CHAPTER 1

Harmonic Oscillators

1.1. Introduction

In every day life we come across numerous things that move. These motions are of two types, *viz. (i)* the motion in which the body moves about a mean position *i.e.* a fixed point and *(ii)* the motion in which the body moves from one place to the other with respect to time. The first type of motion of a body about a mean position is called oscillatory motion. A moving train, flying aeroplane, moving ball etc. correspond to the second type of motion. Examples of oscillatory motion are : an oscillating pendulum, vibrations of a stretched string, movement of water in a cup, vibration of electrons, movement of light in a laser beam etc.

Sometimes both the types of motion are exhibited in the same phenomenon depending on our point of view. The sea waves appear to move towards the beach but the water moves up and down about the mean position. When a stretched rope is displaced, the displacement pulse travels from one end to the other but the material of the rope vibrates about the mean position without travelling forward.

1.2. Sound

In the case of elastic waves the disturbance travels with a velocity depending upon the elastic properties of the medium. These elastic waves are also called sound waves. In every day life sound is referred to the sensation of hearing. When an elastic wave, propagating through a solid, liquid or gas, reaches the ear, the membrane of the ear is set into vibration. The vibrations of the membrane stimulate a nervous response relating to the process of hearing. The human nervous system produces the hearing sensation for frequencies ranging from 16 hertz to 20,000 hertz. For animals the audible frequency range for hearing is different. Beyond these frequency limits, the sound is not audible. The elastic waves relating to frequencies higher than 20,000 hertz are called ultrasonic waves or ultrasonics.

A longitudinal mechanical wave of frequency less than the lower limit of audibility is called *infrasonic wave*. Waves generated by a large source *e.g.*

earthquake waves are infrasonic waves in nature. The production, properties and applications of ultrasonic waves are discussed in Chapter 10. All these mechanical vibrations are simple harmonic in nature.

1.3. Simple Harmonic Motion

Let P be a particle moving on the circumference of a circle of radius a with a uniform velocity v (Fig. 1.1). Let ω be the uniform angular velocity of the particle ($v = a\omega$). The circle along which P moves is called the **circle of reference.** As the particle P moves round the circle continuously with uniform velocity, the foot of the perpendicular M, vibrates along the diameter YY'. If the motion of P is uniform, then the motion of M is periodic *i.e.*, it takes the same time to vibrate once between the points Y and Y'. At any instant the distance of M from the centre O of the circle is called the **displacement.** If the particle moved from X to P in time t, then

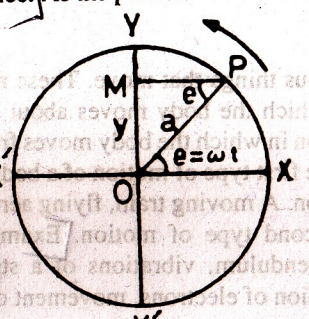


Fig. 1.1.

$$\angle POX = \angle MPO = \theta = \omega t.$$

From the ΔMPO,

$$\sin\theta = \sin\omega t = \frac{OM}{a}$$

or $$OM = y = a \sin \omega t$$

OM is called the displacement of the vibrating particle. The displacement of a vibrating particle at any instant can be defined as its distance from the mean position of rest. The maximum displacement of a vibrating particle is called its **amplitude.**

$$\therefore \quad \text{Displacement} = y = a \sin \omega t \qquad \ldots (1)$$

The rate of change of displacement is called the **velocity** of the vibrating particle.

$$\therefore \quad \text{Velocity} = \frac{dy}{dt} = +a\omega \cos \omega t \qquad \ldots (2)$$

The rate of change of velocity of a vibrating particle is called its acceleration.

$$\therefore \quad \text{Acceleration} = \text{Rate of change of velocity}$$

$$= \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$= \frac{d^2y}{dt^2} = -a\omega^2 \sin \omega t$$

$$= -\omega^2 . a \sin \omega t = -\omega^2 y \qquad \ldots (3)$$

The changes in the displacement, velocity and acceleration of a vibrating particle in one complete vibration are given in the following table.

Angle ωt	*Position of the vibrating particle M*	*Displacement* $y = a \sin \omega t$	*Velocity* $\frac{dy}{dt} = a\omega \cos \omega t$	*Acceleration* $\frac{d^2y}{dt^2} = -a\omega^2 \sin \omega t$
O	O	Zero	$+a\omega$	Zero
$\frac{\pi}{2}$	Y	$+a$	Zero	$-a\omega^2$
π	O	Zero	$-a\omega$	Zero
$\frac{3\pi}{2}$	Y'	$-a$	Zero	$+a\omega^2$
2π	O	Zero	$+a\omega$	Zero

Oscillatory behaviour. At the extreme positions when y is maximum, dy/dt is zero. The acceleration d^2y/dt^2 is maximum and directed towards the mean position. This return force induces a negative velocity. When the displacement y is zero, the velocity dy/dt is maximum and is –ve. When the displacement is negative maximum, the velocity dy/dt is zero and the acceleration is maximum in the positive direction. This return force again induces a velocity in the positive direction which becomes positive maximum when the displacement is zero. The particle overshoots the mean position due to its velocity. The process repeats itself periodically. Thus the system oscillates. In this process, displacement y, velocity dy/dt and acceleration d^2y/dt^2 continuously change with respect to time.

