lb is fired into the block from below with a velocity of 500 ft/sec and comes to rest in the block. (*a*) Find the amplitude of the resulting simple harmonic motion. (*b*) What fraction of the original kinetic energy of the bullet is stored in the harmonic oscillator? Is energy lost in this process? Explain your answer.

16. An automobile can be considered to be mounted on a spring as far as vertical oscillations are concerned. The springs of a certain car are adjusted so that the vibrations have a frequency of 3.0 per second. What is the spring's force constant if the car weighs 3200 lb? What will the vibration frequency be if five passengers, averaging 160 lb each, ride in the car?

17. The scale of a spring balance reading from 0 to 32 lb is 4.0 in. long. A package suspended from the balance is found to oscillate vertically with a frequency of 2.0 oscillations per second. How much does the package weigh?

18. Start from Eq. 15–17 for the conservation of energy (with $\frac{1}{2}kA^2 = E$) and obtain the displacement as a function of the time by integration of Eq. 15–18. Compare with Eq. 15–18.

19. When the displacement is one-half the amplitude, what fraction of the total energy is kinetic and what fraction is potential in simple harmonic motion? At what displacement is the energy half kinetic and half potential?

20. (*a*) Prove that in simple harmonic motion the average potential energy equals the average kinetic energy when the average is taken with respect to time over one period of the motion, and that each average equals $\frac{1}{4}kA^2$. (See Fig. 15–9*a*.) (*b*) Prove that when the average is taken with respect to position over one cycle, the average potential energy equals $\frac{1}{6}kA^2$ and the average kinetic energy equals $\frac{1}{3}kA^2$. (See Fig. 15–9*b*.) (*c*) Explain physically why the two results above (*a* and *b*) are different.

21. (*a*) Show that the general relations for the period and frequency of any simple harmonic motion are

$$T = 2\pi\sqrt{-\frac{x}{a}} \quad \text{and} \quad \nu = \frac{1}{2\pi}\sqrt{-\frac{a}{x}}.$$

(*b*) Show that the general relations for the period and frequency of any simple angular harmonic motion are

$$T = 2\pi\sqrt{-\frac{\theta}{\alpha}} \quad \text{and} \quad \nu = \frac{1}{2\pi}\sqrt{-\frac{\alpha}{\theta}}.$$

22. *Vertical Spring in a Uniform Gravitational Field.* Consider a massless spring of force constant k in a uniform gravitational field. Attach a mass m to the spring. (*a*) Show that if $x = 0$ marks the slack position of the spring, the static equilibrium position is given by $x = mg/k$ (see Fig. 15–24). (*b*) Show that the equation of motion of the mass-spring system is

$$m\frac{d^2x}{dt^2} + kx = mg$$

and that the solution for the displacement as a function of time is $x = A\cos(\omega t + \delta) + mg/k$, where $\omega = \sqrt{k/m}$ as before. (*c*) Show, therefore, that the system has the same ω, v, a, ν, and T in a uniform gravitational field as in the absence of such a field, with the one change that the equilibrium position has been displaced by mg/k. (*d*) Now consider the energy of the system, $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 + mg(h - x) = \text{constant}$, and show that time differentiation leads to the equation of motion of part (*b*). (*e*) Show that when mass falls from $x = 0$ to the static equilibrium position, $x = mg/k$, the loss in gravitational potential energy goes half into a gain in elastic potential energy and half into a gain in kinetic energy. (*f*) Finally, consider the system in motion at the static equilibrium position. Compute separately the change in gravitational potential energy and in elastic potential energy when the mass moves *up* through a displacement A, and when the mass moves *down* through a displacement A. Show that the *total* change in potential energy is the same in each case, namely $\frac{1}{2}kA^2$.

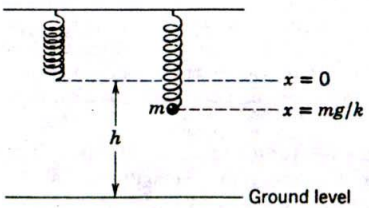


Fig. 15–24

In view of the results (*c*) and (*f*), one can simply ignore the uniform gravitational field in the analysis merely by shifting the reference position from $x = 0$ to $x_0 = x - mg/k = 0$. The new potential energy curve [$U(x_0) = \frac{1}{2}kx_0^2 + \text{constant}$] has the same

parabolic slope as the potential energy curve in the absence of a gravitational field $[U(x) = \frac{1}{2}kx^2]$.

23. A simple pendulum of length 1.00 meter makes 100 complete oscillations in 204 sec at a certain location. What is the value of the acceleration of gravity at this point?

24. What is the length of a simple pendulum whose period is exactly 1 sec at a point where $g = 32.2$ ft/sec^2?

25. Show that the maximum tension in the string of a simple pendulum, when the amplitude θ_m is small, is $mg(1 + \theta_m^2)$. At what position of the pendulum is the tension a maximum?

26. (*a*) What is the frequency of a simple pendulum 2.0 meters long? (*b*) Assuming small amplitudes, what would its frequency be in an elevator accelerating upward at a rate of 2.0 meters/sec^2? (*c*) What would its frequency be in free fall?

27. A simple pendulum of length l and mass m is suspended in a car that is traveling with a constant speed v around a circle of radius R. If the pendulum undergoes small oscillations about its equilibrium position, what will its frequency of oscillation be?

28. What is the period of a pendulum formed by pivoting a meter stick so that it is free to rotate about a horizontal axis passing through its end? Through the 75-cm mark? Through the 60-cm mark?

29. Show that if a uniform stick of length l is mounted so as to rotate about a horizontal axis perpendicular to the stick and at a distance d from the center of mass, the period has a *minimum* value when $d = l/\sqrt{12} = 0.289l$.

30. A disk 1.0 meter in diameter is cut from a metal sheet. The disk is made to swing as a pendulum by drilling a small hole in it and mounting it on a nail driven into a wall. Let l be the distance from the nail to the center of the plate. (*a*) For what value or values of l will the period be 1.7 sec? (*b*) Suppose you want the period to be as small as possible. What value of l would you use?

31. A circular hoop of radius 2.0 ft and weight 8.0 lb is suspended on a horizontal nail. (*a*) What is its frequency of oscillation for small displacements from equilibrium? (*b*) What is the length of the equivalent simple pendulum?

32. A solid sphere of mass 2.0 kg and diameter 0.30 meter is suspended on a wire. Find the period of angular oscillation for small displacements if the torque constant of the wire is 6.0×10^{-3} nt-m/radian.

33. The balance wheel of a watch vibrates with an angular amplitude of π radians and a period of 0.50 sec. Find (*a*) the maximum angular speed of the wheel, (*b*) the angular speed of the wheel when its displacement is $\pi/2$ radians, and (*c*) the angular acceleration of the wheel when its displacement is $\pi/4$ radians.

34. Electrons in an oscilloscope are deflected by two mutually perpendicular electric fields in such a way that at any time t the displacement is given by

$$x = A \cos \omega t, \qquad y = A \cos (\omega t + \alpha).$$

(*a*) Describe the path of the electrons and determine its equation when $\alpha = 0°$. (*b*) When $\alpha = 30°$. (*c*) When $\alpha = 90°$.

35. *Lissajous Figures.* When oscillations at right angles are combined, the frequencies for the motion of the particle in the x- and y-directions need not be equal, so that in the general case Eqs. 15–30 become

$$x = A_x \cos (\omega_x t + \delta) \qquad \text{and} \qquad y = A_y \cos (\omega_y t + \alpha).$$

The path of the particle is no longer an ellipse but is called a *Lissajous curve*, after Jules Antoine Lissajous (1822–1880) who first demonstrated such curves in 1857. (*a*) If

ω_x/ω_y is a rational number, so that the angular frequencies ω_x and ω_y are "commensurable," then the curve is closed and the motion repeats itself at regular intervals of time. Assume $A_x = A_y$ and $\delta = \alpha$ and draw the Lissajous curve for $\omega_x/\omega_y = \frac{1}{2}, \frac{1}{3}$, and $\frac{2}{3}$. (*b*) Let ω_x/ω_y be a rational number, either $\frac{1}{2}, \frac{1}{3}$, or $\frac{2}{3}$ say, and show that the shape of the Lissajous curve depends upon the phase difference $\alpha - \delta$. Draw curves for $\alpha - \delta = 0$, $\pi/4$, and $\pi/2$ radians. (*c*) If ω_x/ω_y is not a rational number, then the curve is "open." Convince yourself that after a long time the curve will have passed through every point lying in the rectangle bounded by $x = \pm A_x$ and $y = \pm A_y$, the particle never passing twice through a given point with the same velocity.

36. (*a*) Show that when $m_2 \to \infty$ in Eq. 15–33, $\mu \to m_1$. (*b*) Show that the effect of a noninfinite wall ($m_2 < \infty$) on the oscillations of a mass m_1 at the end of a spring attached to the wall is to reduce the period, or increase the frequency, of oscillation compared to (*a*). (*c*) Show that when $m_2 = m_1$ the effect is as though the spring were cut in half, each mass oscillating independently about the center of mass at the middle.

37. (*a*) What is the reduced mass of each of the following diatomic molecules: O_2, HCl and CO? Express your answers in atomic mass units, the mass of a hydrogen atom being approximately 1.00 amu. (*b*) An HCl molecule is known to vibrate at a fundamental frequency of $\nu = 8.7 \times 10^{13}$ cycles/sec. What is the effective "force constant" k for the coupling forces between the atoms? In terms of your experience with ordinary springs, would you say that this "molecular spring" is relatively stiff or not?

38. The spring in Fig. 15–17*a* has a force constant $k = 250$ nt/m. Let $m_1 = 1.0$ kg and $m_2 = 3.0$ kg. (*a*) What is the oscillation frequency of the two-body system? (*b*) What is the ratio K_1/K_2 of the kinetic energies of the bodies?

39. Show that the kinetic energy of the two-body oscillator of Fig. 15–17*a* is given by $K = \frac{1}{2}\mu v^2$, where μ is the reduced mass and v ($= v_1 - v_2$) is the relative velocity. It may help to note that linear momentum is conserved while the system oscillates.

40. If the mass of a spring m_s is not negligible but is small compared to the mass m of the object suspended from it, the period of motion is $T = 2\pi\sqrt{(m + m_s/3)/k}$. Derive this result. (*Hint:* The condition $m_s \ll m$ is equivalent to the assumption that the spring stretches uniformly along its length.)

41. (*a*) Attach a solid cylinder to a horizontal massless spring so that it can *roll without slipping* along a horizontal surface, as in Fig. 15–25. The force constant k of the spring is 3.0 nt/meter. If the system is released from rest at a position in which the spring is stretched by 0.25 meter, find the translational kinetic energy and the rotational kinetic energy of the cylinder as it passes through the equilibrium position. (*b*) Show that under these conditions the center of mass of the cylinder executes simple harmonic motion with a period

$$T = 2\pi\sqrt{3M/2k}.$$

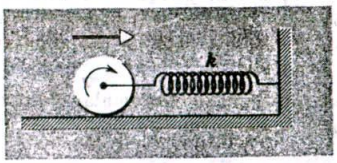


Fig. 15–25

42. Starting from Eq. 15–41, find the velocity v ($= dx/dt$), in forced oscillatory motion. Show that the velocity amplitude is $v_m = F_m/[(m\omega'' - k/\omega'')^2 + b^2]^{\frac{1}{2}}$.

The equations of Section 15–10 are identical in form with those representing an electrical circuit containing a resistance R, an inductance L, and a capacitance C in series with an alternating emf $V = V_m \cos \omega'' t$. Hence, b, m, k, and F_m are analogous to R, L, $1/C$, and V_m, respectively, and x and v are analogous to electric charge q and current i, respectively. In the electrical case the current amplitude i_m, analogous to the velocity amplitude v_m above, is used to describe the quality of the resonance.

43. Two equal masses m and three identical springs of force constant k are arranged as shown in Fig. 15–26. (*a*) Let x_1, x_2 represent the displacement of each mass from its

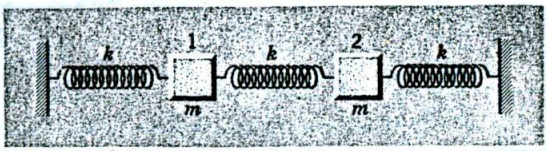


Fig. 15–26

equilibrium position and show that

$$m\frac{d^2x_1}{dt^2} = k(x_2 - 2x_1)$$

and

$$m\frac{d^2x_2}{dt^2} = k(x_1 - 2x_2).$$

(*b*) Find the frequencies of vibration for the system by assuming a solution of the form $x_1 = A_1 \sin \omega t$ and $x_2 = A_2 \sin \omega t$.

Gravitation

CHAPTER 16

16–1 Historical Introduction

Until the seventeenth century the tendency of a body to fall toward the earth, that is, its weight, was regarded as an inherent property of all bodies needing no further explanation. That the weight of a body should be regarded as a force of attraction between the earth and that body was an idea that occurred to Newton and some of his contemporaries, notably Robert Hooke.

The laws governing celestial motions were regarded as quite different from those governing the motions of bodies on the earth. The motion of the heavenly bodies, particularly the planets and the sun, was a subject of much active interest at this time. This subject was discussed by students of natural philosophy at Cambridge in 1664. In 1665, the plague broke out. School was suspended and the students were sent home. One of them was Isaac Newton, then a 23-year-old "scholar of the college."

At home in Woolsthorpe, Newton continued to think about these questions. Apparently he was inspired as he saw an apple fall to the earth from a tree.* It occurred to him that the same force of gravitation which attracts the apple to the earth might also attract the moon to the earth. Newton thought that the centripetal acceleration of the moon in its orbit and the downward acceleration of a body on the earth might have the same

* In a biography of Newton written in 1752 by his friend Stukeley, the author writes of having tea with Newton in a garden under some apple trees, when Newton said that the setting was the same as when he got the idea of gravitation. "It was occasioned by the fall of an apple, as he sat in a contemplative mood . . . ," writes Stukeley.

origin (Fig. 16–1). The very idea that celestial motions and terrestrial motions were subject to similar laws was a break with tradition.

The acceleration of the moon toward the earth can be computed from its period of revolution and the radius of its orbit. We obtain 0.0089 ft/sec^2 (see Example 4, Chapter 4). This value is about 3600 times smaller than g, the acceleration due to gravity on the surface of the earth. Newton sought to account for this difference by assuming that the acceleration of a falling body is inversely proportional to the square of its distance from the earth.

The question of what we mean by "distance from the earth" arises immediately. Newton regarded every particle of the earth as contributing to the gravitational attraction it had on other bodies. The distance and direction of particles of the earth from some object are different for each particle. Newton made the daring assumption that the earth could be treated for such purposes as if all the mass were concentrated at its center.

We could treat the earth as a particle with respect to the sun, for example. It is not obvious at all, however, that we could safely treat the earth as a particle with respect to an apple located only a few feet above its surface. On this assumption, however, a falling body near the earth's surface is a distance of one earth radius from the effective center of attraction of the earth, or 4000 miles. The moon is about 240,000 miles away. The inverse square of the ratio of these distances is $(4000/240{,}000)^2 = 1/3600$, in agreement with the ratio of the accelerations of the moon and apple.

It is believed that Newton made these calculations in 1666. To quote him,

> And in the same year (1666) I began to think of gravity extending to the orb of the moon. . . . I deduced that the forces which keep the planets in their Orbs must [be] reciprocally as the squares of their distances from the centers about which they revolve: and thereby compared the force requisite to keep the Moon in

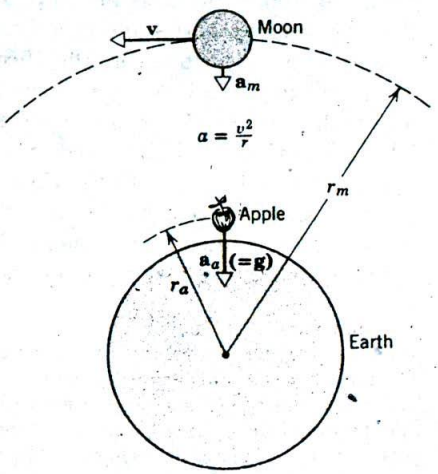


Fig. 16–1 Both the moon and the apple are accelerated toward the center of the earth. The difference in their motions arises because the moon has a tangential velocity v whereas the apple does not.

her Orb with the force of gravity at the surface of the earth and found them answer pretty nearly.

His results were not published until 1687, however, when his *Principia Mathematica* appeared. The reason is thought to be that he was not satisfied at first that he could prove his basic assumption about the earth's acting as a mass particle for objects outside it.* Before he could solve this problem exactly, Newton had to invent the calculus. His proof will be given in Section 16–6.

The *force* on the moon and on the apple depends on the mass of the moon and the mass of the apple, respectively, as well as on that of the earth. Hence, Newton assumed that the gravitational force depended on the masses of the attracting bodies as well as inversely on the square of their distance of separation. He then generalized his concept of gravitational attraction into a law of universal gravitation. He thought that all bodies, no matter where they were located, exerted forces of gravitational attraction upon one another. In order to discover the exact nature of this attractive force, he had to consider bodies of various different masses at significantly different distances from one another. He could not change the distance between the center of the earth and a body on the earth very appreciably, however. It was for this reason that he first compared the motion of the moon and a body on earth. The force between different macroscopic bodies on the earth was so small that it was not detected in Newton's time. Newton apparently realized that this force was small and easily masked by frictional or other forces. Hence, he focused his attention on the motion of the planets in an attempt to confirm his ideas.

The earliest scientific attempts to understand the solar system were made by the Greeks. A detailed description of the conclusions of Greek astronomy was given by Ptolemy (second century). His system is known as the Ptolemaic, or geocentric, theory. It assumes that the earth is stationary at the center of the universe, with the sun, moon, planets, and stars all revolving about the earth in complex orbits. This theory was accepted for almost fifteen centuries and greatly influenced philosophy and literature as well as science. However, the theory was quite complex and could not quantitatively account for an increasing number of observations. In the sixteenth century Copernicus suggested that a simpler description of celestial motions could be given by assuming that the sun was at rest at the center of the universe. In the Copernican or heliocentric theory, the earth was a planet rotating on its axis and revolving about the sun, and the other planets had similar motions (Fig. 16–2).

The growing controversy over the two theories stimulated astronomers

* Some students of history believe that he merely wished to avoid controversy. The publication of his *Theory of Light and Colors* had involved him in bitter arguments. Newton was a shy and introspective man. Bertrand Russell writes of Newton, "If he had encountered the sort of opposition with which Galileo had to contend, it is probable that he would never have published a line." It was Halley, a devoted friend with a great interest in celestial mechanics, who persuaded Newton to publish the *Principia*.