Thus, the velocity of the vibrating particle is maximum (in the direction OY or OY') at the mean position of rest and zero at the maximum positions of vibration. The acceleration of the vibrating particle is zero at the mean position of rest and maximum at the maximum positions of vibration. The acceleration is always directed towards the mean position of rest and is directly proportional to the displacement of the vibrating particle. This type of motion where the acceleration is directed towards a fixed point (the mean position of rest) and is proportional to the displacement of the vibrating particle is called **simple harmonic motion.**

Further,

$$\text{Acceleration} = \frac{d^2y}{dt^2} = -\omega^2 y$$

$$= -\omega^2 \times \text{displacement}$$

Numerically, $$\omega^2 = \frac{\text{Acceleration}}{\text{Displacement}}$$

or $$\omega = 2\pi n = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

or $$\frac{2\pi}{T} = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

or $$T = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{K}.$$

Thus, in general, the time period of a particle vibrating simple harmonically is given by $T = 2\pi \sqrt{K}$ where K is the displacement per unit acceleration.

If the particle P revolves round the circle, n times per second, then the angular velocity ω is given by

$$\omega = 2\pi n = \frac{2\pi}{T} \quad \left(\because n = \frac{1}{T} \text{ where } T \text{ is the time period} \right)$$

or $$y = a \sin 2\pi nt = a \sin 2\pi \frac{t}{T}.$$

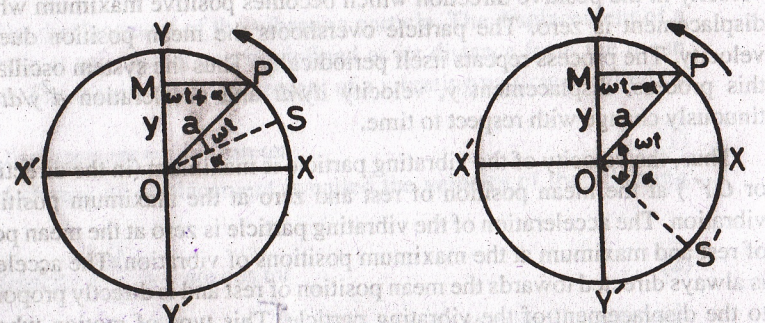


Fig. 1.2.

On the other hand, if the time is counted [Fig. 1.2 *(i)*] from the instant P is at S ($\angle SOX = \alpha$) then the displacement

$$y = a \sin (\omega t + \alpha)$$

$$= a \sin \left(\frac{2\pi t}{T} + \alpha \right)$$

If the time is counted from the instant P is at S' [Fig. 1.2 *(ii)*], then

$$y = a \sin (\omega t - \alpha) = a \sin \left(\frac{2\pi t}{T} - \alpha \right)$$

Phase of the vibrating particle. *(i)* The phase of a vibrating particle is defined as the ratio of the displacement of the vibrating particle at any instant to the amplitude of the vibrating particle *(y/a)* or *(ii)* it is also defined as the fraction of the time interval that has elapsed since the particle crossed the mean position of rest in the positive direction or *(iii)* it is also equal to the angle swept by the radius vector since the vibrating particle last crossed its mean position of rest *e.g.*, in the above equations ωt, $(\omega t + \alpha)$ or $(\omega t - \alpha)$ are called phase angles. The initial phase angle when $t = 0$, is called the epoch angle. α is called the **epoch** angle in the above expressions.

1.4. Differential Equation of SHM

For a particle vibrating simple harmonically, the general equation of displacement is,

$$y = a \sin (\omega t + \alpha) \qquad \ldots (1)$$

Here y is the displacement and a is the amplitude and α is epoch of the vibrating particle.

Differentiating equation (1) with respect to time

$$\frac{dy}{dt} = a\omega \cos (\omega t + \alpha) \qquad \ldots (2)$$

Here dy/dt represents the velocity of the vibrating particle.

Differentiating equation (2) with respect to time

$$\frac{d^2y}{dt^2} = -a\omega^2 \sin (\omega t + \alpha)$$

But $$a \sin (\omega t + \alpha) = y$$

$$\therefore \quad \frac{d^2y}{dt^2} = -\omega^2 y$$

or $$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \ldots (3)$$

Here d^2y/dt^2 represents the acceleration of the particle. Equation (3) represents the differential equation of simple harmonic motion.

It also shows that in any phenomenon where an equation similar to equation (3) is obtained, the body executes simple harmonic motion. The general solution of equation (3) is

$$y = a \sin (\omega t + \alpha)$$

Also the time period of a vibrating particle can be calculated from equation (3).

Numerically $$\omega = \sqrt{\frac{d^2y/dt^2}{y}}$$

or $$\omega = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

or $$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}.$$

1.5. Graphical Representation of SHM

Let P be a particle moving on the circumference of a circle of radius a. The foot of the perpendicular vibrates on the diameter YY'.

$$y = a \sin \omega t = a \sin 2\pi \frac{t}{T}$$

The displacement graph is a sine curve represented by $ABCDE$ (Fig. 1.3).

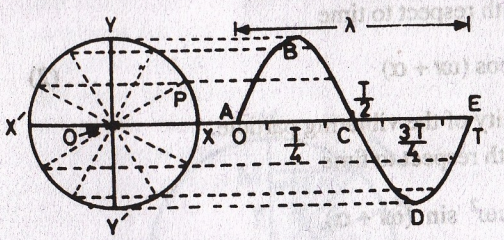


Fig. 1.3. Displacement–Time Curve.

The acceleration of a particle moving with simple harmonic motion is

$$\frac{d^2y}{dt^2} = -a\omega^2 \sin \omega t .$$

The acceleration—time graph is shown in Fig. 1.5. It is a negative sine curve.

The motion of the particle M is simple harmonic.