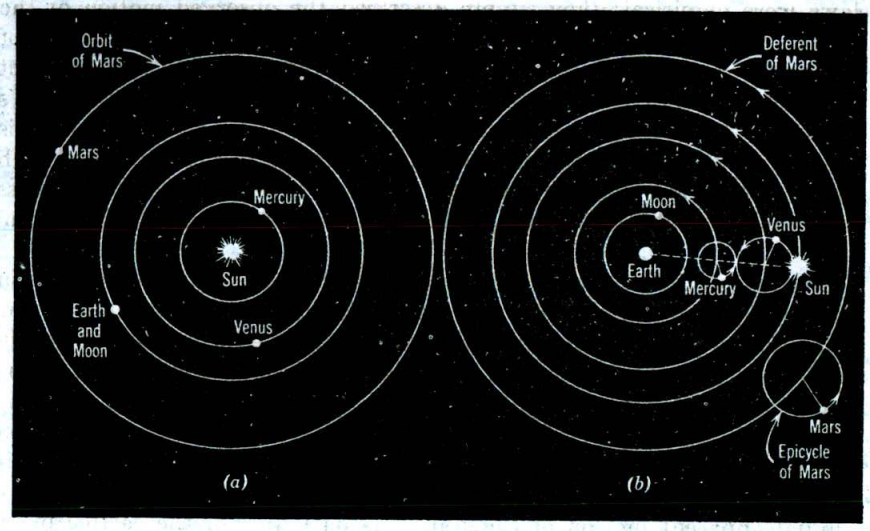


Fig. 16–2 (*a*) The Copernican view of the solar system. The planets move in concentric orbits with the sun at the center. (*b*) The Ptolemaic view of the solar system. Each planet undergoes two simultaneous circular motions. For example, Mars travels about an epicycle while the center of the epicycle travels along a deferent. The earth is at the center of the system. Only the moon and sun have no epicycles.

to obtain more accurate observational data. Such data were compiled by Tycho Brahe (1546–1601), who was the last great astronomer to make observations without the use of a telescope.* His data on planetary motions were analyzed and interpreted for about twenty years by Johannes Kepler (1571–1630), who had been Brahe's assistant. Kepler found important regularities in the motion of the planets. These regularities are known as *Kepler's three laws of planetary motion*

1. All planets move in elliptical orbits having the sun as one focus (the law of orbits).
2. A line joining any planet to the sun sweeps out equal areas in equal times (the law of areas).
3. The square of the period of any planet about the sun is proportional to the cube of the planet's mean distance from the sun (the law of periods).

Kepler's laws lent strong support to the Copernican theory. They showed the great simplicity with which planetary motions could be described when the sun was taken as the reference body. However, these

* The first scientifically useful telescope was built in 1609 by Galileo. With it he discovered the moons of Jupiter and the phases of Venus. Galileo was a strong advocate of the Copernican theory and used his observations to argue in its behalf. Newton, incidentally, invented a telescope, the reflecting type.

laws were empirical; they simply described the observed motion of the planets without any theoretical interpretation. Kepler had no concept of force as a cause of such regularities. In fact, the concept of force was not yet clearly formulated. It was, therefore, a great triumph for Newton's ideas that he could *derive* Kepler's laws from his laws of motion and his law of gravitation. Newton's law of gravitation in this case required each planet to be attracted toward the sun with a force proportional to the mass of the planet and inversely proportional to the square of its distance from the sun.

In this way Newton was able to account for the motion of the planets in the solar system and of bodies falling near the surface of the earth with one common concept. He thereby synthesized into one theory the previously separate sciences of terrestrial mechanics and celestial mechanics. The real scientific significance of Copernicus' work lies in the fact that the heliocentric theory opened the way for this synthesis.* Subsequently, on the assumption that the earth rotates and revolves about the sun, it became possible to explain such diverse phenomena as the daily and the annual apparent motion of the stars, the flattening of the earth from a spherical shape, the behavior of the tradewinds, and many other things that could not have been tied together so simply in a geocentric theory.

It is instructive to review the development of our understanding of the motions of the bodies in the solar system in terms of the program of classical mechanics that we outlined in Chapter 5; see page 80. Historically, there were four "breakthroughs."

1. Copernicus pointed out that the sun and not the earth is the central body of the solar system. In today's language he gave us a reference frame (the sun) much more suitable than the one previously used (the earth) for describing the motions of the solar system. Among other advantages, the Copernican frame, fixed with respect to the sun but not rotating with it, is essentially an *inertial* reference frame; the reference frame fixed to the revolving earth on which we live cannot be so considered for proolems involving planetary motions.
2. Brahe made accurate measurements of the motions of the planets as viewed from the earth. He provided the necessary observational data that made further progress possible.
3. Kepler, studying Brahe's data, deduced from it the three simple empirical laws of planetary motion that we have discussed above. Adopting Copernicus' reference frame, he displayed the kinematic information about planetary motions in simple form.
4. Newton discovered the laws of motion for mechanical systems in general as well as the particular force law that applies to the motions of the planets, namely the law of universal gravitation.

Thus, over a span of about 200 years, we see emerging (1) the appropriate reference frame, (2) accurate kinematical information, (3) the empirical laws of planetary motion, and (4) the general laws of classical mechanics and the force law appropriate to planetary motion.

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* Newton would have been the first to insist that his work was the culmination of the work of others. He once said in a letter to Robert Hooke, "If I have seen further [than others] it is by standing upon the shoulders of Giants." Among these giants we must certainly include Galileo and Kepler.

16–2 The Law of Universal Gravitation

The force between any two particles having masses m_1 and m_2 separated by a distance r is an attraction acting along the line joining the particles and has the magnitude

$$F = G\frac{m_1 m_2}{r^2}, \tag{16–1}$$

where G is a universal constant having the same value for all pairs of particles.

This is Newton's law of universal gravitation. It is important to stress at once many features of this law in order that we understand it clearly.

First, the gravitational forces between two particles are an action-reaction pair. The first particle exerts a force on the second particle that is directed toward the first particle along the line joining the two. Likewise, the second particle exerts a force on the first particle that is directed toward the second particle along the line joining the two. These forces are equal in magnitude but oppositely directed.

The universal constant G must not be confused with the **g** which is the acceleration of a body arising from the earth's gravitational pull on it. The constant G has the dimensions L^3/MT^2 and is a scalar; **g** has the dimensions L/T^2, is a vector, and is neither universal nor constant.

Notice that Newton's law of universal gravitation is not a defining equation for any of the physical quantities (force, mass, or length) contained in it. According to our program for classical mechanics in Chapter 5, force is defined from Newton's second law, $F = ma$. The essence of this law, however, is the assumption that the force on a particle, so defined, can be related in a simple way to measurable properties of the particle and of its environment, that is, the existence of simple force laws is assumed. The law of universal gravitation is such a simple law. The constant G must be found from experiment. Once G is determined for a given pair of bodies, we can use that value in the law of gravitation to determine the gravitational forces between any other pair of bodies.

Notice also that Eq. 16–1 expresses the force between mass *particles*. If we want to determine the force between extended bodies, as for example the earth and the moon, we must regard each body as decomposed into particles. Then the interaction between all particles must be computed. Integral calculus makes such a calculation possible. Newton's motive in developing the calculus arose in part from a desire to solve such problems. In general, it is incorrect to assume that all the mass of a body can be concentrated at its center of mass for gravitational purposes. This assumption is correct for uniform spheres, however, a result that we shall use often and shall prove in Section 16–6.

Implicit in the law of universal gravitation is the idea that the gravitational force between two particles is independent of the presence of other bodies or the properties of the intervening space. The correctness of this idea depends on the correctness of the deductions using it and has so far

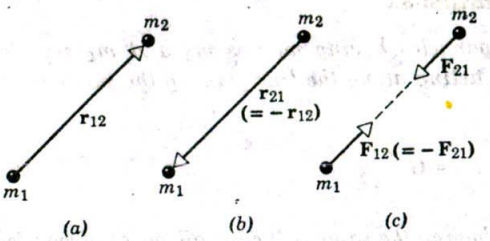


Fig. 16–3 The force exerted on m_2 (by m_1), $\mathbf{F}_{21}$, is directed opposite to the displacement, $\mathbf{r}_{12}$, of m_2 from m_1. The force exerted on m_1 (by m_2), $\mathbf{F}_{12}$, is directed opposite to the displacement, $\mathbf{r}_{21}$, of m_1 from m_2. $\mathbf{F}_{21} = -\mathbf{F}_{12}$, the forces being an action-reaction pair.

been borne out. This fact has been used by some to rule out the possible existence of "gravity screens."

We can express the law of universal gravitation in vector form. Let the displacement vector $\mathbf{r}_{12}$ point *from* the particle of mass m_1 *to* the particle of mass m_2, as Fig. 16–3*a* shows. The gravitational force $\mathbf{F}_{21}$, exerted *on* m_2 *by* m_1, is given in direction and magnitude by the vector relation

$$\mathbf{F}_{21} = -G\frac{m_1 m_2}{r_{12}^3}\mathbf{r}_{12} \qquad (16\text{–}2a)$$

in which r_{12} is the magnitude of $\mathbf{r}_{12}$. The minus sign in Eq. 16–2*a* shows that $\mathbf{F}_{21}$ points in a direction opposite to $\mathbf{r}_{12}$; that is, the gravitational force is *attractive*, m_2 feeling a force directed toward m_1 (see Fig. 16–3). That Eq. 16–2*a* is indeed an inverse square law can be seen by writing it as $\mathbf{F}_{21} = -(Gm_1m_2/r_{12}^2)(\mathbf{r}_{12}/r_{12})$; here the displacement vector divided by its own magnitude, $\mathbf{r}_{12}/r_{12}$, is simply a unit vector $\mathbf{u}_r$ in the direction of the displacement. If we express the relation in scalar form by equating the magnitudes of each side, a factor r_{12} in the numerator cancels one of the factors of r_{12}^3 in the denominator and the inverse square relation of Eq. 16–1 results.

The force exerted *on* m_1 *by* m_2 is clearly

$$\mathbf{F}_{12} = -G\frac{m_2 m_1}{r_{21}^3}\mathbf{r}_{21}. \qquad (16\text{–}2b)$$

Note, in Eqs. 16–2, that $\mathbf{r}_{21} = -\mathbf{r}_{12}$ (see Fig. 16–3*a*,*b*) so that, as we expect, $\mathbf{F}_{12} = -\mathbf{F}_{21}$ (see Fig. 16–3*c*); that is, the gravitational forces acting on the two bodies form an action-reaction pair.

16–3 The Constant of Universal Gravitation, G

To determine the value of G it is necessary to measure the force of attraction between two known masses. The first accurate measurement was made by Lord Cavendish in 1798. Significant improvements were made by Poynting and Boys in the nineteenth century. The present accepted value of G was obtained by P. R. Heyl and P. Chizanowski at the U. S. National Bureau of Standards in 1942. This value is

$$G = 6.673 \times 10^{-11}\ \text{nt-m}^2/\text{kg}^2,$$

accurate to within 0.003×10^{-11} nt-m^2/kg^2. In the British engineering system this value is 3.436×10^{-8} lb-ft^2/slug^2.

The constant G can be determined by the maximum deflection method illustrated in Fig. 16–4. Two small balls, each of mass m, are attached to

the ends of a light rod. This rigid "dumbbell" is suspended, with its axis horizontal, by a fine vertical fiber. Two large balls each of mass M are placed near the ends of the dumbbell on opposite sides. When the large masses are in the positions A and B, the small masses are attracted, by the law of gravitation, and a torque is exerted on the dumbbell rotating it counterclockwise, as viewed from above. When the large masses are in the positions A' and B', the dumbbell rotates clockwise. The fiber opposes these torques as it is twisted. The angle θ through which the fiber is twisted when the balls are moved from one position to the other is measured by observing the deflection of a beam of light reflected from the small mirror attached to it. If the masses and their distances of separation and the torsional constant of the fiber are known, we can calculate G from the measured angle of twist. The force of attraction is very small so that the fiber must have an extremely small torsion constant if we are to obtain a detectable twist. In Example 1 at the end of this section some data are given from which G can be calculated.

The masses in the Cavendish balance of Fig. 16–4 are, of course, not particles but extended objects. Since they are uniform spheres, however, they act gravitationally as though all their mass were concentrated at their centers (Section 16–6).

Because G is so small, the gravitational forces between bodies on the earth's surface are extremely small and can be neglected for ordinary pur-

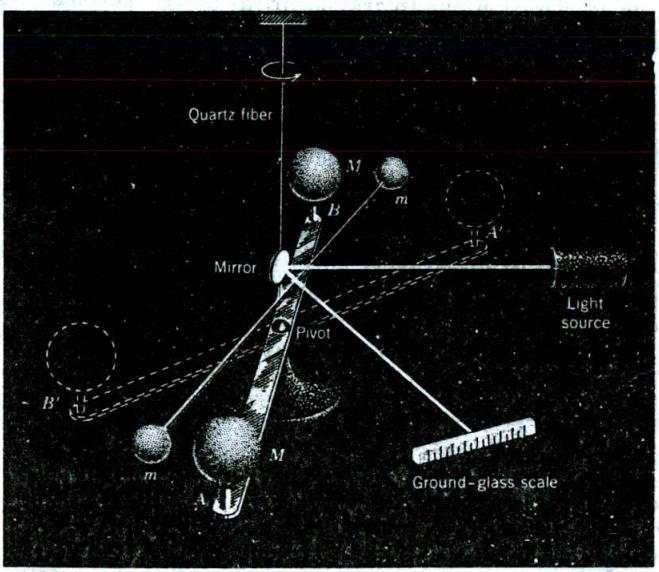


Fig. 16–4 The Cavendish balance, used for experimental verification of Newton's law of universal gravitation. Masses m,m are suspended from a fiber. Masses M,M can rotate on a stationary support. An image of the lamp filament is reflected by the mirror attached to m,m onto the scale so that any rotation of m,m can be measured.

poses. For example, two spherical objects each having a mass of 100 kg (about 220-lb weight) and separated by 1.0 meter at their centers attract each other with a force

$$F = \frac{Gm_1m_2}{r^2} = \frac{(6.67 \times 10^{-11}) \times (100) \times (100)}{(1.0)^2} \text{ nt}$$

$$= 6.7 \times 10^{-7} \text{ nt}$$

or about 1.5×10^{-7} lb! The Cavendish experiment must be a very delicate one indeed.

The large gravitational force which the earth exerts on all bodies near its surface is due to the extremely large mass of the earth. In fact, we can determine the mass of the earth from the law of universal gravitation and the value of G calculated from the Cavendish experiment. For this reason Cavendish is said to have been the first person to "weigh" the earth. Consider the earth, mass M_e, and an object on its surface of mass m. The force of attraction is given both by

$$F = mg$$

and

$$F = \frac{GmM_e}{R_e^2}.$$

Here R_e is the radius of the earth, which is the separation of the two bodies, and g is the acceleration due to gravity at the earth's surface. Combining these equations we obtain

$$M_e = \frac{g R_e^2}{G} = \frac{(9.80 \text{ meters/sec}^2)\ (6.37 \times 10^6 \text{ meters})^2}{6.67 \times 10^{-11} \text{ nt-m}^2/\text{kg}^2} = 5.97 \times 10^{24} \text{ kg}$$

or 6.6×10^{21} tons "weight."

If we were to divide the total mass of the earth by its total volume, we would obtain the average density of the earth. This turns out to be 5.5 gm/cm^3, or about 5.5 times the density of water. The average density of the rock on the earth's surface is much less than this value. We conclude that the interior of the earth contains material of density greater than 5.5 gm/cm^3. From the Cavendish experiment we have obtained information about the nature of the earth's core' (See Question 5 and Problem 35.)

▶ **Example 1.** Let the small spheres of Fig. 16–4 each have a mass of 10.0 gm and let the light rod be 50.0 cm long. The period of torsional oscillation of this system is found to be 769 sec. Then two large fixed spheres each of mass 10.0 kg are placed near each suspended sphere so as to produce the maximum torsion. The angular deflection of the suspended rod is then 3.96×10^{-3} radian and the distance between centers of the large and small spheres is 10.0 cm. Calculate the universal constant of gravitation G from these data.

The period of torsional oscillation is given by Eq. 15–24,

$$T = 2\pi\sqrt{\frac{I}{\kappa}}.$$

For the rigid dumbbell, if we neglect the contribution of the light rod,

$$I = \Sigma mr^2 = (10.0 \text{ gm})(25.0 \text{ cm})^2 + (10.0 \text{ gm})(25.0 \text{ cm})^2$$

or

$$I = 1.25 \times 10^{-3} \text{ kg-m}^2.$$

With $T = 769$ sec, we can obtain the torsional constant κ as

$$\kappa = \frac{4\pi^2 I}{T^2} = \frac{(4\pi^2)(1.25 \times 10^{-3} \text{ kg-m}^2)}{(769 \text{ sec})^2} = 8.34 \times 10^{-8} \text{ kg-m}^2/\text{sec}^2.$$

The relation between the applied torque and the angle of twist is $\tau = \kappa\theta$. We now know κ and the value of θ at maximum deflection.

The torque arises from the gravitational forces exerted by the large spheres on the small ones. This torque will be a maximum for a given separation when the line joining the centers of these spheres is at right angles to the rod. The force on *each* small sphere is

$$F = \frac{GMm}{r^2},$$

and the moment arm for each force is half the length of the rod ($l/2$). Then,

$$\text{torque} = \text{force} \times \text{moment arm}$$

or

$$\tau = 2\frac{GMm}{r^2}\frac{l}{2}.$$

Combining this with

$$\tau = \kappa\theta,$$

we obtain

$$G = \frac{\kappa\theta r^2}{Mml} = \frac{(8.34 \times 10^{-8} \text{ kg-m}^2/\text{sec}^2)(3.96 \times 10^{-3} \text{ radian})(0.100 \text{ meter})^2}{(10.0 \text{ kg})(0.0100 \text{ kg})(0.500 \text{ meter})}$$

$$= 6.63 \times 10^{-11} \text{ nt-m}^2/\text{kg}^2.$$

Notice that this result is about 1% lower than the accepted value. What have we neglected in this calculation that might account for this difference? ◀

16–4 Inertial and Gravitational Mass

The gravitational force on a body is proportional to its mass, as Eq. 16–1 shows. This proportionality between gravitational force and mass is the reason we ordinarily consider the theory of gravitation to be a branch of mechanics, whereas theories of other kinds of force (electromagnetic, nuclear, etc.) may not be.

An important consequence of this proportionality is that we can measure a mass by measuring the gravitational force on it. This can be done by using a spring balance or by comparing the gravitational force on one mass with that on a standard mass, as in a balance; in other words, we can determine the mass of a body by

weighing it. This gives us a more practical and more convenient method of measuring mass than is given by our original definition of mass (Section 5–4).

The question arises whether these two methods really measure the same property. The word mass has been used in two quite different experimental situations. For example, if we try to push a block that is at rest on a horizontal frictionless surface, we notice that it requires some effort to move it. The block seems to be inert and tends to stay at rest, or if it is moving it tends to keep moving. Gravity does not enter here at all. It would take the same effort to accelerate the block in gravity-free space. It is the mass of the block which makes it necessary to exert a force to change its motion. This is the mass occurring in $\mathbf{F} = m\mathbf{a}$ in our original experiments in dynamics. We call this mass m the *inertial mass*. Now there is a different situation which involves the mass of the block. For example, it requires effort just to hold the block up in the air at rest above the earth. If we do not support it, the block will fall to the earth with accelerated motion. The force required to hold up the block is equal in magnitude to the force of gravitational attraction between it and the earth. Here inertia plays no role whatever; the property of material bodies, that they are attracted to other objects such as the earth, does play a role. The force is given by

$$F = G\frac{m'M_e}{R_e^2},$$

where m' is the *gravitational mass* of the block. *Are the gravitational mass m' and the inertial mass m of the block really the same?* Let us look more carefully at this.

Consider two particles A and B of gravitational masses m_A' and m_B' acted on by a third particle C of gravitational mass m_C'. Let the third particle be an equal distance r from the other two. Then, the gravitational force exerted on A by C is

$$F_{AC} = G\frac{m_A'm_C'}{r^2},$$

and the gravitational force exerted on B by C is

$$F_{BC} = G\frac{m_B'm_C'}{r^2}.$$

The ratio of the gravitational forces on A and B is the ratio of their gravitational masses; that is,

$$\frac{F_{AC}}{F_{BC}} = \frac{m_A'}{m_B'}.$$

Now suppose that the third body C is the earth. Then F_{AC} and F_{BC} are what we have called the *weights* of bodies A and B. Hence,

$$\frac{W_A}{W_B} = \frac{m_A'}{m_B'}.$$

Therefore, the law of universal gravitation contains within it the result that the weights of various bodies, at the same place on the earth, are exactly proportional to their *gravitational masses*.