The velocity of a particle moving with simple harmonic motion is

$$v = \frac{dy}{dt} = +a\omega \cos \omega t$$

The velocity—time graph is shown in Fig. 1.4. It is a cosine curve.

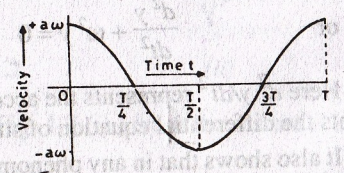


Fig. 1.4. Velocity–Time Curve.

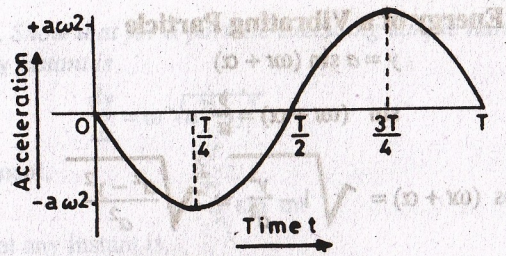


Fig. 1.5. Acceleration—Time Curve.

1.6. Average Kinetic Energy of a Vibrating Particle

The displacement of a vibrating particle is given by

$$y = a \sin (\omega t + \alpha)$$

$$v = \frac{dy}{dt} = a\omega \cos (\omega t + \alpha).$$

If m is the mass of the vibrating particle, the kinetic energy at any instant

$$= \frac{1}{2} m v^2 = \frac{1}{2} m \, . \, a^2\omega^2 \cos^2 (\omega t + \alpha).$$

The average kinetic energy of the particle in one complete vibration

$$= \frac{1}{T} \int_0^T \frac{1}{2} ma^2\omega^2 \cos^2 (\omega t + \alpha) \, dt$$

$$= \frac{1}{T} \frac{ma^2\omega^2}{4} \int_0^T 2 \cos^2 (\omega t + \alpha) \, dt$$

$$= \frac{ma^2\omega^2}{4T} \int_0^T \left[1 + \cos 2 (\omega t + \alpha) \right] dt$$

$$= \frac{ma^2\omega^2}{4T} \left[\int_0^T dt + \int_0^T \cos 2 (\omega t + \alpha) \, dt \right]$$

But, $$\int_0^T \cos 2(\omega t + \alpha) \, dt = 0$$

$\therefore$ Average K. E. $= \frac{ma^2\omega^2}{4T} \cdot T + 0$

$$= \frac{ma^2\omega^2}{4} = \frac{ma^2(4\pi^2 n^2)}{4} = \pi^2 ma^2 n^2$$

where m is the mass of the vibrating particle, a is the amplitude of vibration and n is the frequency of vibration. Also, the average kinetic energy of a vibrating particle is directly proportional to the square of the amplitude.

1.7. Total Energy of a Vibrating Particle

$$y = a \sin(\omega t + \alpha)$$

$$\sin(\omega t + \alpha) = \frac{y}{a}$$

$$\cos(\omega t + \alpha) = \sqrt{1 - \frac{y^2}{a^2}} = \sqrt{\frac{a^2 - y^2}{a^2}}$$

$$= \frac{\sqrt{a^2 - y^2}}{a}$$

Velocity $v = a\omega \cos \omega t = a\omega \frac{\sqrt{a^2 - y^2}}{a} = \omega\sqrt{(a^2 - y^2)}$

$\therefore$ The kinetic energy of the particle at the instant the displacement is y,

$$= \frac{1}{2} m v^2$$

$$= \frac{1}{2} m \cdot \omega^2 (a^2 - y^2)$$

Potential energy of the vibrating particle is the amount of work done in overcoming the force through a distance y.

Acceleration $= -\omega^2 y$

Force $= -m\omega^2 y$

(The $-$ve sign shows that the direction of the acceleration and force are opposite to the direction of motion of the vibrating particle.)

$\therefore$ $$\text{P.E.} = \int_0^y m \cdot \omega^2 y \cdot dy$$

$$= m\omega^2 \cdot \frac{y^2}{2} = \frac{1}{2} m\omega^2 y^2.$$

Total energy of the particle at the instant the displacement is y

$$= \text{K.E.} + \text{P.E.}$$

$$= \tfrac{1}{2} m\omega^2 (a^2 - y^2) + \tfrac{1}{2} m\omega^2 y^2$$

$$= \tfrac{1}{2} m\omega^2 \cdot a^2$$

$$= \tfrac{1}{2} m (2\pi n)^2 a^2 = 2\pi^2 \, ma^2 n^2.$$

As the average kinetic energy of the vibrating particle $= \pi^2 ma^2 n^2$, the average potential energy $= \pi^2 ma^2 n^2$. The total energy at any instant is a constant.

Example 1.1. *Show that for a particle executing simple harmonic motion, its velocity at any instant is*

$$\frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

The displacement,

$$y = a \sin \omega t \quad \ldots (1)$$

The velocity at any instant is,

$$\frac{dy}{dt} = a\omega \cos \omega t \quad \ldots (2)$$

From equation (1)

$$\sin \omega t = \frac{y}{a}$$

$$\cos \omega t = \sqrt{1 - \sin^2 \omega t}$$

$$\cos \omega t = \sqrt{1 - \frac{y^2}{a^2}}$$

$$\frac{dy}{dt} = a\omega \sqrt{1 - \frac{y^2}{a^2}}$$

or $$\frac{dy}{dt} = \omega\sqrt{a^2 - y^2}.$$

Example 1.2. *For a particle vibrating simple harmonically the displacement is* 8 cm *at the instant the velocity is* 6 cm/s *and the displacement is* 6 cm *at the instant the velocity is* 8 cm/s. *Calculate (i) amplitude, (ii) frequency and (iii) time period.*