Now suppose we measure the inertial masses m_A and m_B of the particles A and B by dynamical experiments, perhaps using a spring as in Section 5-4. Having done this, we then let these particles fall to the earth from a given place and measure their accelerations. We find experimentally that objects of different *inertial masses* all fall with the same acceleration g arising from the earth's gravitational pull. But the earth's gravitational pulls on these bodies are their weights, so that

using the second law of motion we obtain

$$W_A = m_A g,$$

$$W_B = m_B g,$$

or

$$\frac{W_A}{W_B} = \frac{m_A}{m_B}.$$

In other words, the weights of bodies at the same place on the earth are exactly proportional to their *inertial masses* as well. Hence, inertial mass and gravitational mass are at least proportional to one another. In fact, they appear to be identical.

Newton devised an experiment to test directly the apparent equivalence of inertial and gravitational mass. If we go back (Section 15–5) and look up the derivation of the period of a simple pendulum, we find that the period (for small angles) was given by

$$T = 2\pi \sqrt{\frac{ml}{m'g}},$$

where m in the numerator refers to the inertial mass of the pendulum bob and m' in the denominator is the gravitational mass of the pendulum bob, such that $m'g$ gives the gravitational pull on the bob. Only if we assume that m equals m', as we did there implicitly, do we obtain the expression

$$T = 2\pi \sqrt{\frac{l}{g}}$$

for the period. Newton made a pendulum bob in the form of a thin shell. Into this hollow bob he put different substances, being careful always to have the same *weight* of substance as determined by a balance. Hence, in all cases the force on the pendulum was the same at the same angle. Because the external shape of the bob was always the same, even the air resistance on the moving pendulum was the same. As one substance replaced another inside the bob, any difference in acceleration could only be due to a difference in the *inertial* mass. Such a difference would show up by a change in the period of the pendulum. But in all cases Newton found the period of the pendulum to be the same, always given by $T = 2\pi \sqrt{l/g}$. Hence, he concluded that $m = m'$ and that inertial and gravitational masses are equivalent.

In 1909, Eötvös devised an apparatus which could detect a difference of 5 parts in 10^9 in gravitational force. He found that equal inertial masses always experienced equal gravitational forces within the accuracy of his apparatus. A refined version of the Eötvös experiment was reported in 1964 by R. H. Dicke and his collaborators, who improved the accuracy of the original experiment by a factor of several hundred.*

In classical physics the equivalence of gravitational and inertial mass was looked upon as a remarkable accident having no deep significance. But in modern physics this equivalence is regarded as a clue leading to a deeper understanding of gravitation (see Sec 16-13). This was, in fact, an important clue leading to the development of the general theory of relativity.

16–5 Variations in Acceleration Due to Gravity

Up to this point we have taken the acceleration due to gravity g as a constant. From Newton's law of gravitation, however, it is apparent that

* See "The Eötvös Experiment," by R. H. Dicke, *Scientific American*, December 1961, for an elegant review of this subject.

g will vary with altitude, that is, with distance from the center of the earth. We have already pointed this out specifically in the moon-apple discussion. Let us compute the change in g that occurs as we proceed outward from the earth's surface. From Eq. 16–1,

$$F = G\frac{m_1 m_2}{r^2},$$

we obtain, on differentiating with respect to r,

$$dF = -2\frac{Gm_1 m_2}{r^3}dr.$$

Combining these two equations, we obtain

$$\frac{dF}{F} = -2\frac{dr}{r}.$$

Therefore, the fractional change in F is twice the fractional change in r. The minus sign indicates that the force decreases as the separation distance increases. If we let m_1 be the earth's mass and m_2 the object's mass, the gravitational force on the object attributable to the earth is

$$F = m_2 g$$

directed toward the earth. If we differentiate this expression, we obtain

$$dF = m_2\, dg,$$

and on dividing this equation by the previous one we find that

$$\frac{dF}{F} = \frac{dg}{g} = -2\frac{dr}{r}. \qquad (16\text{–}3)$$

For example, in going up 10 miles from the earth's surface, r changes from about 4000 miles to 4010 miles, a relative increase of 1/400. Therefore,* g must change by about $-1/200$ over this distance, or from about 980 cm/sec^2 to about 975 cm/sec^2. Hence, g is really very nearly constant near the earth's surface at a given latitude. At higher altitudes, such as those for a typical satellite orbit or for the moon's orbit, g drops appreciably, as Table 16–1 shows.

Measurements of **g** are an essential source of information about the shape of the earth. To define the problem more closely we usually consider not the earth itself but an imaginary closed surface called the *geoid*. Over the oceans the geoid is defined to coincide with mean sea level, whereas over the continents it is defined as a continuation of this level; in principle the position of the geoid can be found by digging small sea-level canals across the continents and noting the mean water

* Equation 16–3 is a differential expression and is exact. The corresponding expression obtained when dr is replaced by a finite change Δr is a good approximation, provided that $\Delta r/r$ is very small.

Table 16–1

VARIATION OF g WITH ALTITUDE AT 45° LATITUDE

Altitude, meters	g, meters/sec^2	Altitude, meters	g, meters/sec^2
0	9.806	32,000	9.71
1,000	9.803	100,000	9.60
4,000	9.794	500,000	8.53
8,000	9.782	1,000,000[1]	7.41
16,000	9.757	380,000,000[2]	0.00271

[1] Typical satellite orbit altitude (= 620 miles).
[2] Radius of moon's orbit (= 240,000 miles).

level. The geoid is a surface of constant gravitational potential; at any point the direction of a plumb line is at right angles to it.

The ancient Greeks believed the earth to be round and one of them, Eratosthenes (*c* 276–194 BC), measured the radius of the earth on the assumption that it is a sphere. He obtained a value of 7400 km, which is to be compared with the modern value of 6371 km. This basic information about the shape of the earth was gradually forgotten and was not rediscovered until the great voyages of exploration of the fifteenth century.

Later it was learned by measurement that, to a good second approximation, the geoid is not a sphere but is an ellipsoid of revolution, flattened along the earth's rotational axis and bulging at the equator. The equatorial radius, in fact, exceeds the polar radius by 21 km. This flattening is caused by centrifugal effects in the rotating, plastic earth. The geoidic surface is not exactly ellipsoidal, lying outside the ellipsoid of closest fit under mountain masses and inside it over the oceans.

The fact that the equator is farther from the center of the earth than are the poles means that there should be a steady increase in the measured value of g as one goes from the equator (latitude 0°) to either pole (latitude 90°). This is shown in Table 16-2. As Example 2 shows, however, about half of this variation can be accounted for by another effect, namely, the change in the effective value of g caused by the earth's rotation. If the earth were rotating fast enough, for example,

Table 16–2

VARIATION OF g WITH LATITUDE AT SEA LEVEL

Latitude	g, meters/sec^2	Latitude	g, meters/sec^2
0°	9.78039	50°	9.81071
10°	9.78195	60°	9.81918
20°	9.78641	70°	9.82608
30°	9.79329	80°	9.83059
40°	9.80171	90°	9.83217

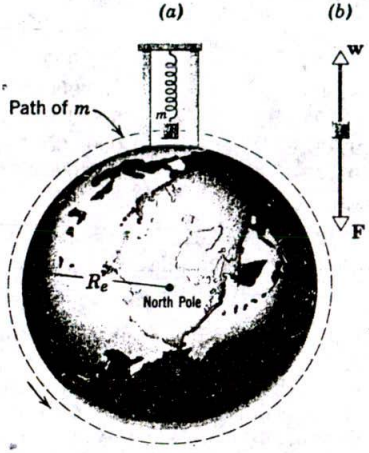


Fig. 16–5 Example 2. Effect of the earth's rotation on the weight of a body as measured by a spring balance.

objects on its surface at the equator would seem to be weightless, which means that the effective value of $g = (W/m)$ would be zero. For all rotational speeds less than this critical value, g has a definite nonzero value which is, however, less than the value it would have at the same point on a nonrotating earth.

In 1959, it was observed that the orbit of the Vanguard artificial earth satellite, calculated using values of **g** based on an ellipsoidal geoid, did not agree exactly with the observed orbit. It was concluded that the geoid is best approximated not by an ellipsoid of revolution but by a slightly pear-shaped figure, the small end of the "pear" being in the northern hemisphere and extending about 15 meters above the reference ellipsoid. The motion of a satellite is governed at all times by the value of **g** at its position. Thus an artificial earth satellite forms a useful probe to explore the values of **g** near the surface of the earth and from this to deduce information about the shape of the geoid. These studies* are continuing as of this date.

▶ **Example 2.** *Effect on g of the rotation of the earth.* Figure 16–5 is a schematic view of the earth looking down on the North Pole. In it we show an enlarged view of a body of mass m hanging from a spring balance at the equator. The forces on this body are the upward pull of the spring balance w, which is the apparent weight of the body, and the downward pull of the earth's gravitational attraction $F = GmM_e/R_e^2$. This body is not in equilibrium because it experiences a centripetal acceleration a_R as it rotates with the earth. There must, therefore, be a net force acting on the body toward the center of the earth. Consequently,

* See, for example, "Satellite Orbits and Their Geophysical Implications," by D. G. King-Hele, *Contemporary Physics*, April 1961.

the force F of gravitational attraction (the true weight of the body) must exceed the upward pull of the balance w (the apparent weight of the body).

From the second law of motion we obtain

$$F - w = ma_R,$$

$$\frac{GM_e m}{R_e^2} - mg = ma_R,$$

$$g = \frac{GM_e}{R_e^2} - a_R \qquad \text{at the equator.}$$

At the poles $a_R = 0$ so that

$$g = \frac{GM_e}{R_e^2} \qquad \text{at the poles.}$$

This is the value of g we would obtain anywhere (assuming a spherical earth) were the rotation of the earth to be neglected.

Actually the centripetal acceleration is not directed in toward the center of the earth other than at the equator. It is directed perpendicularly in toward the earth's axis of rotation at any given latitude. The detailed analysis is, therefore, really a two-dimensional one. However, the extreme case is at the equator. There

$$a_R = \omega^2 R_e = \left(\frac{2\pi}{T}\right)^2 R_e = \frac{4\pi^2 R_e}{T^2},$$

in which ω is the angular speed of the earth's rotation, T is the period, and R_e is the radius of the earth. Using the values

$$R_e = 6.37 \times 10^6 \text{ meters},$$

$$T = 8.64 \times 10^4 \text{ sec},$$

we obtain

$$a_R = 0.0336 \text{ meter/sec}^2.$$

Referring to Table 16–2, we see that this effect is enough to account for more than half the difference between the observed values of g at low and high latitudes. ◀

16–6 Gravitational Effect of a Spherical Distribution of Mass

We have already used the fact that a large sphere attracts particles outside it, just as though the mass of the sphere were concentrated at its center. Let us now prove this result.

Consider a uniformly dense spherical shell whose thickness t is small compared to its radius r (Fig. 16–6). We seek the gravitational force it exerts on an external particle P of mass m.

We assume that each particle of the shell exerts on P a force which is proportional to the mass of the small part, inversely proportional to the square of the distance between that part of the shell and P, and directed along the line joining them. We must then obtain the resultant force on P attributable to all parts of the spherical shell.

The small part of the shell at A attracts m with a force $\mathbf{F}_1$. A small part of equal mass at B, equally far from m but diametrically opposite A, attracts m with a force $\mathbf{F}_2$. The resultant of these two forces on m is $\mathbf{F}_1 + \mathbf{F}_2$. Notice,

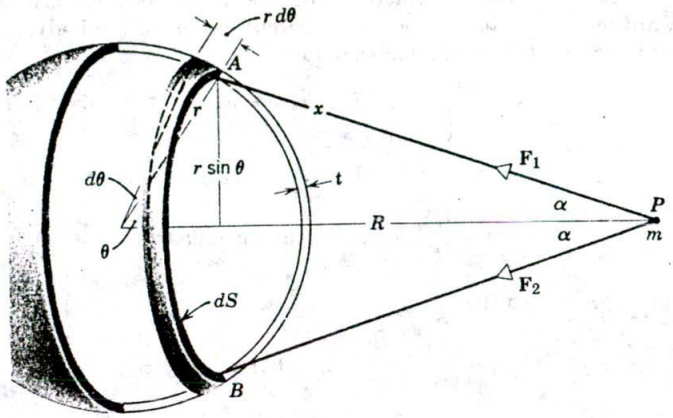


Fig. 16–6 Gravitational attraction of a section dS of a spherical shell of matter on a particle of mass m.

however, that the vertical components of these two forces cancel one another and that the horizontal components, $F_1 \cos \alpha$ and $F_2 \cos \alpha$, are equal. By dividing the spherical shell into pairs of particles like these, we can see at once that all transverse forces on m cancel in pairs. A small mass in the upper hemisphere exerts a force having an upward component on m that will annul the downward component of force exerted on m by an equal symmetrically located mass in the lower hemisphere of the shell. To find the resultant force on m arising from the shell, we need consider only horizontal components.

Let us take as our element of mass of the shell a circular strip labeled dS in the figure. Its length is $2\pi(r \sin \theta)$, its width is $r\,d\theta$, and its thickness is t. Hence, it has a volume

$$dV = 2\pi t r^2 \sin \theta \, d\theta.$$

Let us call the density ρ, so that the mass within the strip is

$$dM = \rho \, dV = 2\pi t \rho r^2 \sin \theta \, d\theta.$$

The force exerted by dM on the particle of mass m at P is horizontal and has the value

$$dF = G \frac{m \, dM}{x^2} \cos \alpha$$

$$= 2\pi G t \rho m r^2 \frac{\sin \theta \, d\theta}{x^2} \cos \alpha. \tag{16–4}$$

The variables x, α, and θ are related. From the figure we see that

$$\cos \alpha = \frac{R - r \cos \theta}{x}. \tag{16–5}$$

Since, by the law of cosines,

$$x^2 = R^2 + r^2 - 2Rr \cos \theta, \tag{16–6}$$

we have

$$r \cos \theta = \frac{R^2 + r^2 - x^2}{2R}. \tag{16–7}$$

On differentiating Eq. 16–6, we obtain

$$2x\,dx = 2Rr \sin\theta\,d\theta$$

or

$$\sin\theta\,d\theta = \frac{x}{Rr}\,dx. \tag{16–8}$$

We now put Eq. 16–7 into Eq. 16–5 and then put Eqs. 16–5 and 16–8 into Eq. 16–4. As a result we eliminate θ and α and obtain

$$dF = \frac{\pi G t\rho m r}{R^2}\left(\frac{R^2 - r^2}{x^2} + 1\right)dx.$$

This is the force exerted by the circular strip dS on the particle m.

We must now consider every element of mass in the shell and sum up over all the circular strips in the entire shell. This operation is an integration over the shell with respect to the variable x. But x ranges from a minimum value of $R - r$ to a maximum value $R + r$.

Since

$$\int_{R-r}^{R+r}\left(\frac{R^2 - r^2}{x^2} + 1\right)dx = 4r,$$

we obtain the resultant force

$$F = \int_{R-r}^{R+r} dF = G\frac{(4\pi r^2\rho t)m}{R^2} = G\frac{Mm}{R^2}, \tag{16–9}$$

where

$$M = (4\pi r^2 t\rho)$$

is the total mass of the shell. This is exactly the same result we would obtain for the force between *particles* of mass M and m separated a distance R. We have proved, therefore, that *a uniformly dense spherical shell attracts an external mass point as if all its mass were concentrated at its center.*

A solid sphere can be regarded as composed of a large number of concentric shells. If each spherical shell has a uniform density, even though different shells may have different densities, the same result applies to the solid sphere. Hence, a body like the earth, the moon, or the sun, to the extent that they are such spheres, may be regarded gravitationally as point particles to bodies outside them.

Notice that our proof applies only to spheres and then only when the density is constant over the sphere or a function of radius alone.

An interesting result of some significance is the force exerted by a spherical shell on a particle *inside* it. This force is *zero*. To prove this we refer to Fig. 16–7, where m is shown inside the shell. Notice that R is now smaller than r. The limits of our integration over x are now $r - R$ to $R + r$. But

$$\int_{r-R}^{R+r}\left(\frac{R^2 - r^2}{x^2} + 1\right)dx = 0,$$

so that $F = 0$.

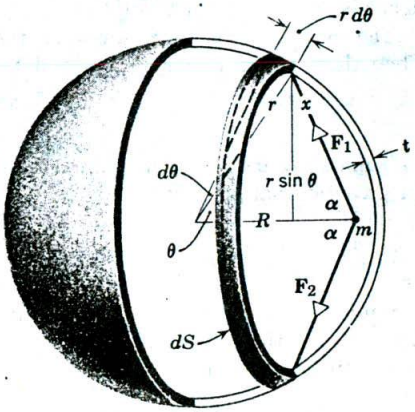


Fig. 16–7 Gravitational attraction of a section dS of a spherical shell of matter on a particle of mass m. Here the particle is inside the shell.

This last result, although not obvious, is plausible because the mass elements of the shell on opposite sides of m now exert forces of opposite directions on m inside. The total annulment depends on the fact that the force varies precisely as an inverse square of the separation distance of two particles. (See Problem 10.) Important consequences of this result will be discussed in the chapters on electricity. There we shall see that the electrical force between charged particles also depends inversely on the square of the distance between them. A consequence of interest in gravitation is that the gravitational force exerted by the earth on a particle decreases as the particle goes deeper into the earth, *assuming a constant density for the earth*, for the portions of matter in shells external to the position of the particle exert no force on it, the force becoming zero at the center of the earth. Hence, g would be a maximum at the earth's surface and decrease both outward and inward from that point *if the earth had constant density*. Can you imagine a spherically symmetric distribution of the earth's mass which would not give this result? (See Problem 35.)

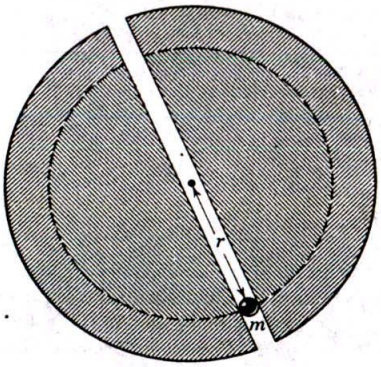


Fig. 16–8 Example 3. Particle moving in a tunnel through the earth.

▶ **Example 3.** Suppose a tunnel could be dug through the earth from one side to the other along a diameter, as shown in Fig. 16–8.

(*a*) Show that the motion of a particle dropped into the tunnel is simple harmonic motion. Neglect all frictional forces and assume that the earth has a uniform density.

The gravitational attraction of the earth for the particle at a distance r from the center of the earth arises entirely from that portion of matter of the earth in shells internal to the position of the particle. The external shells exert no force on the particle. Let us assume that the earth's density is uniform at the value ρ. Then the mass inside a sphere of radius r is

$$M' = \rho V' = \rho \frac{4\pi r^3}{3}.$$

This mass can be treated as though it were concentrated at the center of the earth for gravitational purposes. Hence, the force on the particle of mass m is

$$F = \frac{-GM'm}{r^2}.$$

The minus sign is used to indicate that the force is attractive and directed toward the center of the earth.

Substituting for M', we obtain

$$F = -G\frac{(\rho 4\pi r^3)m}{3r^2} = -\left(G\rho\frac{4\pi m}{3}\right)r = -kr.$$

Here $G\rho 4\pi m/3$ is a constant, which we have called k. The force is, therefore, proportional to the displacement r but oppositely directed. This is exactly the criterion for simple harmonic motion.

(*b*) If mail were delivered through this chute, how much time would elapse between deposit at one end and delivery at the other end?