The velocity of a particle executing SHM,

$$v = \frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

In the first case,

$$v_1 = \omega\sqrt{a^2 - y_1^2}$$

Here $v_1 = 6$ cm/s, $y_1 = 8$ cm

$\therefore$ $$6 = \omega\sqrt{a^2 - 64} \quad \ldots (1)$$

In the second case,

$$v_2 = \omega\sqrt{a^2 - y_2^2}$$

Here $v_2 = 8$ cm/s, $y_2 = 6$ cm

$\therefore$ $$8 = \omega\sqrt{a^2 - 36} \quad \ldots (2)$$

Dividing (2) by (1) and squaring

$$\frac{64}{36} = \frac{a^2 - 36}{a^2 - 64}$$

$$a = 10 \text{ cm.}$$

The amplitude of vibration $= 10$ cm.

Substituting the value of

$a = 10$ cm in equation (1)

$$6 = \omega \sqrt{100 - 64}$$

$\omega = 1$ radian/s

Frequency $$n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \text{ hertz}$$

Time period $$T = \frac{1}{n} = 2\pi \text{ seconds.}$$

Example 1.3. *Show that for a particle executing SHM, the instantaneous velocity is* $\omega\sqrt{a^2 - y^2}$ *and instantaneous acceleration is* $-\omega^2 y$.

For a particle executing SHM,

$$y = a \sin(\omega t + \alpha) \quad \ldots (1)$$

The instantaneous velocity,

$$v = \frac{dy}{dt} = + a\omega \cos(\omega t + \alpha) \quad \ldots (2)$$

From equation (1)

$$\sin(\omega t + \alpha) = \frac{y}{a}$$

$$\cos(\omega t + \alpha) = \sqrt{1 - \sin^2(\omega t + \alpha)}$$

$$= \sqrt{1 - \frac{y^2}{a^2}}$$

$$\therefore \quad v = a\omega \sqrt{1 - \frac{y^2}{a^2}}$$

$$v = \omega \sqrt{a^2 - y^2} \quad \ldots (3)$$

The instantaneous acceleration,

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = -a\omega^2 \sin(\omega t + \alpha)$$

$$= -\omega^2 [a \sin(\omega t + \alpha)]$$

$$= -\omega^2 y \quad \ldots (4)$$

Example 1.4. *The motion of a particle in SHM is given by*

$$x = a \sin \omega t$$

If it has a speed u when the displacement is x_1 *and speed v when the displacement is* x_2. *Show that the amplitude of vibration is*

$$a = \left[\frac{v^2 x_1^2 - u^2 x_2^2}{v^2 - u^2}\right]^{\frac{1}{2}}$$

(Utkal, 1989)

Here $x = a \sin \omega t$

$$u = \frac{dx_1}{dt} = \omega \sqrt{a^2 - x_1^2} \quad \ldots (i)$$

and $$v = \frac{dx_2}{dt} = \omega \sqrt{a^2 - x_2^2} \quad \ldots (ii)$$

Squaring and dividing

$$\frac{u^2}{v^2} = \frac{a^2 - x_1^2}{a^2 - x_2^2}$$

$$u^2 a^2 - u^2 x_2^2 = v^2 a^2 - v^2 x_1^2$$

$$a^2 [v^2 - u^2] = v^2 x_1^2 - u^2 x_2^2$$

$$a = \left[\frac{v^2 x_1^2 - u^2 x_2^2}{v^2 - u^2}\right]^{\frac{1}{2}} \quad \ldots (iii)$$

Example 1.5. *A particle performs simple harmonic motion given by the equation*

$$y = 20 \sin[\omega t + \alpha]$$

If the time period is 30 seconds *and the particle has a displacement of 10* cm *at t=0, find (i) epoch ; (ii) the phase angle at t=5* seconds, *and (iii) the phase difference between two positions of the particle 15* seconds *apart.*

Here $y = 20 \sin(\omega t + \alpha)$

$T = 30$ s

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{30} = \frac{\pi}{15} \text{ radians/s}$$

(*i*) At $t = 0, \quad y = 10$ cm

$$\therefore \quad 10 = 20 \sin\left(\frac{\pi}{15} \times 0 + \alpha\right)$$

or $\sin \alpha = 0{\cdot}5$

or $\alpha = \frac{\pi}{6}$ radian

(*ii*) At $t = 5$ s,

The phase angle $= (\omega t + \alpha)$

$$= \left(\frac{\pi}{15} \times 5 + \frac{\pi}{6}\right)$$

$$= \frac{\pi}{2}$$

(*iii*) At $t = 0$

the phase angle $\theta_1 = \frac{\pi}{6}$

At $t = 15$

the phase angle $\theta_2 = (\omega t + \alpha)$

$$\theta_2 = \left(\frac{\pi}{15} \times 15 + \frac{\pi}{6}\right)$$

$$\theta_2 = \frac{7\pi}{6}.$$

The phase difference $\theta_2 - \theta_1 = \frac{7\pi}{6} - \frac{\pi}{6} = \pi$ **radians.**

Example 1.6. *A particle executes simple harmonic motion given by the equation*

$$y = 12 \sin\left(\frac{2\pi t}{10} + \frac{\pi}{4}\right)$$

Calculate (i) amplitude; (ii) frequency; (iii) epoch; (iv) displacement at $t = 1{\cdot}25$ s, *(v) velocity at* $t = 2{\cdot}5$ s *and (vi) acceleration at* $t = 5$ s.