The period of this simple harmonic motion is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3m}{G\rho 4\pi m}} = \sqrt{\frac{3\pi}{G\rho}}.$$

Let us take $\rho = 5.51 \times 10^3$ kg/meter^3 and $G = 6.67 \times 10^{-11}$ nt-m^2/kg^2. This gives

$$T = \sqrt{\frac{3\pi}{G\rho}} = \sqrt{\frac{3\pi}{(6.67 \times 10^{-11})(5.51 \times 10^3)}} \text{ sec} = 5050 \text{ sec} = 84.2 \text{ min}.$$

The time for delivery is one-half period, or about 42 min. Notice that this time is independent of the mass of the mail.

The earth does not really have a uniform density. Suppose ρ were some function of r, rather than a constant. What effect would this have on our problem? ◀

16–7 The Motions of Planets and Satellites

The motions of bodies in the solar system can be deduced from the laws of motion and the law of universal gravitation. As Kepler pointed out (see page 385), all planets move in elliptical orbits, the sun being at one focus. We can learn a lot about planetary motion by considering the special case of circular orbits. We shall neglect the forces between planets, considering only the interaction between the sun and a given planet. These considerations apply equally well to the motion of a satellite (natural or artificial) about a planet.

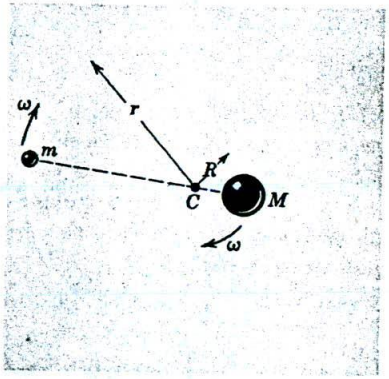


Fig. 16–9 Two bodies moving in circular orbits under the influence of each other's gravitational attraction. They both have the same angular velocity ω.

Consider two spherical bodies of masses M and m moving in circular orbits under the influence of each other's gravitational attraction. The center of mass of this system of two bodies lies along the line joining them at a point C such that $mr = MR$ (Fig. 16–9). If there are no external forces acting on this system, the center of mass has no acceleration. In this case we choose C to be the origin of our reference frame. The large body of mass M moves in an orbit of constant radius R and the small body of mass m in an orbit of constant radius r, both having the same angular velocity ω. In order for this to happen, the gravitational force acting on each body must provide the necessary centripetal acceleration. Since these gravitational forces are simply an action-reaction pair, the centripetal

forces must be equal but oppositely directed. That is, $m\omega^2 r$ (the magnitude of the centripetal force exerted by M on m) must equal $M\omega^2 R$ (the magnitude of the centripetal force exerted by m on M). That this is so follows at once, for $mr = MR$ so that $m\omega^2 r = M\omega^2 R$. The specific requirement, then, is that the gravitational force on either body must equal the centripetal force needed to keep it moving in its circular orbit, that is,

$$\frac{GMm}{(R+r)^2} = m\omega^2 r. \tag{16–10}$$

If one body has a much greater mass than the other, as in the case of the sun and a planet, its distance from the center of mass is much smaller than that of the other body. Let us therefore assume that R is negligible compared to r. Equation 16–10 then becomes

$$GM_s = \omega^2 r^3,$$

where M_s is the mass of the sun. If we express the angular velocity in terms of the period of the revolution, $\omega = 2\pi/T$, we obtain

$$GM_s = \frac{4\pi^2 r^3}{T^2}. \tag{16–11}$$

This is a basic equation of planetary motion; it holds also for elliptical orbits if we define r to be the semi-major axis of the ellipse. Let us consider some of its consequences.

One immediate consequence of Eq. 16–11 is that it predicts Kepler's third law of planetary motion in the special case of circular orbits. For we can express Eq. 16–11 as

$$T^2 = \frac{4\pi^2}{GM_s} r^3.$$

Notice that the mass of the planet is not involved in this expression. Here, $4\pi^2/GM_s$ is a constant, the same for all planets.

When the period T and radius r of revolution are known for any planet, Eq. 16–11 can be used to determine the mass of the sun. For example, the earth's period is

$$T = 365 \text{ days} = 3.15 \times 10^7 \text{ sec},$$

and its orbital radius is

$$r = 93 \text{ million miles} = 1.5 \times 10^{11} \text{ meters}.$$

Hence,

$$M_s = \frac{4\pi^2 r^3}{GT^2} = \frac{(4\pi^2)(1.5 \times 10^{11} \text{ meters})^3}{(6.67 \times 10^{-11} \text{ nt-m}^2/\text{kg}^2)(3.15 \times 10^7 \text{ sec})^2} \cong 2.0 \times 10^{30} \text{ kg}.$$

The mass of the sun is thus about 300,000 times the mass of the earth. The error made in neglecting R compared to r is seen to be trivial, for

$$R = \frac{m}{M} r = \frac{1}{300{,}000} r \cong 300 \text{ miles}; \qquad \frac{R}{r} 100\% \cong \frac{1}{3000} \text{ of } 1\%.$$

In a similar manner we can determine the mass of the earth from the period and radius of the moon's orbit about the earth. (See Problem 22.)

If we know the mass of the sun M_s and the period of revolution T of any planet about it, we can determine the radius of the planet's orbit r from Eq. 16–11. Since the period is easily obtained from astronomical observations, this method of determining a planet's distance from the sun is fairly reliable.

Equation 16–11 holds also for the motion of artificial satellites about the earth; we need only substitute the mass of the earth M_e for M_s in that equation.

Kepler's second law of planetary motion (see page 385) must, of course, hold for circular orbits. In such orbits both ω and r are constant so that equal areas are swept out in equal times by the line joining a planet and the sun. For the exact elliptical orbits, however, or for any orbit in general, both r and ω will vary. Let us consider this case.

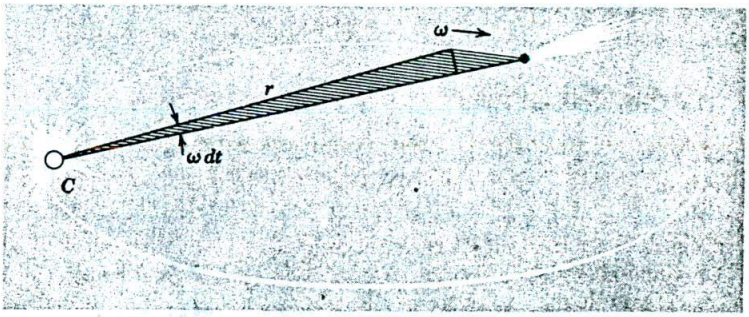


Fig. 16–10 A comet moving along an elliptical path with the sun C at the focus of the ellipse. In time dt the comet sweeps out an angle $d\theta = \omega dt$.

Figure 16–10 shows a particle revolving about C along some arbitrary path. The area swept out by the radius vector in a very short time interval Δt is shown shaded in the figure. This area, neglecting the small triangular region at the end, is one-half the base times the altitude or approximately $\frac{1}{2}(r\omega\,\Delta t) \cdot r$. This expression becomes more exact in the limit as $\Delta t \to 0$, the small triangle going to zero more rapidly than the large one. The rate at which area is being swept out instantaneously is therefore

$$\lim_{\Delta t \to 0} \frac{\frac{1}{2}(r\omega\,\Delta t)(r)}{\Delta t} = \tfrac{1}{2}\omega r^2$$

But $m\omega r^2$ is simply the angular momentum of the particle about C. Hence, Kepler's second law, which requires that the rate of sweeping out of area $\frac{1}{2}\omega r^2$ be constant, is entirely equivalent to the statement that *the angular momentum of any planet about the sun remains constant.* The angular momentum of the particle about C cannot be changed by a force on it directed toward C. Kepler's second law would, therefore, be valid for any *central force*, that is, any force directed toward the sun. The exact nature of this force—how it depends on distance of separation or other properties of the bodies—is not revealed by this law.

It is Kepler's first law that requires the gravitational force to depend exactly on the inverse square of the distance between two bodies, that is, on $1/r^2$. Only such a force, it turns out, can yield planetary orbits which are elliptical with the sun at one focus.

Newton's laws of motion and his law of universal gravitation are in almost complete agreement with astronomical observations.* In our calculation we considered the motion of a planet about the sun as a "two-body" problem. However, we saw that the motion of the sun could be neglected while retaining a high degree of accuracy because of the large ratio of solar mass to planetary mass. This reduced the problem to that of motion of one body about a center of force. If we had required greater accuracy, we would have had to include the sun's motion in our problem (see Problem 39). In fact, for an exact treatment we would have to take into account the effect of other planets and satellites on the motion of sun and planet. This "many-body" problem is quite formidable, but it can be solved by approximation methods to a high degree of accuracy. The results of such calculations are in excellent agreement with astronomical observations.

16–8 The Gravitational Field

A basic fact of gravitation is that two masses exert forces on one another. We can think of this as a direct interaction between the two mass particles, if we wish. This point of view is called *action-at-a-distance*, the particles interacting even though they are not in contact. Another point of view is the *field* concept which regards a mass particle as modifying the space around it in some way and setting up a *gravitational field.* This field then acts on any other mass particle in it, exerting the force of gravitational attraction on it. The field, therefore, plays an intermediate role in our thinking about the forces between mass particles. According to this view we have two separate parts to our problem. First, we must determine the field established by a given distribution of mass particles; and secondly, we must calculate the force that this field exerts on another mass particle placed in it.

For example, consider the earth as an isolated mass. If a body is now brought in the vicinity of the earth, a force is exerted on it. This force has a definite direction and magnitude at each point in space. The direction is

* The major axis of the elliptical orbit of Mercury rotates slightly more than that predicted by Newtonian mechanics when the perturbing influence of other planets is included. This effect is accounted for in the general theory of relativity.

radially in toward the center of the earth and the magnitude is mg. We can, therefore, associate with each point near the earth a vector $\mathbf{g}$ which is the acceleration that a body would experience if it were released at this point. We call $\mathbf{g}$ the *gravitational field strength* at the point in question. Since

$$\mathbf{g} = \frac{\mathbf{F}}{m}, \tag{16–12}$$

we may define gravitational field strength at any point as the gravitational force per unit mass at that point. We calculate the force from the field simply by multiplying $\mathbf{g}$ by the mass m of the particle placed at any point.

The gravitational field is an example of a *vector field*, each point in this field having a vector associated with it. There are also scalar fields, such as the temperature field in a heat-conducting solid. The gravitational field arising from a fixed distribution of matter is also an example of a *stationary field*, because the value of the field at a given point does not change with time.

The field concept is particularly useful for understanding electromagnetic forces between moving electric charges. It has distinct advantages, both conceptually and in practice, over the action-at-a-distance concept. The field concept was not used in Newton's day. It was developed much later by Faraday for electromagnetism and only then applied to gravitation. Subsequently, this point of view was adopted for gravitation in the general theory of relativity. The chief purpose of introducing it here is to give the student an early familiarity with a concept that proves to be important in the development of physical theory.

▶ **Example 4.** In Chapter 15 we derived the formula for the period of a simple pendulum, $T = 2\pi\sqrt{l/g}$. Keeping in mind that the earth's gravitational field is not uniform over large distances, as was assumed for small distances, what is the longest period a simple pendulum could have in the vicinity of the earth's surface?

The formula $T = 2\pi\sqrt{l/g}$, although not applicable when g varies over the pendulum's path, suggests that we increase the length of the pendulum. Let us make the length infinite. The pendulum bob would then travel along the arc of a circle of infinite radius, that is, along a straight line, as shown in Fig. 16–11. The direction of the earth's gravitational field is everywhere radially in toward the center of the earth, so that its direction changes along the arc. Let us assume that the bob of mass m has an amplitude that is small compared to the radius of the earth. Then the bob is always a distance R_e, the earth's radius, from the center of the earth, to a good approximation. Then the force F on m is

$$F = \frac{GM_e m}{R_e^2} = mg,$$

where M_e is the mass of the earth. This force is directed toward the earth's center as shown. The component of this vector force along x, the line of motion of the bob, is

$$F_x = F\cos\theta = -F\frac{x}{R_e} = -\frac{GM_e m}{R_e^3}x,$$

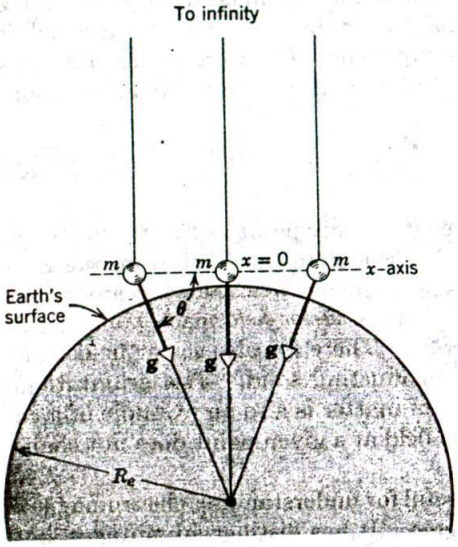


Fig. 16–11 Example 4. A simple pendulum suspended at infinity.

where the minus sign indicates that the force is directed opposite to the displacement from $x = 0$. We can write this as

$$F_x = -kx,$$

where $k = GM_e m/R_e^3$, a constant.

The formula for the period of a simple harmonic oscillator is $T = 2\pi\sqrt{m/k}$. Hence,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{GM_e m/R_e^3}} = 2\pi\sqrt{\frac{R_e}{GM_e/R_e^2}} = 2\pi\sqrt{\frac{R_e}{g}},$$

because g at the earth's surface equals GM_e/R_e^2. Putting in $R_e = 6.37 \times 10^6$ meters and $g = 9.80$ meters/sec^2, we obtain $T = 84.3$ min as the longest period of a simple pendulum in the vicinity of the earth's surface. (See Problem 14.) ◀

16–9 Gravitational Potential Energy

In Chapter 8 we discussed the gravitational potential energy of a particle (mass m) and the earth (mass M). We considered only the special case in which the particle remains close to the earth so that we could assume the gravitational force acting on the particle to be constant for all positions of the particle. In this section we remove that restriction and consider particle-earth separations that may be appreciably greater than the earth's radius.

Equation 8-5*b*, which we may write as,

$$\Delta U = U_b - U_a = -W_{ab}, \qquad (8\text{–}5b)$$

defines the change ΔU in the potential energy of any system, in which a conservative force (gravity, say) acts, as the system changes from con-

figuration a to configuration b. W_{ab} is the work done by that conservative force as the system changes.

The potential energy of the system in any arbitrary configuration b is (see Eq. 8–5b)

$$U_b = -W_{ab} + U_a. \tag{16–13}$$

To give a value to U_b we must (arbitrarily) choose configuration a to be some agreed-upon reference configuration and we must assign to U_a some (arbitrarily) agreed-upon value, usually zero.

In Chapter 8 we chose as a reference configuration for the earth-particle system that in which the particle is resting on the surface of the earth and we assigned to this configuration the potential energy $U_a = 0$. When the particle is at a height y above the surface of the earth, the potential energy U $(= U_b)$ is given from Eq. 16–13 as

$$U = -W_{ab} + 0 = -(-mg)(y) = mgy.$$

The conservative force in question, gravity, points down and has the value $(-mg)$; the displacement of the particle $(+y)$ points up from the reference level; hence the difference in sign for these quantities.

For the more general case, in which the restriction $y \ll R$ (in which R is the radius of the earth) is not imposed, we find it convenient to select a different reference configuration, namely that in which the particle and the earth are infinitely far apart. We assign the value zero to the potential energy of the system in this configuration. Thus the zero-potential-energy configuration is also the zero-force configuration. We made a similar choice when we defined the zero-energy configuration of a spring to be its normal unstressed state, for which the restoring force is zero.

When the particle of mass m is a distance r from the center of the earth, the system potential energy is given by Eq. 16–13 as

$$U(r) = -W_{\infty r} + 0 \tag{16–14}$$

in which $W_{\infty r}$ is the work done by the conservative force (gravity) on the particle as the particle moves in from infinity to a distance r from the center of the earth. For simplicity we assume for the present that the particle moves toward the earth along a radial line. The gravitational force $F(r)$ acting on the particle (assuming $r \geqslant R$) will then be $-GMm/r^2$, the minus sign indicating an attractive force, that is, a force that pulls the particle toward the earth. We may then find $U(r)$ from Eq. 16–14 as

$$\begin{aligned} U(r) &= -W_{\infty r} \\ &= -\int_{\infty}^{r} F(r)\,dr \\ &= -\int_{\infty}^{r} \left(-\frac{GMm}{r^2}\right) dr = -\frac{GMm}{r}\Bigg|_{\infty}^{r} \\ &= -\frac{GMm}{r}. \end{aligned} \tag{16–15}$$

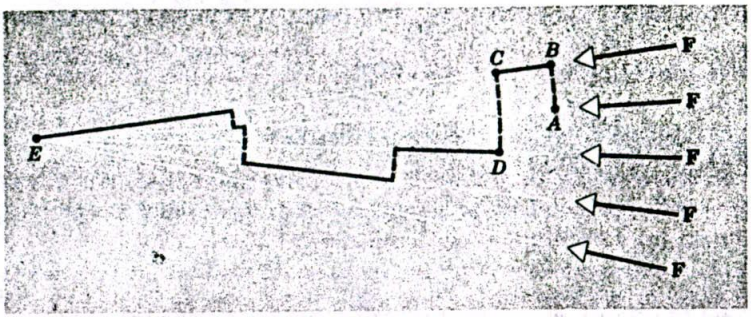


Fig. 16–12 Work done in taking a mass from A to E is independent of the path.

The minus sign indicates that the potential energy is negative at any finite distance; that is, the potential energy is zero at infinity and decreases as the separation distance decreases. This corresponds to the fact that the gravitational force exerted on the particle by the earth is attractive. As the particle moves in from infinity, the work $W_{\infty r}$ done by this force on the particle is positive, which means, from Eq. 16–14, that $U(r)$ is negative.

Equation 16–15 holds no matter what path is followed by the particle in moving in from infinity to radius r. We can show this by breaking up any arbitrary path into infinitesimal steplike portions, which are drawn alternately along the radius and perpendicular to it (Fig. 16–12). No work is done along perpendicular segments, like AB, because along them the force is perpendicular to the displacement. But the work done along the radial parts of the path, such as BC, adds up to the work done in going directly along a radial path, such as AE. The work done in moving the particle between two points in a gravitational field is, therefore, independent of the actual path connecting these points. Hence, the gravitational force is a *conservative* force.

Equation 16–15 shows that the potential energy of the particles M and m is a characteristic of the system $M + m$. The potential energy is a property of the *system* of bodies, rather than of either body alone. The potential energy changes whether M or m is displaced; each is in the gravitational field of the other. Nor does it make any sense to assign part of the potential energy to M and part of it to m. Often, however, we *do* speak of the potential energy of a body m (planet or stone, say) in the gravitational field of a much more massive body M (sun or earth, respectively). The justification for speaking as though the potential energy belongs to the planet or to the stone alone is this: When the potential energy of a system of two bodies changes into kinetic energy, the lighter body gets most of the kinetic energy. The sun is so much more massive than a planet that the sun receives hardly any of the kinetic energy; and the same is true for the earth in the earth-stone system.

We can derive the gravitational force from the potential energy. The

relation for spherically symmetric potential energy functions is $F = -dU/dr$; see Eq. 8–7. This relation is the converse of Eq. 16–15. From it we obtain

$$F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) = -\frac{GMm}{r^2}. \qquad (16\text{–}16)$$

The minus sign here shows that the force is an attractive one, directed inward along a radius opposite to the radial displacement vector.

We can, if we wish, associate a scalar field with gravitation. We first define the *gravitational potential* V as the *gravitational potential energy per unit mass* of a body in a gravitational field. Then,

$$V = \frac{U(r)}{m} = -\frac{GM}{r} \qquad (16\text{–}17)$$

Associated with every point in the space around a mass M we then have a number, the gravitational potential. This gives us a *scalar field*, potential being a scalar quantity. To determine the force exerted by this field on a mass particle m placed in it, we simply compute $-dV/dr$ at the point in question and multiply by m. The force has a magnitude $-m\,dV/dr$ and a direction radially in toward the center of force M.