Here $y = 12 \sin\left(\frac{2\pi t}{10} + \frac{\pi}{4}\right)$ . . . (1)

The displacement equation is

$$y = a \sin(\omega t + \alpha) \quad \ldots (2)$$

Comparing equations (1) and (2)

(*i*) Amplitude $a = 12$ units

(*ii*) $\omega = \frac{2\pi}{10}$

Frequency $n = \frac{\omega}{2\pi} = \frac{1}{10} = \mathbf{0{\cdot}1\ hertz}$

(*iii*) Epoch, $\alpha = \frac{\pi}{4}$

(*iv*) When $t = 1{\cdot}25$ s

$$y = 12 \sin\left(\frac{2\pi \times 1{\cdot}25}{10} + \frac{\pi}{4}\right)$$

$$y = 12 \sin \pi/2$$

or $y = \mathbf{12\ units}$

(*v*) At $t = 2{\cdot}5$ s

$$\text{Velocity} = \frac{dy}{dt} = a\omega \cos(\omega t + \alpha)$$

$$\frac{dy}{dt} = 12 \times \frac{2\pi}{10} \cos\left[\frac{2\pi}{10} \times 2{\cdot}5 + \frac{\pi}{4}\right]$$

$$\frac{dy}{dt} = \mathbf{-5{\cdot}552\ units.}$$

The – ve sign shows that the velocity is directed towards the mean position.

(*vi*) At $t = 5$ s

$$\text{Acceleration} = \frac{d^2y}{dt^2} = -a\omega^2 \sin(\omega t + \alpha)$$

$$\frac{d^2y}{dt^2} = -12 \times \left(\frac{2\pi}{10}\right)^2 \sin\left(\frac{2\pi}{10} \times 5 + \frac{\pi}{4}\right)$$

$$= -0{\cdot}48\,\pi^2 \sin\left(\pi + \frac{\pi}{4}\right) = \mathbf{3{\cdot}35\ units.}$$

Example 1.7. *A simple harmonic motion is represented by*

$$y = 10 \sin\left(10t - \frac{\pi}{6}\right)$$

where y is measured in metres, *t in* seconds *and the phase angle in* radians. *Calculate :*

(i)the frequency;

(ii)the time period;

(iii)the maximum displacement;

(iv)the maximum velocity;

(v)the maximum acceleration, and;

(vi)displacement, velocity and acceleration at time, t=0 and t=1 second.

Here $y = 10 \sin\left(10t - \frac{\pi}{6}\right)$ . . . (1)

The displacement equation is

$$y = a \sin(\omega t + \alpha) \quad \ldots (2)$$

(*i*) From (1) and (2)

$\omega = 10$

But $\omega = 2\pi n$

$\therefore$ $2\pi n = 10$

or $n = \dfrac{10}{2\pi}$ hertz

$n = \mathbf{1{\cdot}6\ hertz.}$

(*ii*) Time period,

$$T = \frac{1}{n} = \frac{2\pi}{10}$$

or $T = 0{\cdot}63$ s

(*iii*) Maximum displacement,

$a = \mathbf{10\ m}$

(*iv*) Velocity, $\dfrac{dy}{dt} = a\omega \cos(\omega t + \alpha)$

Therefore, maximum velocity

$$\frac{dy}{dt} = a\omega$$

But $a = 10$ m

and $\omega = 10$

$\therefore$ $\dfrac{dy}{dt} = 10 \times 10 = \mathbf{100\ m/s}$

(*v*) Acceleration, $\dfrac{d^2y}{dt^2} = -a\omega^2 \sin(\omega t + \alpha)$

Maximum acceleration

$$\frac{d^2y}{dt^2} = -a\omega^2$$

or $\dfrac{d^2y}{dt^2} = -10 \times (10)^2 = \mathbf{-1000\ m/s^2}$

– ve sign shows that the acceleration is directed towards the mean position.

(*vi*) From equation (1)

(*a*) At $t = 0$

$$y = 10 \sin\left(-\frac{\pi}{6}\right)$$

$y = \mathbf{-5\ m}$

$$\frac{dy}{dt} = a\omega \cos(\omega t + \alpha)$$

At $t = 0$

$$\frac{dy}{dt} = a\omega \cos\alpha$$

$$= 10 \times 10 \cos\left(-\frac{\pi}{6}\right)$$

$$= 100 \times 0{\cdot}866 = \mathbf{86{\cdot}6}\ \text{m/s}$$

$$\frac{d^2y}{dt^2} = -\ a\omega^2 \sin\alpha$$

or $\dfrac{d^2y}{dt^2} = -a\omega^2 \sin\left(-\dfrac{\pi}{6}\right)$

or $\dfrac{d^2y}{dt^2} = a\omega^2 \sin\left(\dfrac{\pi}{6}\right)$

$= 10 \times 100 \times 0{\cdot}5 = \mathbf{500\ m/s^2}.$

(*b*) From equation (1)

At $t = 1$,

Displacement

$$y = 10 \sin\left(10 - \frac{\pi}{6}\right)$$

$$y = 10 \sin\left(\frac{60 - 3{\cdot}142}{6}\right)$$

$$y = 10 \sin\left(\frac{56{\cdot}858}{6}\right)$$

$y = 10 \sin(3\pi)$ (approximately)

or $y = 10 \sin\pi = \mathbf{0.}$

Velocity

$$\frac{dy}{dt} = a\omega \cos\left(10 - \frac{\pi}{6}\right)$$

or $\dfrac{dy}{dt} = a\omega \cos\pi$ (approximately)