▶ **Example 5.** *Escape velocity.* We can readily find the gravitational potential energy of a particle of mass m at the surface of the earth as (Eq. 16–15) $U(R) = -GM_em/R_e$. The amount of work required to move a body from the surface of the earth to infinity is GM_em/R_e, or about 6.0×10^7 joules/kg. If we could give a projectile more than this energy at the surface of the earth, then, neglecting the resistance of the earth's atmosphere, it would escape from the earth never to return. As it proceeds outward its kinetic energy decreases and its potential energy increases, but its speed is never reduced to zero. The critical initial speed, called the escape speed v_0, such that the projectile does not return, is given by

$$\tfrac{1}{2}mv_0^2 = \frac{GM_em}{R_e}$$

or

$$v_0 = \sqrt{2\frac{GM_e}{R_e}} = 7.0 \text{ miles/sec (25,000 miles/hr)} = 11.2 \text{ km/sec}$$

Should a projectile be given this initial speed, it would escape from the earth. For initial speeds less than this the projectile will return. Its kinetic energy becomes zero at some finite distance from the earth and the projectile falls back to earth.* ◀

The lighter molecules in the earth's upper atmosphere can attain high enough speeds by thermal agitation to escape into outer space. Hydrogen gas, which must have been present in the earth's atmosphere a long time ago, has now disappeared from it. Helium gas escapes at a steady rate, much of it resupplied by radioactive decay from the earth's crust. The escape speed for the sun is much too great to

* We have ignored the forces exerted on the projectile by bodies other than the earth. At sufficiently great distances from the earth we must take into account the gravitational forces arising from the moon, the planets, the sun etc., so that we can no longer use the simple "two-body" result. A projectile can escape from the earth by being "captured" by another astronomical body, for example, in such "many-body" cases.

allow hydrogen to escape from its atmosphere. On the other hand, the speed of escape on the moon is so small that it can hardly keep any atmosphere at all. (See Problem 23).

16–10 Potential Energy for Many-Particle Systems

If two particles are separated by a distance r, their potential energy is given from Eq. 16–14 as

$$U(r) = -W_{\infty r} \qquad (16\text{–}14)$$

in which $W_{\infty r}$ is the work done *by the gravitational force* as the particles move from an infinite separation to separation r. We now give another interpretation to $U(r)$.

Let us balance out the gravitational force by an *external force* applied by some external agent and let us arrange it so that, at all times, this external force is equal and opposite to the gravitational force for each particle. The work done by the *external* force as the particles move from an infinite separation to separation r is not $W_{\infty r}$ but $-W_{\infty r}$; this follows because the displacements are the same but the forces are equal and opposite. Thus we may interpret Eq. 16–14 as follows: *The potential energy of a system of particles is equal to the work that must be done by an external agent to assemble the system, starting from the standard reference configuration.*

Thus, if you lift a stone of mass m a distance of y above the earth's surface, you are the external agent (separating earth and stone) and the work you do in "assembling the system" is $+mgy$, which is also the potential energy. Similarly, the work done *by an external agent* as a body of mass m moves in from infinity to a distance r from the earth is *negative* because the agent must exert a restraining force on the body; this is in agreement with Eq. 16–14.

These considerations hold for systems that contain more than two particles. Consider three bodies of masses m_1, m_2, and m_3. Let them initially be infinitely far from one another. The problem is to compute the work done by an external agent to bring them into the positions shown in Fig. 16–13. Let us first bring m_2 in toward m_1 from an infinite separation to the separation $\mathbf{r}_{12}$. The work done against the gravitational force exerted by m_1 on m_2 is $-Gm_1m_2/r_{12}$. Now let us bring m_3 in from infinity to the separation $\mathbf{r}_{13}$ from m and $\mathbf{r}_{23}$ from m_2. The work done against the gravitational force exerted by m_1 on m_3 is $-Gm_1m_3/r_{13}$ and that against the gravitational force exerted by m_2 on m_3 is $-Gm_2m_3/r_{23}$. The total work done in assembling this system is the total potential energy of the system

$$= \left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right)$$

Notice that no vector operations are needed in this procedure.

No matter how we assemble the system, that is, regardless of which bodies are moved or which paths are taken, we always find this same

amount of work required to being the bodies into the configuration of Fig. 16–13 from an initial infinite separation. The potential energy must, therefore, be associated with the system rather than with any one or two bodies. If we wanted to separate the system into three isolated masses once again, we would have to supply an amount of energy

$$+\left(\frac{Gm_1m_2}{r_{12}}+\frac{Gm_1m_3}{r_{13}}+\frac{Gm_2m_3}{r_{23}}\right)$$

This energy may be regarded as a sort of binding energy holding the mass particles together in the configuration shown.

Just as we can associate elastic potential energy with the compressed or stretched configuration of a spring holding a mass particle, so we can associate gravitational potential energy with the configuration of a system of mass particles held together by gravitational forces. Similarly, if we want to think of the elastic potential energy of a particle as being stored in the spring, so we can think of the gravitational potential energy as being stored in the gravitational field of the system of particles. A change in either configuration results in a change of potential energy.

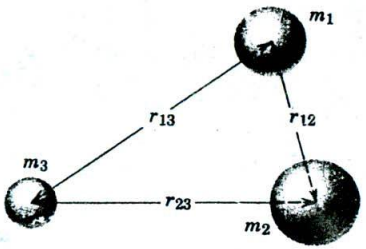


Fig. 16–13 Three masses m_1, m_2, and m_3 brought together from infinity.

These concepts occur again when we meet forces of electric or magnetic origin, or, in fact, of nuclear origin. Their application is rather broad in physics. The advantage of the energy method over the dynamical method is derived from the fact that the energy method uses scalar quantities and scalar operations rather than vector quantities and vector operations. When the actual forces are not known, as is often the case in nuclear physics, the energy method is essential.

▶ **Example 6.** What is the binding energy of the earth-sun system?. Neglect the presence of other planets or satellites.

For simplicity assume that the earth's orbit about the sun is circular at a radius r_{es}. The work done against the gravitational force to bring the earth and sun from an infinite separation to a separation r_{es} is

$$-G\frac{M_sM_e}{r_{es}} \cong -5.0 \times 10^{33} \text{ joules},$$

where we take $M_s \cong 330{,}000M_e$, $M_e = 6.0 \times 10^{24}$ kg, $r_{es} = 150 \times 10^9$ meters. The minus sign indicates that the force is attractive, so that work is done by the gravitational force. It would take an equivalent amount of work by an outside agent to separate these bodies completely from rest. Because the kinetic energy of the earth in its orbit is half the magnitude of the potential energy of the earth-

sun system, only half of this work is needed to break up the system, so that the effective binding energy, assuming that the earth-sun-system is at rest after breakup, is about 2.5×10^{33} joules.

What effect does the presence of the moon and other planets have on the energy binding the earth to the solar system? ◀

16–11 Energy Considerations in the Motions of Planets and Satellites

Consider again the motion of a body of mass m (planet or satellite, say) about a massive body of mass M (sun or earth, say). We shall consider M to be at rest in an inertial reference frame with the body m moving about it in a circular orbit. The potential energy of the system is

$$U(r) = -\frac{GMm}{r},$$

where r is the radius of the circular orbit. The kinetic energy of the system is

$$K = \tfrac{1}{2}m\omega^2 r^2$$

the sun being at rest. From the equation preceding Eq. 16–11 we obtain

$$\omega^2 r^2 = \frac{GM}{r},$$

so that

$$K = \frac{1}{2}\frac{GMm}{r}.$$

The total energy is

$$E = K + U = \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r} = -\frac{GMm}{2r}. \qquad (16\text{–}18)$$

This energy is constant and is negative. Now the kinetic energy can never be negative, but from Eq. 16–18 we see that it must go to zero as the separation goes to infinity. The potential energy is always negative, except for its zero value at infinite separation. The meaning of the total negative energy, then, is that the system is a closed one, the planet m always being bound to the attracting solar center M and never escaping from it (Fig. 16–14).

Even when we consider elliptical orbits, in which r and ω vary, the total energy is negative. It is also constant, corresponding to the fact that gravitational forces are conservative. Hence, both the total energy and the total angular momentum are constant in planetary motion. These quantities are often called *constants of the motion*. We obtain the actual orbit of a planet with respect to the sun by starting with these conservation relations and eliminating the time variable by use of the laws of dynamics and gravitation. The result is that planetary orbits are elliptical.

In the earlier theories of the atom, as in the Bohr theory of the hydrogen

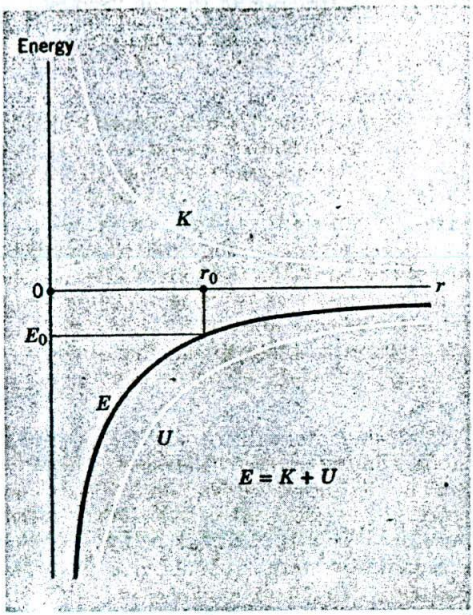


Fig. 16–14 Kinetic energy K, potential energy U, and total energy $E = U + K$ of a body in circular planetary motion. A planet with total energy $E_0 < 0$ will remain in an orbit of radius r_0. The farther the planet is from the sun, the greater (that is, less negative) its (constant) total energy E. To escape from the center of force and still have kinetic energy at infinity, it would need positive total energy.

atom, these identical mechanical relations are used in describing the motion of an electron about an attracting nuclear center. These same relations are used for open orbits (total energy positive) as in the experiments of Rutherford on the scattering of charged nuclear particles. Central forces, and particularly inverse square forces, are encountered often in physical systems.

16–12 The Earth as an Inertial Reference Frame

In describing the experiments which were fundamental to our definitions of force and mass, we had to assume some reference frame relative to which accelerations could be measured. If the reference frame itself were erratically accelerated, we would not observe any regularity in our measured accelerations. As a matter of fact, our laboratory experiments are performed in a reference frame which is fixed to the earth. We have already discussed the effect that the rotation of the earth about its own axis has on our measurements. What effect does the motion of the earth as a whole about the sun, or some other cosmic body, have?

The acceleration of the earth with respect to the sun is $\omega^2 r$ or about 6×10^{-3} meter/sec^2. It would seem at first that this acceleration, small as it is, might prove disturbing in experiments involving small forces. That this is not the case, however, follows from the universality of the law of gravitation. Not only the earth but also the masses we use in our laboratory apparatus are accelerated toward the sun at practically the same rate.

Let us compute the error made in neglecting the earth's orbital acceleration. The acceleration of the earth toward the sun is k/r^2 where r is the distance from the center of the sun and the center of the earth and k is GM_s. Consider now a body on that side of the earth most distant from the sun. We can imagine that we are weighing it on a spring scale, for example. Then, the part of its acceleration

toward the sun which is due to the gravitational attraction of the sun itself is

$$\frac{k}{(r + r_0)^2} = \frac{k}{r^2}\left(1 - \frac{2r_0}{r} + \cdots + \text{much smaller terms}\right),$$

where r_0 is the radius of the earth. The *difference* between the acceleration of the earth due to the sun's attraction (that is, k/r^2) and the acceleration of the apparatus due to the sun's attraction (the expression above) would be less than $(k/r^2)(2r_0/r)$. But $2r_0/r$ is about 10^{-4}. The difference, then, would be less than 10^{-4} of the earth's acceleration, or less than 10^{-6} meter/sec^2. The relative acceleration of the body and the earth due to the sun's attraction is about one-ten-millionth as strong as the gravitational acceleration of the body due to the earth. The moon has a similar effect of comparable magnitude on the body. Hence, only if we were measuring to one part in a million would we need to consider seriously the accelerating nature of a reference frame attached to the earth. For almost all practical purposes the earth is good enough as an inertial reference frame.

16–13 The Principle of Equivalence

Consider two reference frames: (1) a nonaccelerating (inertial) reference frame S in which there is a uniform gravitational field and (2) a reference frame S' which is accelerating uniformly with respect to an inertial frame but in which there is no gravitational field. In his general theory of relativity, Albert Einstein showed that two such frames are exactly equivalent physically. That is, experiments carried out under the same conditions in these two frames should give the same results. This is the *principle of equivalence.*

Suppose that a space ship is at rest in an inertial reference frame S in which there is a uniform gravitational field, say at the surface of the earth. Inside the spaceship objects, such as an apple, which are released will fall with an acceleration, say $\mathbf{g}$, in the gravitational field; objects which are at rest—such as an astronaut sitting on the floor or a package on a spring scale attached to the ceiling—will experience a force, exerted by the floor or the spring respectively, opposing their weight.

Now suppose that the rockets are turned on and that the spaceship proceeds to a region of outer space where there is no gravitational field. Let the acceleration of the spaceship, our new frame S', be $\mathbf{a} = -\mathbf{g}$ with respect to the inertial reference frame S; that is, the ship is accelerating away from the earth beyond the region where the earth's field (or any other gravitational field) is appreciable. The conditions in the spaceship will now be similar to those in a spaceship at rest on the surface of the earth. Inside the ship, if the astronaut releases an apple, it will accelerate downward relative to the spaceship with an acceleration $\mathbf{g}$. In fact, since all bodies that are free of any forces move with uniform velocity relative to the inertial frame S, all such bodies appear to fall with the *same* acceleration $\mathbf{g}$ with respect to the spaceship, S'. Furthermore, objects which are at rest relative to the spaceship—such as an astronaut sitting on the floor or a package on a spring scale attached to the ceiling—will experience forces indistinguishable from the forces which balanced their weight in the case when the spaceship was at rest in a gravitational field in S.

Indeed, if the astronaut did not know that rockets were accelerating his ship from S, he would be justified in concluding that he was in a gravitational field—a field whose pull accelerated the falling apple in S' and whose pull required that a balancing force be applied to the package (the tension in the spring) and to the spaceman (the normal force of the floor) to keep them at rest in S'. The astronaut simply could not tell the difference, from observations in his own frame, between a situation in which his ship was accelerating relative to an inertial frame in a region having no gravitational field and a situation in which the spaceship was unacceler-

ated in an inertial frame in which a uniform gravitational field existed. The two situations are exactly equivalent.

Einstein pointed out that, from the principle of equivalence, it follows that one cannot speak of the absolute acceleration of a reference frame, only a relative one, just as it followed from the special theory of relativity that one cannot speak of the absolute velocity of a reference frame, only a relative one. It also follows from the principle of equivalence that inertial mass and gravitational mass are equal. For all bodies which are free of any forces will move with uniform velocity relative to an inertial reference frame no matter what their inertial masses are, and they should, therefore, all have the same acceleration relative to an accelerated reference frame. Hence, from the principle of equivalence of S and S', all bodies should fall with the same acceleration in a homogeneous gravitational field.

From the discussion so far we see that a uniform gravitational field can be imitated by a "field of acceleration." Indeed, a uniform gravitational field can be "transformed away" by transforming to a reference frame accelerating in the direction of the field with an acceleration equal in magnitude to that due to the field. In this new frame a particle whose motion was originally subject to a gravitational field is now a free particle. For example, in an artificial earth satellite an apple released by an astronaut will not fall relative to the satellite and the astronaut himself will be free of the forces which countered the pull of gravity before launching, so that he feels weightless. In general, however, gravitational fields, such as that of the earth, are not uniform through all space, so that one cannot replace the gravitational fields throughout space simply by transforming to a single reference frame accelerating with respect to the source of the field. One would need a different accelerated frame at each point in space to imitate the entire gravitational field.

QUESTIONS

1. If the force of gravity acts on all bodies in proportion to their masses, why doesn't a heavy body fall faster than a light body?

2. Modern observational astronomy and navigation procedures make use of the geocentric (or Ptolemaic) point of view (by using the rotating "celestial sphere"). Is this wrong? If not, then what criterion determines the system (the Copernican or Ptolemaic) we use? When would we use the heliocentric (or Copernican) system?

3. How does the weight of a body vary en route from the earth to the moon? Would its mass change?

4. Would we have more sugar to the pound at the pole or the equator?

5. Does the concentration of the earth's mass near its center change the variation of g with height compared with a homogeneous sphere? How?

6. Because the earth bulges near the equator, the source of the Mississippi River, although high above sea level, is nearer to the center of the earth than is its mouth. How can the river flow "uphill"?

7. The earth is an oblate spheroid because of the "flattening" effect of the earth's rotation. Is a degree of latitude larger or smaller near either pole than near the equator? Why?

8. Why can we learn more about the shape of the earth by studying the motion of an artificial satellite than by studying the motion of the moon?

9. How can one determine the mass of the moon?

10. One clock is based on an oscillating spring, the other on a pendulum. Both are taken to Mars. Will they keep the same time there that they kept on Earth? Will they agree with each other? Explain. Mars has a mass 0.1 that of Earth and a radius half as great.

11. From Kepler's second law and observations of the sun's motion as seen from the earth, we can conclude that the earth is closer to the sun during winter in the Northern hemisphere than during summer. Explain.

12. Does the law of universal gravitation require the planets of our solar system to have the actual orbits observed? Would planets of another star, similar to our Sun, have the same orbits? Suggest factors that might have determined the special orbits observed.

13. The gravitational force exerted by the sun on the Moon is greater than (about twice as great as) the gravitational force exerted by the earth on the moon. Why then doesn't the moon escape from the earth (during a solar eclipse, for example)?

14. Explain why the following reasoning is wrong. "The sun attracts all bodies on the earth. At midnight, when the sun is directly below, it pulls on an object in the same direction as the pull on the earth on that object; at noon, when the sun is directly above, it pulls on an object in a direction opposite to the pull of the earth. Hence, all objects should be heavier at midnight (or night) than they are at noon (or day)."

15. The gravitational attraction of the sun and the moon on the earth produces tides. The sun's tidal effect is about half as great as that of the moon's. The direct pull of the sun on the earth, however, is about 175 times that of the moon. Why is it then that the moon causes the larger tides?

16. If lunar tides slow down the rotation of the earth (owing to friction), the angular momentum of the earth decreases. What happens to the motion of the moon as a consequence of the conservation of angular momentum? Does the sun (and solar tides) play a role here?

17. Would you expect the total energy of the solar system to be constant? The total angular momentum? Explain your answers.

18. Does a rocket really need the escape speed of 25,000 miles/hr initially to escape from the earth?

19. Objects at rest on the earth's surface move in circular paths with a period of 24 hr. Are they "in orbit" in the sense that an earth satellite is in orbit? Why not? What would the length of the "day" have to be to put such objects in true orbit?

20. Neglecting air friction and technical difficulties, can a satellite be put into an orbit by being fired from a huge cannon at the earth's surface? Explain.

21. Can a satellite move in a stable orbit in a plane not passing through the earth's center? Explain.

22. As measured by an observer on earth would there be any difference in the periods of two satellites, each in a circular orbit near the earth in an equatorial plane, but one moving eastward and the other westward?

23. After Sputnik I was put into orbit we were told that it would not return to earth but would burn up in its *descent.* Considering the fact that it did not burn up in its *ascent,* how is this possible?

24. Show that a satellite may speed down: that is, show that if frictional forces cause a satellite to lose total energy, it will move into an orbit closer to the earth and may have increased kinetic energy.

25. An artificial satellite is in a circular orbit about the earth. How will its orbit change if one of its rockets is momentarily fired (*a*) toward the earth, (*b*) away from the earth, (*c*) in a forward direction, (*d*) in a backward direction, (*e*) at right angles to the plane of the orbit?

26. Inside a space ship what difficulties would you encounter in walking? In jumping? In drinking?

27. If a planet of given density were made larger, its force of attraction for an object on its surface would increase because of the planet's greater mass but would decrease because of the greater distance from the object to the center of the planet. Which effect predominates?

28. Consider a hollow spherical shell. How does the gravitational potential inside compare with that on the surface? What is the gravitational field strength inside?

29. A stone is dropped along the center of a deep vertical mine shaft. Assume no air resistance but consider the earth's rotation. Will the stone continue along the center of the shaft? If not, describe its motion.

30. Can one regard gravity as a "fictitious" force arising from the acceleration of one's reference frame relative to an inertial reference frame, rather than a "real" force?

PROBLEMS

1. At what altitude above the earth's surface would the acceleration of gravity be about 16 ft/sec^2?

2. The distinction between mass and weight drew attention when Jean Richer in 1672 took a pendulum clock from Paris to Cayenne, French Guiana, for use in astronomical observations and found that it lost 2.5 min each day. It was already known from Huygens' work that the period of a pendulum of given length was proportional to $1/\sqrt{g}$. If $g = 980.9$ cm/sec^2 in Paris, what is g in Cayenne?

3. What is the period of a "seconds pendulum" (period = 2 sec on earth) on the surface of the moon?

4. How far from the earth must a body be along a line toward the sun so that the sun's gravitational pull balances the earth's? The sun is 9.3×10^7 miles away and its mass is $3.24 \times 10^5 M_e$.

5. A body is suspended on a spring balance in a ship sailing along the equator with a speed v. Show that the scale reading will be very close to $W_0(1 \pm 2\omega v/g)$, where ω is the angular speed of the earth and W_0 is the scale reading when the ship is at rest. Explain the plus or minus.

6. A scientist is making a precise measurement of g at a certain point in the Indian Ocean (on the equator) by timing the swings of a pendulum of accurately known construction. To provide a stable base the measurements are conducted in a submerged submarine. It is observed that slightly different result for g is obtained when the submarine is moving eastward through the point than when it is moving westward, the speed in each case being 10 miles/hr. Account for this difference and calculate the effect, in parts per million, that it has on the value of g.

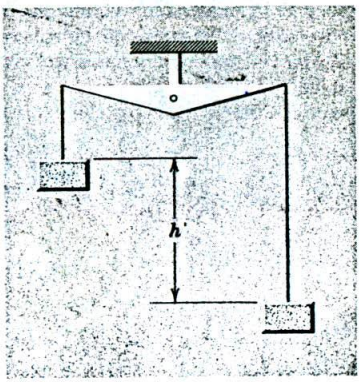


Fig. 16–15

7 Masses, assumed equal, hang from strings of different lengths on a balance at the surface of the earth, as shown in Fig. 16–15. If the strings have negligible mass and differ in length by h, (*a*) show that the error in weighing, $W' - W$, is given by $W' - W = 8\pi G\rho mh/3$ and (*b*) find the difference in length which will give an error of one part in a million. Take mass of each body to be m and the density of the earth, ρ, to be 5.5 gm/cm^3.

8. With what speed would mail pass through the center of the earth if it were delivered by the chute of Example 3?

9. (*a*) Show that in a chute dug through the earth along any chord line, rather than along a diameter, the motion of an object will be simple harmonic. (*b*) Find the period

of the motion. (*c*) Will the object attain the same speed along the chord line as it does along a diameter?

10. Consider a mass particle at a point P anywhere inside a spherical shell of matter. Assume the shell is of uniform thickness and density. Construct a narrow double cone with apex at P intercepting areas A_1 and A_2 on the shell (Fig. 16–16). (*a*) Show that the resultant gravitational force exerted on the particle at P by the intercepted mass elements is zero. (*b*) Show then that the resultant gravitational force of the entire shell on an internal particle is zero everywhere.

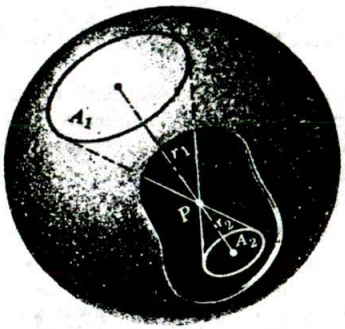


Fig. 16–16

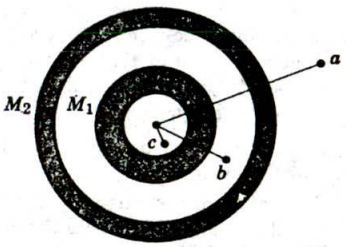


Fig. 16–17

11. Two concentric spherical shells of uniform density of mass M_1 and M_2 are situated as shown in Fig. 16–17. Find the force on a particle of mass m when (*a*) the particle is located at $r = a$, (*b*) the particle is located at $r = b$, and (*c*) the particle is located at $r = c$. The distance r is measured from the center of the shells.

12. The following problem is from the 1946 "Olympic" examination of Moscow State University (see Fig. 16–18): A spherical hollow is made in a lead sphere of radius R, such that its surface touches the outside surface of the lead sphere and passes through its center. The mass of the sphere before hollowing was M. With what force, according to the law of universal gravitation, will the hollowed lead sphere attract a small sphere of mass m, which lies at a distance d from the center of the lead sphere on the straight line connecting the centers of the spheres and of the hollow?

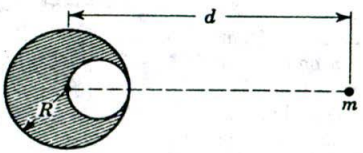


Fig. 16–18

13. (*a*) With what horizontal speed must a satellite be projected at 100 miles above the surface of the earth so that it will have a circular orbit about the earth? Take the earth's radius as 4000 miles. (*b*) What will be the period of rotation?

14. Use qualitative arguments to explain why the following four periods are equal (all are 84 min): (*a*) time of revolution of a satellite just above the earth's surface; (*b*) period of oscillation of mail in a tunnel through the earth; (*c*) period of simple

pendulum having a length equal to the earth's radius in a uniform field 9.8 nt/kg; (*d*) period of an infinite simple pendulum in the earth's real gravitational field.

15. A projectile is fired vertically from the earth's surface with an initial speed of 10 km/sec. Neglecting atmospheric retardation, how far above the surface of the earth would it go? Take the earth's radius as 6400 km.

16. Does it take more energy to get a satellite up to 1000 miles above the earth than to put it in orbit there? 2000 miles? 3000 miles? Take the earth's radius to be 4000 miles.

17. Two earth satellites, A and B, each of mass m, are to be launched into (nearly) circular orbits about the earth's center. Satellite A is to orbit at an altitude of 4000 miles. Satellite B is to orbit at an altitude of 12,000 miles. The radius of the earth R_e is 4000 miles (Fig. 16–19). (*a*) What is the ratio of the potential energy of satellite B to that of satellite A, in orbit? (Explain the result in terms of the work required to get each satellite from its orbit to infinity.) (*b*) What is the ratio of the kinetic energy of satellite B to that of satellite A, in orbit? (*c*) Which satellite has the greater total energy if each has a mass of 1.0 slug? By how much?

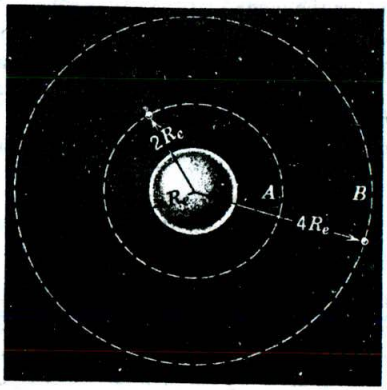


Fig. 16–19

18. (*a*) Can a satellite be sent out to a distance where it will revolve about the earth with an angular velocity equal to that at which the earth rotates, so that it remains always above the same point on the earth? (*b*) Must the plane of its orbit be an equatorial plane? (*c*) What would be the radius of such an orbit?

19. Consider two satellites A and B of equal mass m, moving in the same circular orbit of radius r around the earth E but in *opposite* senses of rotation and therefore on a collision course (see Fig. 16–20). (*a*) In terms of G, M_e, m, and r, find the total mechanical energy $E_A + E_B$ of the two-satellite-plus-earth system before collision. (*b*) If the collision is completely inelastic so that wreckage remains as one piece of tangled material

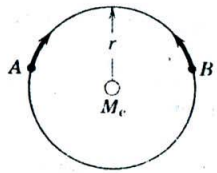


Fig. 16–20

(mass = $2m$), find the total mechanical energy immediately after collision. (*c*) Describe the subsequent motion of the wreckage.

20. A satellite travels initially in an approximately circular orbit 400 miles above the surface of the earth; its mass is 217 kg. (*a*) Determine its speed. (*b*) Determine its period. For various reasons the satellite loses mechanical energy at the (average) rate of 1.42×10^5 joules per complete orbital revolution. Adopting the reasonable approximation that the trajectory is a "circle of slowly diminishing radius," (*c*) determine the distance from the *surface* of the earth, the speed, and the period of the satellite at the end of its 1500th orbital revolution. (*d*) What is the average retarding force? (*e*) Is angular momentum conserved?

21. The mean distance of Mars from the sun is 1.524 times that of Earth from the sun. Find the number of years required for Mars to make one revolution about the sun.

22. Determine the mass of the earth from the period T and the radius r of the moon's orbit about the earth; $T = 27.3$ days and $r = 2.39 \times 10^5$ miles.

23. (*a*) Show that to escape from the atmosphere of a planet a necessary condition for a molecule is that it have a speed such that $v^2 > 2GM/r$, where M is the mass of the planet and r is the distance of the molecule from the center of the planet. (*b*) Determine the escape speed from the earth for an atmospheric particle 1000 km above the earth's surface. (*c*) Do the same for the moon and the sun.

24. Mars has a mean diameter of 4200 miles, Earth one of 7900 miles. The mass of Mars is $0.11M_e$. (*a*) How does the mean density of Mars compare with that of Earth? (*b*) What is the value of g on Mars? (*c*) What is the escape velocity on Mars?

25. A pair of stars rotates about a common center of mass. One of the stars has a mass M which is twice as large as the mass m of the other, that is, $M = 2m$. Their centers are a distance d apart, d being large compared to the size of either star. (*a*) Derive an expression for the period of rotation of the stars about their common center of mass in terms of d, m, and G. (*b*) Compare the angular momenta of the two stars about their common center of mass by calculating the ratio L_m/L_M. (*c*) Compare the kinetic energies of the two stars by calculating the ratio K_m/K_M.

26. In a double star, two stars of mass 3×10^{30} kg each rotate about their common center of mass, 10^{11} meters away. (*a*) What is their common angular speed? (*b*) Suppose that a meteorite passes through this center of mass moving at right angles to the line joining the stars. What must its speed be if it is to escape from the gravitational field of the double star?

27. Three identical bodies of mass M are located at the vertices of an equilateral triangle with side L. At what speed must they move if they all revolve under the influence of one another's gravity in a circular orbit circumscribing the triangle while still preserving the equilateral triangle?

28. Two particles of mass m and M are initially at rest an infinite distance apart. Show that at any instant their relative velocity of approach attributable to gravitational attraction is $\sqrt{2G(M + m)/d}$, where d is their separation at that instant.

29. A particle of mass m is subject to an attractive central force of magnitude k/r^2, k being a constant. If at the instant when the particle is at an extreme position in its closed orbit, at a distance a from the center of force, its speed is $\sqrt{k/2ma}$, find the other extreme position.

30. What is the percentage change in the acceleration of the earth toward the sun from a total eclipse of the sun to the point where the moon is on a side of the earth directly opposite the sun?

31. An 800-kg mass and a 600-kg mass are separated by 0.25 meter. (*a*) What is the gravitational field strength due to these masses at a point 0.20 meter from the 800-kg mass and 0.15 meter from the 600-kg mass? (*b*) What is the gravitational potential at this point due to these same masses?

32. Masses of 200 and 800 gm are 12 cm apart. (*a*) Find the gravitational force on a unit mass at a point on the line joining the masses 4.0 cm from the 200-gm mass. (*b*) Find the potential energy per unit mass at that point. (*c*) How much work is needed to move this unit mass to a point 4.0 cm from the 800-gm mass along the line of centers?

33. (*a*) Write an expression for the potential energy of a body of mass m in the gravitational field of the earth and moon. Let M_e be the earth's mass, M_μ the moon's mass (where $M_e = 81M_\mu$), R the distance from the earth's center, and r the distance from the moon's center. The distance between earth and moon is about 240,000 miles. (*b*) At what point or points will the gravitational field strength attributable to the earth and moon be zero? (*c*) What will the potential energy and field strength be for m when it is on the earth's surface? The moon's surface? Are these answers unique?

34. A sphere of matter, radius a, has a concentric cavity, radius b, as shown in cross section in Fig. 16–21. (*a*) Sketch the gravitational force F exerted by the sphere on a

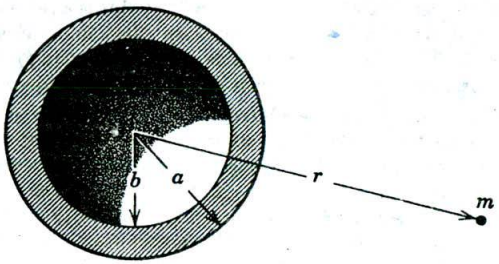


Fig. 16–21

particle of mass m, located a distance r from the center of the sphere, as a function of r in the range $0 \leq r \leq \infty$. Consider points $r = 0$, b, a, and ∞ in particular. (*b*) Sketch the corresponding curve for the potential energy $U(r)$ of the system. (*c*) From these graphs, how would you obtain graphs of the gravitational field strength and the gravitational potential due to the sphere?

35. The variation of g in the earth's interior is given in the accompanying table. The earth's radius is 6400 km.

Depth, km	g, meters/sec^2	Depth, km	g, meters/sec^2
0	9.82	1400	9.88
33	9.85	1600	9.86
100	9.89	1800	9.85
200	9.92	2000	9.86
300	9.95	2200	9.90
413	9.98	2400	9.98
600	10.01	2600	10.09
800	9.99	2800	10.26
1000	9.95	2900	10.37
1200	9.91	4000	8.00

Within the earth's central core (below 2900 km) the values of g diminish monotonically (not linearly) from 10.37 meters/sec^2 to zero. The actual variation of g below 4000 km is uncertain. (*a*) Plot qualitatively g versus r (where r *is the distance from the earth's center*) from 0 to 6400 km. (*b*) Explain carefully how the earth's density must vary as we proceed from its surface to its center in order to account for this variation of g. (*c*) Take $\rho = 1$ at the surface (its average value is actually 3.0 gm/cm^3), and plot qualitatively ρ versus r. Assume throughout that ρ and g are spherically symmetrical.

36. In addition to acceleration toward the earth's axis of rotation and toward the sun, a frame of reference attached to the earth is accelerated toward the center of the galaxy. Take the period of the sun's rotation about the galactic center to be 2.5×10^8 yr and its distance from the center to be 2.4×10^{20} meters and compare these three accelerations for a point on the equator.

37. *Foucault Pendulum.* A pendulum whose upper end is attached so as to allow the pendulum to swing freely in any direction can be used to repeat an experiment first shown publicly by Foucault in Paris in 1851. If the pendulum is set oscillating, the plane of oscillation slowly rotates with respect to a line drawn on the floor, even though the tension in the wire supporting the bob and the gravitational pull of the earth on the bob lie in a vertical plane. (*a*) Show that this is a result of the fact that the earth is not an inertial reference frame. (*b*) Show that for a Foucault pendulum at a latitude angle θ, the period of rotation of the plane is $(24/\sin\theta)$ hr. (*c*) Explain in simple terms the result at $\theta = 90°$ (the poles) and $\theta = 0°$ (the equator).

38. Physicists have speculated about the possible existence of bodies with negative mass; for such hypothetical bodies it is postulated that m in the formulas of physics should be replaced by $-m$. Suppose that two particles, of mass $+m$ and $-m$ respectively, are placed a distance d apart. Show (*a*) the force acting on each and (*b*) the acceleration of each. Describe the expected motion, assuming that both particles are initially at rest, and show that this motion does not violate the laws of conservation of linear momentum or of mechanical energy. Such negative-mass particles have not been found.

39. (*a*) Show that the two-body problem of Section 16–7 can be simplified to a one-body problem by use of the reduced mass concept of Section 15–8. That is, show that if we use $\mu = mM/(m + M)$ instead of m, we may solve for the motion of m relative to M exactly as though M were the origin of our inertial reference frame. (*b*) Show that the assumption made in Section 16–7 that R is negligibly small compared to r is equivalent to assuming that the reduced mass μ is equal to m. (*c*) Compare μ for the earth-sun system with the earth's mass; compare μ for the moon-earth system with the moon's mass. (*d*) If we were to use the reduced mass μ of the two-body system instead of m, how would this affect the equations of Section 16–7?

Fluid Statics

CHAPTER 17

17–1 Fluids

It is customary to classify matter, viewed macroscopically, into solids and fluids. A *fluid* is a substance that can *flow*. Hence the term fluid includes liquids and gases. Such classifications are not always clearcut. Some fluids, such as glass or pitch, flow so slowly that they behave like solids for the time intervals that we usually work with them. Plasma, which is highly ionized gas, does not fit easily into any of these categories; it is often called a "fourth state of matter" to distinguish it from the solid, the liquid, and the gaseous state. Even the distinction between a liquid and a gas is not clearcut because, by changing the pressure and temperature properly, it is possible to change a liquid (water, say) into a gas (steam, say) without the appearance of a meniscus and without boiling; the density and viscosity change in a continuous manner throughout the process.* In this text, however, we will define a fluid as it is ordinarily understood, and we will be interested only in those properties of fluids connected with their ability to flow. Therefore, the same basic laws control the static and dynamic behavior of both liquids and gases in spite of the differences between them that we observe at ordinary pressures.

For solids, which have a definite size and shape, we formulated the mechanics of rigid bodies, modified by the laws of elasticity for bodies that cannot be considered perfectly rigid. Since fluids change their shape readily and, in the case of gases, have a volume equal to that of the container in which they are confined, we must develop new techniques for

* Pressures higher than the so-called critical point pressure must be employed to do this; for H_2O the critical point pressure is 218 atmospheres.

solving problems in fluid mechanics. Our applications of mechanics to continuous media, both solids and fluids, are based on Newton's laws of motion combined with the appropriate force laws. For fluids, as for solids, however, we find it convenient to develop special formulations of these basic laws.

17–2 Pressure and Density

There is a difference in the way a surface force acts on a fluid and on a solid. For a solid there are no restrictions on the direction of such a force, but for a fluid at rest the surface force must always be directed at right angles to the surface. For a fluid at rest cannot sustain a tangential force; the fluid layers would simply slide over one another when subjected to such a force. Indeed, it is the inability of fluids to resist such tangential forces (or shearing stresses) that gives them their characteristic ability to change their shape or to flow.

It is convenient, therefore, to describe the force acting on a fluid by specifying the *pressure* p, which is defined as the magnitude of the *normal* force per unit surface area. Pressure is transmitted to solid boundaries or across arbitrary sections of fluid *at right angles* to these boundaries or sections at every point. Pressure is a scalar quantity. Some common units of pressure are lb/in.^{2}, nt/meter^2, dynes/cm^2, bars (1 bar = 10^6 dynes/cm^2), atmospheres (1 atm = 14.7 lb/in.^{2}) and mm-Hg (760 mm-Hg = 1 atm).