$= 10 \times 10 \times (-1) = \mathbf{-100\ m/s.}$

Acceleration

$$\frac{d^2y}{dt^2} = -a\omega^2 \sin\left(10 - \frac{\pi}{6}\right)$$

$= -a\omega^2 \sin\pi$ (approximately) $= \mathbf{0.}$

1.8. Energy of Vibration

For a vibrating system, the work done is equal to the product of the force and the displacement while the power required is the work done per second. Let the displacement of the vibrating system at any instant be given by

$$y = a \sin (\omega t - \alpha)$$

and let the periodic force be

$$F = F_0 \sin \omega t$$

The work done for a displacement dy

$$dW = F \times dy$$

The total work done

$$W = \int F dy$$

$$W = \int F_0 \sin \omega t \, [a \cos (\omega t - \alpha] \, d (\omega t)$$

Work done per cycle of motion,

$$W = F_0 a \int_0^{2\pi} \sin \omega t \cos (\omega t - \alpha) \, d (\omega t)$$

But $\cos (\omega t + \alpha) = \cos \omega t \cos \alpha + \sin \omega t \sin \alpha$

$$W = F_0 a \int_0^{2\pi} \sin \omega t \, (\cos \omega t \cos \alpha + \sin \omega t \sin \alpha) \, d (\omega t)$$

$$W = F_0 a \cos \alpha \int_0^{2\pi} \sin \omega t \cos \omega t \, d(\omega t) + F_0 a \sin \alpha \int_0^{2\pi} \sin^2 \omega t \, d(\omega t)$$

$$W = F_0 a \cos \alpha \left[\frac{\sin^2 \omega t}{2} \right]_0^{2\pi} + F_0 a \sin \alpha \left[\frac{\omega t}{2} - \frac{\sin 2\omega t}{4} \right]_0^{2\pi}$$

or $W = [F_0 a \sin \alpha] \, [\pi]$

$W = \pi F_0 a \sin \alpha.$

Example 1.8. *The force and displacement of a simple dynamic system undergoing sinusoidal excitation are given by the equations*

$$F = 10 \sin \left(\frac{\pi t}{10} \right) \text{newtons}$$

$$y = 0{\cdot}1 \sin \left(\frac{\pi t}{10} - \frac{\pi}{6} \right) \text{metres}$$

Calculate the work done by the excitation force in (i) 20 seconds, *and (ii)* 1 minute.

The work done per cycle by the excitation force is given by

$$W = \pi F_0 a \sin \alpha$$

Here F_0 is the maximum force and a is the maximum displacement.

Here $$F = 10 \sin \left(\frac{\pi t}{10} \right) \quad \ldots (1)$$

$$y = 0{\cdot}1 \sin \left(\frac{\pi t}{10} - \frac{\pi}{6} \right) \quad \ldots (2)$$

Also $$F = F_0 \sin \omega t \quad \ldots (3)$$

and $$y = a (\sin \omega t - \alpha) \quad \ldots (4)$$

From these equations

$$F_0 = 10 \text{ newtons}$$

$$a = 0{\cdot}1 \text{ metre}$$

$$\alpha = \frac{\pi}{6}$$

and $$\omega = \frac{\pi}{10}$$

Work done per cycle,

$$W = \pi F_0 a \sin \alpha$$

$$W = \pi \times 10 \times 0{\cdot}1 \times 0{\cdot}5$$

$$W = 0{\cdot}5 \, \pi \text{ joules per cycle}$$

or the angular frequency

$$\omega = \frac{\pi}{10} \text{ radians/s}$$

$\therefore$ Time period $$T = \frac{2\pi}{\omega}$$

$$= \frac{2\pi \times 10}{\pi} = \mathbf{20 \text{ seconds.}}$$

(*i*) As the time taken to complete one cycle is 20 seconds.

The work done in 20 seconds

= work done per cycle = $\mathbf{0{\cdot}5 \, \pi}$ **joules.**

(*ii*) In one minute, three cycles are completed.

Therefore, work done in one minute

$= 3 \times 0{\cdot}5 \, \pi$ joules = $\mathbf{1{\cdot}5 \, \pi}$ **joules.**

Example 1.9. *Show that the mean kinetic and potential energies of non-dissipative simple harmonic vibrating systems are equal.*

For free vibration in the absence of damping, the displacement at any instant is given by

$$y = a \sin \omega t$$

$$\frac{dy}{dt} = a\omega \cos \omega t$$

Kinetic energy $= \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$

$$KE = \tfrac{1}{2} m (a^2\omega^2 \cos^2 \omega t)$$

or $$KE = \tfrac{1}{2} Ka^2 \cos^2 \omega t \quad \ldots (1)$$

Here $K = m\omega^2$

or $$\omega^2 = \frac{K}{m}$$

Here K is the force per unit displacement

Potential energy $= \frac{1}{2} Ky^2$

$$PE = \tfrac{1}{2} Ka^2 \sin^2 \omega t \quad \ldots (2)$$

Total kinetic energy for one complete cycle

$$= \int_0^T \tfrac{1}{2} Ka^2 \cos^2 \omega t \, dt$$

$$= \tfrac{1}{4} Ka^2 T \quad \ldots (3)$$

Total potential energy for one complete cycle

$$= \int_0^T \tfrac{1}{2} Ka^2 \sin^2 \omega t \, dt$$

$$= \tfrac{1}{4} Ka^2 T \quad \ldots (4)$$

Hence the mean potential and kinetic energies are equal.

Example 1.10. *The equation of a progressive wave is given by*

$$y = 10 \sin (0{\cdot}5x - 200\, t)$$

Where x and y are in cm *and t is in* seconds

Calculate the

(a) amplitude

(b) frequency and

(c) velocity

of the wave. Also calculate the maximum velocity of a particle of the medium.