A fluid under pressure exerts a force on any surface in contact with it. Consider a closed surface containing a fluid (Fig. 17–1). An element of the surface can be represented by a vector $\Delta\mathbf{S}$ whose magnitude gives the area of the element and whose direction is taken to be the outward normal to the surface of the element. Then the force $\Delta\mathbf{F}$ exerted by the fluid against this surface element is

$$\Delta\mathbf{F} = p\,\Delta\mathbf{S}.$$

Since $\Delta\mathbf{F}$ and $\Delta\mathbf{S}$ have the same direction, the pressure p can be written as

$$p = \frac{\Delta F}{\Delta S}.$$

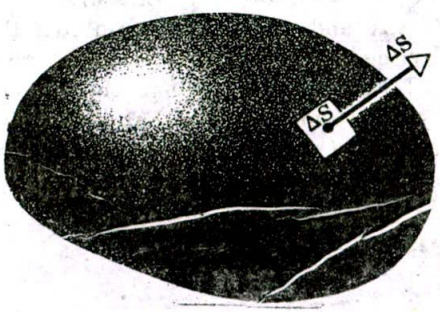


Fig. 17–1 An element of surface ΔS can be represented by a vector $\Delta\mathbf{S}$, equal to its area in magnitude and normal to it in direction.

The pressure so defined may depend on the size of the element of area ΔS that we choose. To avoid this difficulty we refine our definition of pressure by taking a small element of surface containing a point in question and considering this quotient as the element shrinks to the point. Then the pressure at the point is given as

$$p = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S}.$$

The pressure may vary from point to point on the surface.

The density ρ of a homogeneous fluid (its mass divided by its volume) may depend on many factors, such as its temperature and the pressure to which it is subjected. For liquids the density varies very little over wide ranges in pressure and temperature, and we can safely treat it as a constant for our present purposes; see entries under *Water* in Table 17–1. The density of a gas, however, is very sensitive to changes in temperature and pressure; see entries under *Air* in Table 17–1. This table shows the range of densities that occur in nature.

Table 17–1

DENSITIES OF SOME MATERIALS AND OBJECTS IN KG/METER^3

Interstellar space	$10^{-18} \times 10^{-21}$
Best laboratory vacuum	$\sim 10^{-16}$
Hydrogen: at 0° C and 1.0 atm	9.0×10^{-2}
Air: at 0° C and 1.0 atm	1.3
at 100°C and 1.0 atm	0.95
at 0° C and 50 atm	6.5
Styrofoam	$\sim 1 \times 10^{2}$
Ice	0.92×10^{3}
Water: at 0° C and 1.0 atm	1.000×10^{3}
at 100° C and 1.0 atm	0.958×10^{3}
at 0° C and 50 atm	1.002×10^{3}
Aluminum	2.7×10^{3}
Mercury	1.36×10^{4}
Platinum	2.14×10^{4}
The earth: average density	5.52×10^{3}
density of core	9.5×10^{3}
density of crust	2.8×10^{3}
The sun: average density	1.4×10^{3}
density at center	$\sim 1.6 \times 10^{5}$
White dwarf stars (central densities)	$10^{8} - 10^{15}$
A uranium nucleus	$\sim 10^{17}$

17–3 The Variation of Pressure in a Fluid at Rest

If a fluid is in equilibrium, every portion of the fluid is in equilibrium. Let us consider a small element of fluid volume submerged within the body

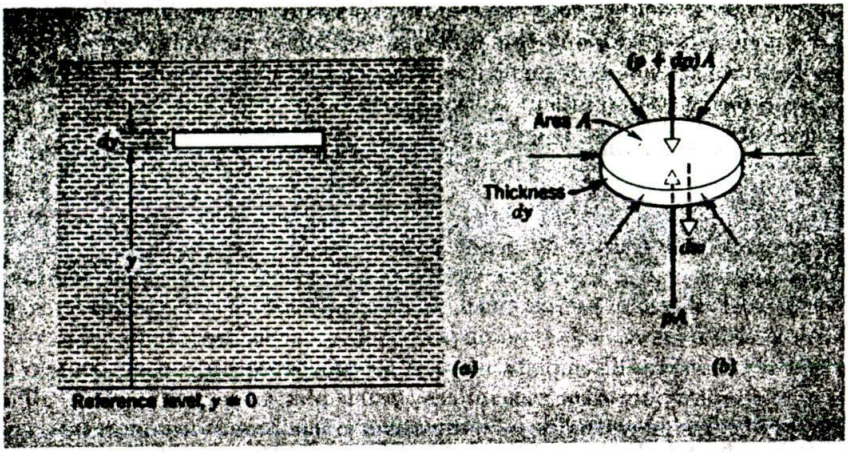


Fig. 17–2 (*a*) A small volume element of fluid at rest. (*b*) The forces on the element.

of the fluid. Let this element have the shape of a thin disk and be a distance y above some reference level, as shown in Fig. 17–2*a*. The thickness of the disk is dy and each face has an area A. The mass of this element is $\rho A\, dy$ and its weight is $\rho g A\, dy$. The forces exerted on the element by the surrounding fluid are perpendicular to its surface at each point (Fig. 17–2*b*).

The resultant horizontal force is zero, for the element has no horizontal acceleration. The horizontal forces are due only to the pressure of the fluid, and by symmetry the pressure must be the same at all points within a horizontal plane at y.

The fluid element is also unaccelerated in the vertical direction, so that the resultant vertical force on it must be zero. However, the vertical forces are due not only to the pressure of the fluid on its faces but also to the weight of the element. If we let p be the pressure on the lower face and $p + dp$ the pressure on its upper face, the upward force is pA (exerted on the lower face) and the downward force is $(p + dp)A$ (exerted on the upper face) plus the weight of the element dw. Hence, for vertical equilibrium

$$\begin{aligned} pA &= (p + dp)A + dw \\ &= (p + dp)A + \rho g A\, dy, \end{aligned}$$

and

$$\frac{dp}{dy} = -\rho g. \tag{17-1}$$

This equation tells us how the pressure varies with elevation above some reference level in a fluid in static equilibrium. As the elevation increases (dy positive), the pressure decreases (dp negative). The cause of this pressure variation is the weight per unit cross-sectional area of the layers

of fluid lying between the points whose pressure difference is being measured.

The quantity ρg is often called the *weight density* of the fluid; it is the weight per unit volume of the fluid. For water, for example, the weight density is 62.4 lb/ft^3.

If p_1 is the pressure at elevation y_1 and p_2 the pressure at elevation y_2 above some reference level, integration of Eq. 17–1 gives

$$\int_{p_1}^{p_2} dp = -\int_{y_1}^{y_2} \rho g \, dy$$

or

$$p_2 - p_1 = -\int_{y_1}^{y_2} \rho g \, dy. \quad (17\text{–}2)$$

For liquids ρ is practically constant because liquids are nearly incompressible; and differences in level are rarely so great that any change in g need be considered. Hence, taking ρ and g as constants, we obtain

$$p_2 - p_1 = -\rho g(y_2 - y_1) \quad (17\text{–}3)$$

for a homogeneous liquid.

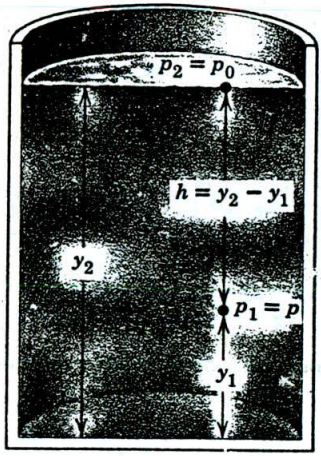


Fig. 17–3 A liquid whose top surface is open to the atmosphere.

If a liquid has a free surface, this is the natural level from which to measure distances. To change our reference level to the top surface, we take y_2 to be the elevation of the surface, at which point the pressure p_2 acting on the fluid is usually that exerted by the earth's atmosphere p_0. We take y_1 to be at any level and we represent the pressure there as p. Then,

$$p_0 - p = -\rho g(y_2 - y_1).$$

But $y_2 - y_1$ is the depth h below the surface at which the pressure is p (see Fig. 17–3), so that

$$p = p_0 + \rho g h. \quad (17\text{–}4)$$

This shows clearly that the pressure is the same at all points at the same depth.

For gases ρ is comparatively small and the difference in pressure at two points is usually negligible (see Eq. 17–3). Thus, in a vessel containing a gas the pressure can be taken as the same everywhere. However, this is not the case if $y_2 - y_1$ is very great. The pressure of the air varies greatly as we ascend to great heights in the atmosphere. In fact, in such cases the density ρ varies with altitude and ρ must be known as a function of y before we can integrate Eq. 17–2.

▶ **Example 1.** We can get a reasonable idea of the variation of pressure with altitude in the earth's atmosphere if we assume that the density ρ is proportional to

the pressure. This would be very nearly true if the temperature of the air remained the same at all altitudes. Using this assumption, and also assuming that the variation of g with altitude is negligible, find the pressure p at an altitude y above sea level.

From Eq. 17–1 we have

$$\frac{dp}{dy} = -\rho g.$$

Since ρ is proportional to p, we have

$$\frac{\rho}{\rho_0} = \frac{p}{p_0},$$

where ρ_0 and p_0 are the known values of density and pressure at sea level. Then,

$$\frac{dp}{dy} = -g\rho_0 \frac{p}{p_0},$$

so that

$$\frac{dp}{p} = -\frac{g\rho_0}{p_0}\, dy.$$

Integrating this from the value p_0 at the point $y = 0$ (sea level) to the value p at the point y (above sea level), we obtain

$$\ln \frac{p}{p_0} = -\frac{g\rho_0}{p_0} y$$

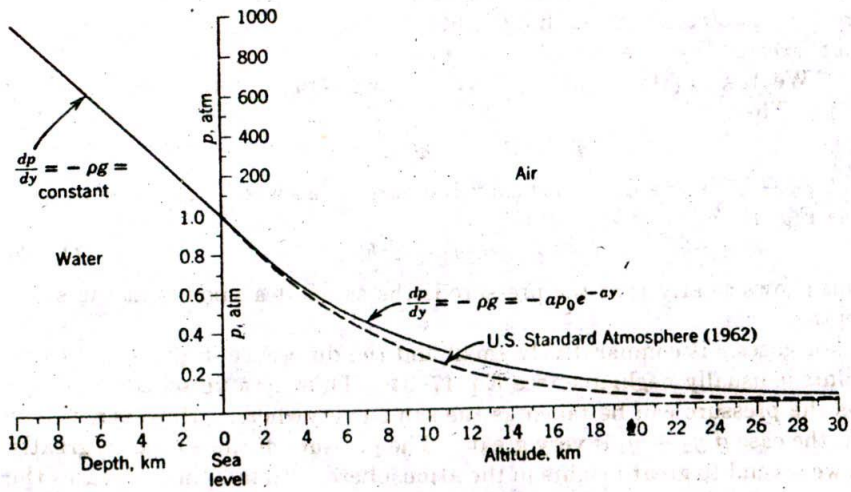


Fig. 17–4 Example 1. Variation of pressure with altitude in air and with depth in water. Note that the pressure scales are different for altitude and depth. The solid line for air is calculated on the assumption that the air has a constant temperature and that g does not change with altitude. The dashed line (the U.S. Standard Atmosphere—1962) is a more refined calculation in which these assumptions are not made.

or

$$p = p_0 e^{-g(\rho_0/p_0)y}.$$

However,

$$g = 9.80 \text{ meters/sec}^2, \qquad \rho_0 = 1.20 \text{ kg/meters}^2 \text{ (at } 20°\text{C)},$$

$$p_0 = 1.01 \times 10^5 \text{ nt/meter}^2,$$

so that

$$g\frac{\rho_0}{p_0} = 1.16 \times 10^{-4} \text{ meter}^{-1} = 0.116 \text{ km}^{-1}.$$

Hence,

$$p = p_0 e^{-ay}.$$

where $a = 0.116 \text{ km}^{-1}$.

We have seen that because liquids are almost imcompressible the lower layers are not noticeably compressed by the weight of the upper layers superimposed on them and the density ρ is practically constant at all levels. For gases at uniform temperature the density ρ of any layer is proportional to the pressure p at that layer. The variation of pressure with distance above the bottom of the fluid for a gas is different from that for a liquid. Figure 17–4 shows the pressure distribution in water and in air. ◀

Equation 17–3 gives the relation between the pressures at any two points in a fluid, regardless of the shape of the containing vessel. For no matter what the shape of the containing vessel, two points in the fluid can be connected by a path made up of vertical and horizontal steps. For example, consider points A and B in the homogeneous liquid contained in the U-tube of Fig. 17–5*a*. Along the zigzag path from A to B there is a difference in pressure $\rho g y'$ for each vertical segment of length y', whereas along each horizontal segment there is no change in pressure. Hence, the difference in pressure $p_B - p_A$ is ρg times the algebraic sum of the vertical segments from A to B, or $\rho g(y_2 - y_1)$.

If the U-tube contains different imiscible liquids, say a dense liquid in the right tube and a less dense one in the left tube, as shown in Fig. 17–5*b*,

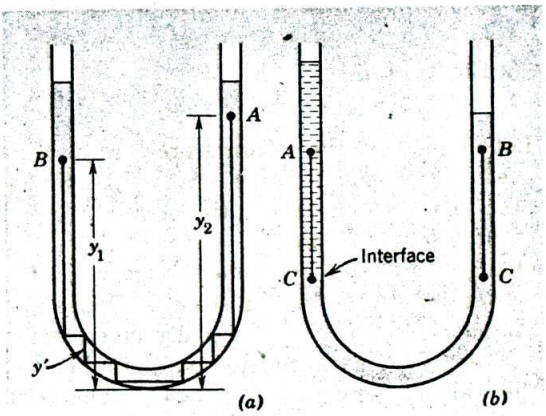


Fig. 17–5 (*a*) The difference in pressure between two points A and B in a homogeneous liquid depends only on their difference in elevation $y_2 - y_1$. (*b*) Two points A and B at the same elevation can be at different pressures if the densities there differ.

the pressure can be different at the same level on different sides. In the figure the liquid surface is higher in the left tube than in the right. The pressure at A will be greater than at B. The pressure at C is the same on both sides, but the pressure falls less from C to A than from C to B, for a column of liquid of unit cross-sectional area connecting A and C will weigh less than a corresponding column connecting B and C.

▶ **Example 2.** A U-tube is partly filled with water. Another liquid, which does not mix with water, is poured into one side until it stands a distance d above the water level on the other side, which has meanwhile risen a distance l (Fig. 17–6). Find the density of the liquid relative to that of water.

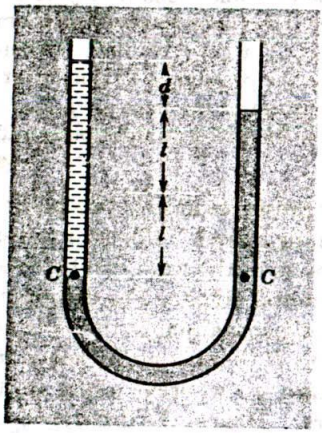


Fig. 17–6 Example 2.

In Fig. 17–6 points C are at the same pressure. Hence, the pressure drop from C to each surface is the same, for each surface is at atmospheric pressure.

The pressure drop on the water side is $\rho_w g 2l$; the $2l$ comes from the fact that the water column has risen a distance l on one side and fallen a distance l on the other side, from its initial position. The pressure drop on the other side is $\rho g(d + 2l)$, where ρ is the density of the unknown liquid. Hence,

$$\rho_w g 2l = \rho g(d + 2l)$$

and

$$\frac{\rho}{\rho_w} = \frac{2l}{(2l + d)}.$$

The ratio of the density of a substance to the density of water is called the *relative density* (or the *specific gravity*) of that substance. ◀

17–4 Pascal's Principle and Archimedes' Principle

Figure 17–7 shows a liquid in a cylinder that is fitted with a piston to which we may apply an external pressure p_0. The pressure p at any arbitrary point P a distance h below the upper surface of the liquid is given by Eq. 17–4, or

$$p = p_0 + \rho g h.$$

Let us increase the external pressure by an arbitrary amount Δp_0 (which need not be small compared to p_0). Since liquids are virtually incompressible, the density ρ in the preceding equations remains essentially constant during the process. The equation shows that, to this extent, the change in pressure Δp at the arbitrary point P is equal to Δp_0. This result was stated by the French scientist Blaise Pascal (1623–1662) and is called *Pascal's principle*. It is usually given as follows: Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel. This result is a necessary consequence of the laws of fluid mechanics, rather than an independent principle.

Although we often assume liquids to be incompressible, they are, in fact, slightly compressible. This means that a change of pressure applied to one portion of a liquid propagates through the liquid as a wave at the speed of sound in that liquid. Once the disturbance has died out and equilibrium is established, it is found that Pascal's principle is valid. The principle holds for gases with slight complications of interpretation caused by the large volume changes that may occur when the pressure on a confined gas is changed.

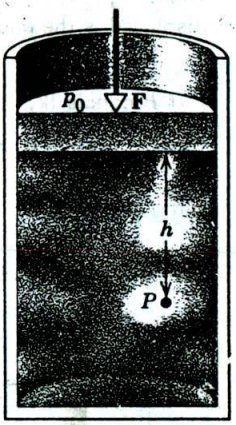


Fig. 17–7 A fluid in a cylinder fitted with a movable piston. The pressure at any point P is due not only to the weight of the fluid above the level of P but also to the force exerted by the piston.

Archimedes' principle is also a necessary consequence of the laws of fluid statics. When a body is wholly or partly immersed in a fluid (either a liquid or a gas) at rest, the fluid exerts pressure on every part of the body's surface in contact with the fluid. The pressure is greater on the parts immersed more deeply. The resultant of all the forces is an upward force called the *buoyancy* of the immersed body. We can determine the magnitude and direction of this resultant force quite simply as follows.

The pressure on each part of the surface of the body certainly does not depend on the material the body is made of. Let us suppose, then, that the body, or as much of it as is immersed, is replaced by fluid like the surroundings. This fluid will experience the pressures that acted on the immersed body (Fig. 17–8) and will be at rest. Hence, the resultant upward force on it will equal its weight and will act vertically upward through its center of gravity. From this follows *Archimedes' principle*, namely, that a body wholly or partly immersed in a fluid is buoyed up with a force equal to the weight of the fluid displaced by the body. We have seen that the force acts vertically up through the center of gravity of the fluid before its displacement. The corresponding point in the immersed body is called its *center of buoyancy*.

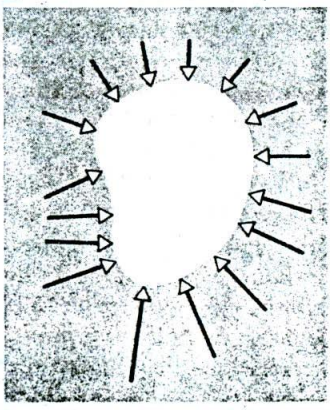


Fig. 17–8 Archimedes' principle.

▶ **Example 3.** What fraction of the total volume of an iceberg is exposed? The density of ice is $\rho_i = 0.92$ gm/cm^3 and that of sea water is $\rho_w = 1.03$ gm/cm^3. The weight of the iceberg is

$$W_i = \rho_i V_i g,$$

where V_i is the volume of the iceberg; the weight of the volume V_w of sea water displaced is the buoyant force

$$B = \rho_w V_w g.$$

But B equals W_i, for the iceberg is in equilibrium, so that

$$\rho_w V_w g = \rho_i V_i g,$$

and

$$\frac{V_w}{V_i} = \frac{\rho_i}{\rho_w} = \frac{0.92}{1.03} = 89\%.$$

The volume of water displaced V_w is the volume of the submerged portion of the iceberg, so that 11% of the iceberg is exposed. ◀

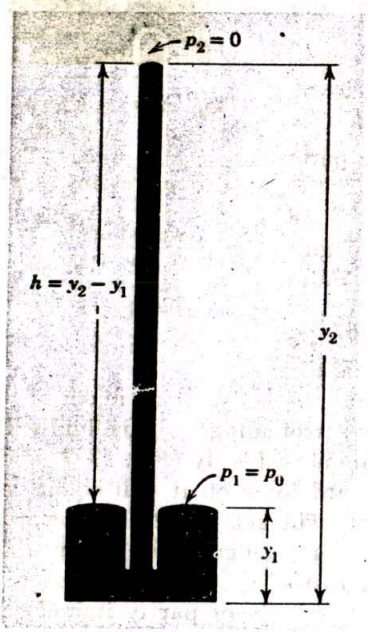


Fig. 17–9 The Torricelli barometer.