(Bhagalpur, 1990)

$$y = 10 \sin (0{\cdot}5\, x - 200\, t)$$

$$y = -10 \sin (200\, t - 0{\cdot}5\, x) \quad \ldots (i)$$

The displacement is

$$y = a \sin (\omega t + \alpha) \quad \ldots (ii)$$

Comparing *(i)* and *(ii)*

$$a = -10 \text{ cm}$$

or $$a = 10 \text{ cm} \quad \ldots (iii)$$

$$\omega = 200$$

$$\omega = 2\pi n$$

Frequency $$n = \frac{\omega}{2\pi}$$

$$= \frac{200}{2\pi} = \frac{\mathbf{100}}{\pi} \textbf{ Hz} \quad \ldots (iv)$$

From equation *(i)*

$$y = -10 \sin (0{\cdot}5) (400\, t - x) \quad \ldots (v)$$

Also

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \ldots (vi)$$

Comparing *(v)* and *(vi)*

$$v = \mathbf{400 \text{ cm/s}}$$

and $$\frac{2\pi}{\lambda} = 0{\cdot}5$$

$$\lambda = 4\pi \text{ cm}$$

$$\lambda = \mathbf{12{\cdot}56 \text{ cm}}$$

Maximum velocity

$$\frac{dy}{dt} = a\omega$$

$$= 10 \times 200$$

$$= \mathbf{2000 \text{ cm/s.}}$$

Example 1.11. *Write down the equation for a wave travelling along the negative Z direction and having an amplitude* 0·01 m, *frequency* 550 Hz *and speed* 330 m/s. *How would the equation change if a wave with the same parameters was travelling along the positive Z direction. Justify your answer.*

(I.A.S.)

Here $y = a \sin \frac{2\pi}{\lambda} (vt - \ldots)$

The wave is travelling in the positive z direction.

Here $a = 0{\cdot}01$ m

$\nu = 550$ Hz

$v = 330$ m/s

$$\lambda = \frac{v}{\nu} = \frac{330}{550}$$

$$\lambda = 0{\cdot}6 \text{ m}$$

$$\therefore \quad y = 0{\cdot}01 \sin\left(\frac{2\pi}{0{\cdot}6}\right)(330\,t - z) \qquad \ldots (i)$$

The wave is travelling along + z direction.

For a wave travelling in $-z$ direction,

$$y = 0{\cdot}01 \sin\left(\frac{2\pi}{0{\cdot}6}\right)(330t + z) \qquad \ldots (ii)$$

For $t = 0$, from equation (i)

$$y = -0{\cdot}01 \sin\left(\frac{2\pi}{0{\cdot}6}\right)z \qquad \ldots (iii)$$

The wave is travelling along $+z$ direction.

Similarly for t=0, from equation *(ii)*

$$y = +0{\cdot}01 \sin\left(\frac{2\pi}{0{\cdot}6}\right)z \qquad \ldots . (iv)$$

The wave is travelling along $-z$ direction.

Example 1.12. *A particle of mass* 0·8 kg *is executing S.H.M. with an amplitude of* 1 m *and periodic time* $\frac{11}{7}$ s. *Calculate the velocity and the kinetic energy of the particle when the displacement is* 0·6 m.

Here, $m = 0{\cdot}8$ kg

$y = 0{\cdot}6$ m

$a = 1$ m

$T = \frac{11}{7}$ s

$$\omega = \frac{2\pi}{T} = \frac{2 \times 22 \times 7}{7 \times 11} = 4 \text{ radians/s}$$

$y = a \sin \omega t$

$$\sin \omega t = \frac{y}{a} = \frac{0{\cdot}6}{1} = 0{\cdot}6$$

$$\cos \omega t = \sqrt{1 - \sin^2 \omega t} = \sqrt{1 - (0{\cdot}6)^2} = 0{\cdot}8$$

(*i*) Velocity

$$\frac{dy}{dt} = a\omega \cos \omega t = 1 \times 4 \times 0{\cdot}8 \qquad \ldots (i)$$

$$= 3{\cdot}2 \text{ m/s}$$

(*ii*) Kinetic energy

$$= \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 = \frac{1}{2} \times 1 \times (3{\cdot}2)^2$$

$$= 5{\cdot}12 \text{ J.}$$

1.9. Oscillations with One Degree of Freedom

A pendulum of a clock, a loaded spring and LC circuit have one degree of freedom. In the case of simple pendulum, the swing depends upon the angular

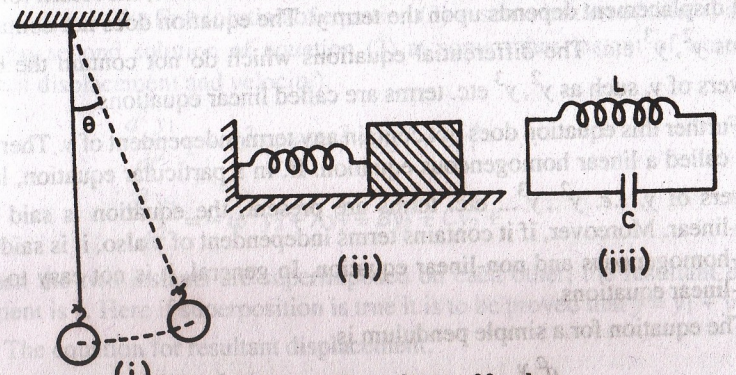


Fig. 1.6. Systems with one degree of freedom.

displacement made by the string with the vertical direction. In the case of a loaded spring, the displacement of the mass, and in the case of LC circuit the charge on the condenser plate describe the nature of the oscillations (Fig. 1.6). These oscillations take place about the mean position. All these systems have one degree of freedom.