17–5 Measurement of Pressure

Evangelista Torricelli (1608–1647) devised a method for measuring the pressure of the atmosphere by his invention of the mercury barometer in 1643. The mercury barometer is a long glass tube that has been filled with mercury and then inverted in a dish of mercury, as in Fig. 17–9. The space above the mercury column contains only mercury vapor whose pressure is so small at ordinary temperatures that it can be neglected. It is easily shown (see Eq. 17–3) that the atmospheric pressure p_0 is

$$p_0 = \rho g h.$$

Most pressure gauges use atmospheric pressure as a reference level and measure the difference between the actual pressure and atmospheric pressure, called the *gauge pressure*. The actual pressure at a point in a fluid is called the *absolute pressure*. Gauge pressure is given either above or below atmospheric pressure. A gauge that reads pressures below atmospheric is usually called a vacuum gauge.

The pressure of the atmosphere at any point is numerically equal to the weight of a column of air of unit cross-sectional area extending from that point to the top of the atmosphere. The atmospheric pressure at a point, therefore, decreases with altitude. There are variations in atmospheric pressure from day to day since the atmosphere is not static. The mercury column in the barometer will have a height of about 76 cm at sea level, varying with the atmospheric pressure. A pressure equivalent to that

exerted by exactly 76 cm of mercury at 0°C under standard gravity, $g = 32.174$ ft/sec^2 = 980.665 cm/sec^2, is called *one atmosphere* (1 atm). The density of mercury at this temperature is 13.5950 gm/cm^3. Hence, one atmosphere is equivalent to

$$\begin{aligned} 1 \text{ atm} &= (13.5950 \text{ gm/cm}^3)(980.665 \text{ cm/sec}^2)(76.00 \text{ cm}) \\ &= 1.013 \times 10^5 \text{ nt/meter}^2 \\ &= 2116 \text{ lb/ft}^2 \\ &= 14.70 \text{ lb/in.}^2 \end{aligned}$$

Often pressures are specified by giving the height of mercury column, at 0°C under standard gravity, which exerts the same pressure. This is the origin of the expression "centimeters of mercury (cm-Hg)" or "inches of mercury (in-Hg)" pressure. Pressure is the ratio of force to area, however, and not a length.

Torricelli was Galileo's successor as professor of mathematics at the Accademia in Florence. He described his experiments with the mercury barometer in two letters of 1644 to his friend M. A. Ricci in Rome. In them he says that the aim of his investigation was "not simply to produce a vacuum, but to make an instrument which shows the mutations of the air, now heavier and dense, and now lighter and thin." On hearing of the Italian experiments, Blaise Pascal, in France, reasoned that if the mercury column was held up simply by the pressure of the air, the column ought to be shorter at a high altitude. He tried it on a church steeple in Paris, but desiring more decisive results, he wrote to his brother-in-law to try the experiment on the Puy de Dôme, a high mountain in Auvergne. There was a difference of 3 inches in the height of the mercury, "which ravished us with admiration and astonishment." Pascal himself constructed a barometer using red wine and a glass tube 46 feet long.

The chief significance of these experiments at the time was the realization it brought that an evacuated space could be created. Aristotle believed that a vacuum could not exist, and as late a writer as Descartes held the same view. For two thousand years philosophers spoke of the horror that nature had for empty space—the *horror vacui*. Because of this horror nature was said to prevent the formation of a vacuum by laying hold of anything nearby and with it instantly filling up any vacuated space. Hence, the mercury or wine should fill up the inverted tube because "nature abhorred a vacuum." The experiments of Torricelli and Pascal showed that there were limitations to nature's ability to prevent a vacuum. They created a sensation at the time. The goal of producing a vacuum became more of a practical reality through the development of pumps by Otto von Guericke in Germany around 1650 and by Robert Boyle in England around 1660. Even though these pumps were relatively crude, they did provide a tool for experimentation. With a pump and a glass jar, an experimental space could be provided in which to study how the properties of heat, light, sound, and later electricity and magnetism, are affected by an increasingly rarefied atmosphere. Although even today we cannot completely remove every trace of gas from a closed vessel, these seventeenth-century experimenters freed science from the bugaboo of *horror vacui* and spurred efforts to create highly evacuated systems.

Except for the telescope, no scientific discovery of the seventeenth century excited wonder and curiosity to a greater degree than did the experiments with the barometer and the air pump.

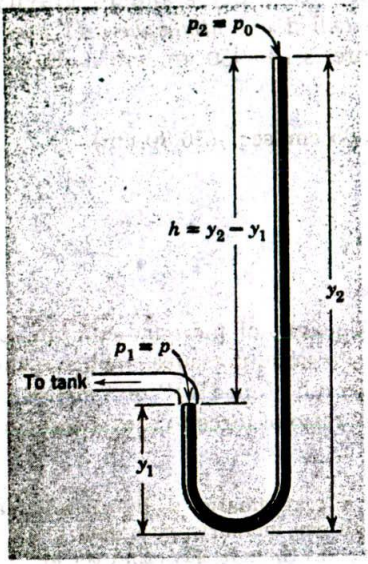


Fig. 17–10 The open-tube manometer, as used to measure the pressure in a tank.

The open-tube manometer (Fig. 17–10) measures gauge pressure. It consists of a U-shaped tube containing a liquid, one end of the tube being open to the atmosphere and the other end being connected to the system (tank) whose pressure p we want to measure. From Eq. 17–4 we obtain

$$p - p_0 = \rho g h.$$

Thus the gauge pressure, $p - p_0$, is proportional to the difference in height of the liquid columns in the U-tube. If the vessel contains gas under high pressure, a dense liquid like mercury is used in the tube; water can be used when low gas pressures are involved.

▶ **Example 4.** An open-tube mercury manometer (Fig. 17–10) is connected to a gas tank. The mercury is 39.0 cm higher on the right side than on the left when a barometer nearby reads 75.0 cm-Hg. What is the absolute pressure of the gas? Express the answer in cm-Hg, atm, and lb/in.^{2}.

The gas pressure is the pressure at the top of the left mercury column. This is the same as the pressure at the same horizontal level in the right column. The pressure at this level is the atmospheric pressure (75.0 cm-Hg) plus the pressure exerted by the extra 39.0-cm column of Hg, or (assuming standard values of mercury density and gravity) a total of 114 cm-Hg. Therefore, the absolute pressure of the gas is

$$114 \text{ cm-Hg} = \tfrac{114}{76} \text{ atm} = 1.50 \text{ atm} = (1.50)(14.7) \text{ lb/in.}^2$$
$$= 22.1 \text{ lb/in.}^2.$$

What is the gauge pressure of the gas? ◀

QUESTIONS

1. (*a*) Two bodies (for example, balls) have the same shape and size but one is denser than the other. Assuming the air resistance to be the same on each, show that when they are released simultaneously from the same height the heavier body will get to the ground first. (*b*) Two bodies (for example, raindrops) have the same shape and density but one is larger than the other. Assuming the air resistance to be proportional to the body's speed through the air, which body will fall faster?

2. Water is poured to the same level in each of the three vessels shown, all of the same base area (Fig. 17–11). If the pressure is the same at the bottom of each vessel, the

force experienced by the base of each vessel is the same. Why then do the three vessels have different weights when put on a scale? This apparently contradictory result is commonly known as the "hydrostatic paradox."

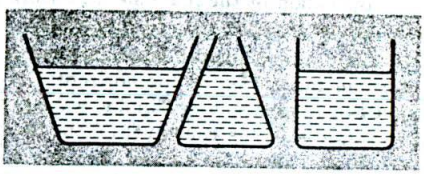


Fig. 17–11

3. Can a mountain climber climb high enough so that the atmospheric pressure is reduced to one-half of its sea-level value?

4. (*a*) An ice cube is floating in a glass of water. When the ice melts, will the water level rise? Explain. (*b*) If the ice cube contains a piece of lead, the water level will fall when the ice melts. Explain.

5. Does Archimedes' law hold in a vessel in free fall? In a satellite moving in a circular orbit? Explain.

6. A spherical bob made of cork floats half submerged in a pot of tea at rest on the earth. Will the cork float or sink aboard a spaceship coasting in free space? On the surface of Jupiter?

7. Two bodies of equal weight and volume and having the same shape, except that one has an opening at the bottom and the other is sealed, are immersed to the same depth in water. Is less work required to immerse one than the other? If so, which one and why?

8. A ball floats on the surface of water in a container exposed to the atmosphere. Will the ball remain immersed at its former depth or will it sink or rise somewhat if (*a*) the container is covered and the air is removed, (*b*) the container is covered and the air is compressed?

9. Explain why an inflated balloon will rise to a definite height once it starts to rise, whereas a submarine will always sink to the bottom of the ocean once it starts to sink, if no changes are made. How then can a submarine stay at a definite level under the water?

10. A soft plastic bag weighs the same when empty as when filled with air at atmospheric pressure. Why? Would the weights be the same if measured in a vacuum?

11. A leaky tramp steamer that is barely able to float in the North Sea steams up the Thames estuary toward the London docks. It sinks before it arrives. Why?

12. Is it true that a floating object will only be in stable equilibrium if its center of buoyancy lies above its center of gravity? Illustrate with examples.

13. Very often a sinking ship will turn over as it becomes immersed in water. Why?

14. A barge filled with scrap iron is in a canal lock. If the iron is thrown overboard, what happens to the water level in the lock?

15. A bucket of water is suspended from a spring balance. Does the balance reading change when a piece of iron suspended from a string is immersed in the water? When a piece of cork is put in the water?

16. Explain why a uniform wooden stick which will float horizontally if it is not loaded will float vertically if enough weight is added to one end. See Problem 8.

17. A solid cylinder is placed in a container in contact with the base. When liquid is poured into the container, none of it goes beneath the solid, which remains closely in contact with the base. Is there a buoyant force on the solid? Explain.

18. Estimate with some care the buoyant force exerted by the atmosphere on you.

19. An open-tube manometer has one tube twice the diameter of the other. Explain how this would affect the operation of the manometer. Does it matter which end is connected to the chamber whose pressure is to be measured?

20. An open bucket of water is on a smooth plane inclined at an angle α to the horizontal. Find the equilibrium inclination to the horizontal of the free surface of the water when (*a*) the bucket is held at rest, $a = 0$ and $v = 0$; (*b*) the bucket is allowed to slide down at constant speed, $a = 0$, $v =$ constant; (*c*) the bucket slides down without restraint, $a =$ constant. If the plane is curved so that $a \neq$ constant, what will happen?

21. If a U-tube containing water is rotated about a vertical axis through the center of one limb, the water level will fall in one limb and rise in the other compared to the rest position. Explain carefully. See Problem 22.

22. Explain how it can be that pressure is a scalar quantity when forces, which are vectors, can be produced by the action of pressures.

PROBLEMS

1. (*a*) Find the pressure, in lb/in.^{2}, 500 ft below the surface of the ocean. The relative density of sea water is 1.03. (*b*) Find the pressure in the atmosphere 10 miles above sea level.

2. A simple U-tube contains mercury. When 13.6 cm of water is poured into the right arm, how high does the mercury rise in the left arm from its initial level?

3. In 1654 Otto von Guericke, burgomeister of Magdeburg and inventor of the air pump, gave a demonstration before the Imperial Diet in which two teams of eight horses could not pull apart two evacuated brass hemispheres. (*a*) Show that the force F required to pull apart the hemispheres is $F = \pi R^2 P$ where R is the (outside) radius of the hemispheres and P is the difference in pressure outside and inside the sphere (Fig. 17–12). (*b*) Taking R equal to 1 ft and the inside pressure as 0.1 atm, what force would the team of horses have had to exert to pull apart the hemispheres? (*c*) Why were two teams of horses used? Would not one team prove the point just as well?

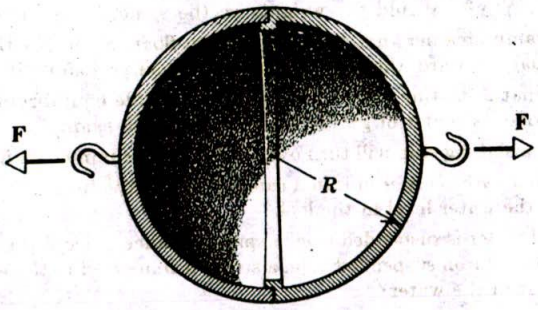


Fig. 17–12

4. The height at which the pressure in the atmosphere is just $1/e$ that at sea level is called the *scale height* of the atmosphere at sea level. (*a*) Show that the scale height H at sea level is also the height of an atmosphere that has the same density everywhere as at sea level and that will exert the same pressure at sea level as the actual infinite atmosphere does. (*b*) Show that the scale height at sea level is 8.6 km.

5. Water stands at a depth D behind the vertical upstream face of a dam, as shown in Fig. 17-13. Let W be the width of the dam. (*a*) Find the resultant force exerted on

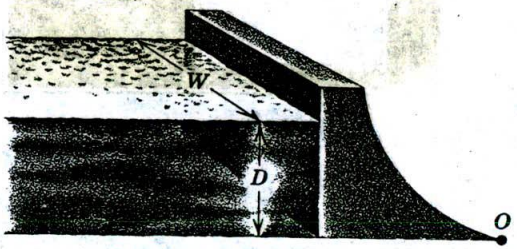


Fig. 17–13

the dam by the water and the torque exerted about O by this force. (*b*) What is the line of action of the equivalent resultant force?

6. A swimming pool has the dimensions 80 ft × 30 ft × 8.0 ft. When it is filled with water, what is the force on the bottom? On the ends? On the sides?

7. A U-tube is filled with a single homogeneous liquid. The liquid is temporarily depressed in one side by a piston. The piston is removed and the level of the liquid in each side oscillates. Show that the period of oscillation is $\pi\sqrt{2L/g}$ where L is the total length of the liquid in the tube.

8. A cylindrical wooden rod is loaded with lead at one end so that it floats upright in water as in Fig. 17–14. The length of the submerged portion is $l = 8.0$ ft. The rod is set into vertical oscillation. (*a*) Show that the oscillation is simple harmonic. (*b*) Find the period in seconds of the oscillation. Neglect the fact that the water has a damping effect on the motion.

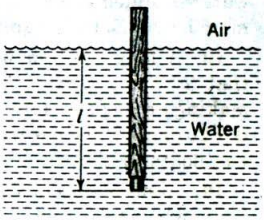


Fig. 17–14

9. Two identical cylindrical vessels with their bases at the same level each contain a liquid of density ρ. The area of either base is A, but in one vessel the liquid height is h_1 and in the other h_2. Find the work done by gravity in equalizing the levels when the two vessels are connected.

10. The surface of contact of two fluids of different densities that are at rest and do not mix is horizontal. Prove this general result (*a*) from the fact that the potential energy of a system must be a minimum in stable equilibrium; (*b*) from the fact that at any two points in a horizontal plane in either fluid the pressures are equal.

11. A piston of small cross-sectional area a is used in the hydraulic press to exert a small force f on the enclosed liquid. A connecting pipe leads to a larger piston of cross-

sectional area A (Fig. 17–15). (*a*) What force F will the larger piston sustain? (*b*) If the small piston has a diameter of 1.5 in. and the large piston one of 21 in., what weight on the small piston will support 2.0 tons on the large piston?

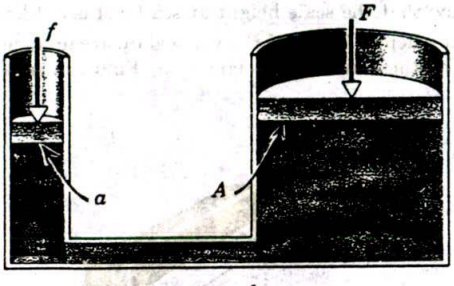


Fig. 17–15

12. What is the minimum area of a block of ice 1.0 ft thick floating on water that will hold up an automobile weighing 2500 lb? Does it matter where the car is placed on the block of ice?

13. An iron casting weighs 60 lb in air and 40 lb in water. What is the volume of cavities in the casting?

14. A hollow spherical iron shell floats almost completely submerged in water. If the outer diameter is 2.0 ft and the relative density of iron is 7.8, find the inner diameter.

15. A block of wood floats in water with two-thirds of its volume submerged. In oil it has 0.90 of its volume submerged. Find the density of the wood and the oil.

16. A block of wood weighs 8.0 lb and has a relative density of 0.60. It is to be loaded with lead so that it will float in water with 0.90 of its volume immersed. What weight of lead is needed (*a*) if the lead is on top of the wood? (*b*) if the lead is attached below the wood?

17. A cube floating on mercury has one-fourth of its volume submerged. If enough water is added to cover the cube, what fraction of its volume will remain immersed in mercury? Does the answer depend on the shape of the body?

18. A long uniform wooden bar with square cross section floats on water. Show that (*a*) if the relative density of the wood is 0.50, the bar will float with all four faces making an angle of 45° with the water surface but (*b*) if the relative density of the wood is 0.20 or 0.80, the bar will float with two faces parallel to the water surface. (*Hint:* The bar floats in the equilibrium position for which the potential energy is a minimum.)

19. Assume the density of brass weights to be 8.0 gm/cm^3 and that of air to be 0.0012 gm/cm^3. What per cent error arises from neglecting the buoyancy of air in weighing an object of mass m gm and relative density ρ on a beam balance?

20. (*a*) Consider a container of fluid subject to a *vertical upward* acceleration a. Show that the pressure variation with depth in the fluid is given by

$$p = \rho h(g + a),$$

where h is the depth and ρ is the density. (*b*) Show also that if the fluid as a whole undergoes a *vertical downward* acceleration a, the pressure at a depth h is given by

$$p = \rho h(g - a).$$

(*c*) What is the state of affairs in free fall?

21. (*a*) Consider the horizontal acceleration of a mass of liquid in an open tank. Acceleration of this kind causes the liquid surface to drop at the front of the tank and to rise at the rear. Show that the liquid surface slopes at an angle θ with the horizontal, where $\tan \theta = a/g$, a being the horizontal acceleration. (*b*) How does the pressure vary with depth?

22. (*a*) A fluid mass is rotating at constant angular velocity ω about the central vertical axis of a cylindrical container. Show that the variation of pressure in the radial direction is given by

$$\frac{dp}{dr} = \rho\omega^2 r.$$

(*b*) Take $p = p_c$ at the axis of rotation ($r = 0$) and show that the pressure p at any point r is

$$p = p_c + \tfrac{1}{2}\rho\omega^2 r^2.$$

(*c*) Show that the liquid surface is of paraboloidal form (Fig. 17–16); that is, a vertical cross section of the surface is the curve $y = \omega^2 r^2/2g$. (*d*) Show that the variation of pressure with depth is $dp = \rho g\, dh$.

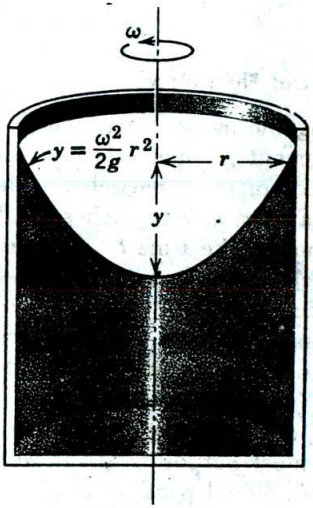


Fig. 17–16