For an oscillatory system with one degree of freedom, the displacement of the "moving particle" depends upon the simple harmonic relation

$$y = a \sin (\omega t + \alpha).$$

In the case of a simple pendulum, at any instant of time t, y refers to the displacement of the bob from the mean position. In the case of a loaded spring at any instant of time t, y refers to the displacement of the oscillating mass from the mean position. In the case of the oscillatory LC circuit, at any instant of time t, y refers to the charge on the condenser.

An oscillatory system will continue to oscillate for an infinite time according to the equation

$$y = a \sin (\omega t + \alpha).$$

Damped oscillations. In actual practice the oscillatory system experiences frictional or resistive forces. Due to these reasons the oscillations get damped. In the case of a pendulum, the amplitude decreases due to the resistance offered by air. In the case of LC circuit, the resistance of the circuit produces damping.

1.10. Linearity and Superposition Principle

The differential equation of SHM is given by

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

In this equation, d^2y/dt^2 is proportional to $-y$. The time periods for a simple pendulum, mass and springs, bar pendulum, Kater's pendulum, LC circuit etc. are derived on the basis of this equation. In all these cases, the return force per unit displacement depends upon the term y. The equation does not contain the terms y^2, y^3 etc. The differential equations which do not contain the higher powers of y, such as y^2, y^3 etc. terms are called linear equations.

Further this equation does not contain any term independent of y. Therefore, it is called a linear homogeneous equation. If, in a particular equation, higher powers of y, *i.e.* y^2, y^3... etc. terms are present, the equation is said to be non-linear. Moreover, if it contains terms independent of y also, it is said to be non-homogeneous and non-linear equation. In general, it is not easy to solve non-linear equations.

The equation for a simple pendulum is

$$M\frac{d^2y}{dt^2} = -Mg\sin\theta \quad \ldots(1)$$

It is assumed that amplitude is small and $\sin\theta = \theta$.

Equation $M\dfrac{d^2y}{dt^2} = -Mg\,\theta$ is linear.

However, $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} \ldots$

Taking this value of $\sin\theta$

$$M\frac{d^2y}{dt^2} = -Mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right) \quad \ldots(2)$$

Equation (2) is non-linear for it contains terms of θ^3, θ^5 etc.

Linear homogeneous equations. One of the important properties of linear homogeneous equations is that the sum of any two solutions is a solution by itself. This property is not true in the case of non-linear equations.

Consider the differential equation,

$$\frac{d^2y}{dt^2} = -\omega^2 y + Ay^2 + By^3 + Cy^4 \text{ etc.} \quad \ldots(3)$$

If the values of the constants, A, B, C etc. in equation (3) are zero or they can be taken sufficiently near to zero, then equation (3) becomes

$$\frac{d^2y}{dt^2} = -\omega^2 y \quad \ldots(4)$$

This equation is linear and homogeneous.

If in equation (3), constants A, B, C etc. are not zero, then the equation is non-linear.

Suppose, y_1 is the first solution of equation (3) at some instant of time t_1 and y_2 is the second solution of equation (3) at some other instant of time t_2 (different displacement and velocity).

$$\therefore \quad \frac{d^2y_1}{dt^2} = -\omega^2 y_1 + Ay_1^2 + By_1^2 + Cy_1^4 + \ldots \quad \ldots(5)$$

$$\frac{d^2y_2}{dt^2} = -\omega^2 y_2 + Ay_2^2 + By_2^3 + Cy_2^4 + \ldots \quad \ldots(6)$$

When the two instants are superimposed on each other, the resultant displacement is y. Here if superposition is true it is to be proved that $y = y_1 + y_2$.

$\therefore$ The equation for resultant displacement,

$$\frac{d^2y}{dt^2} = \frac{d^2}{dt^2}(y_1 + y_2)$$

$$= -\omega^2(y_1 + y_2) + A(y_1 + y_2)^2 + B(y_1 + y_2)^2 + C(y_1 + y_2)^4 + \ldots \quad \ldots(7)$$

Adding equations (5) and (6)

$$\frac{d^2y_1}{dt^2} + \frac{d^2y_2}{dt^2} = -\omega^2(y_1 + y_2) + A(y_1^2 + y_2^2) + B(y_1^3 + y_2^3) + C(y_1^4 + y_4^4) + \ldots \quad \ldots(8)$$

The equations (7) and (8) are identical, only if

$$\frac{d^2}{dt^2}(y_1 + y_2) = \frac{d^2y_1}{dt^2} + \frac{d^2y_2}{dt^2} \quad \ldots(9)$$

$$-\omega^2(y_1 + y_2) = -\omega^2 y_1 - \omega^2 y_2 \quad \ldots(10)$$

$$A(y_1^2 + y_2^2) = A(y_1 + y_2)^2 \quad \ldots(11)$$

$$B(y_1^3 + y_2^3) = B(y_1 + y_2)^3 \quad \ldots(12)$$

$$C(y_1^4 + y_2^4) = C(y_1 + y_2)^4 \quad \ldots(13)$$

Equations (9) and (10) are true. But equations (11), (12) and (13) are true only, if

$$A = 0, \qquad B = 0. \qquad C = 0$$

When A, B, C etc. are zero, the equations become linear. Hence superposition principle is true only in the case of homogeneous linear equations. Also the sum of any two solutions is also a solution of the homogeneous linear equation.

All harmonic oscillators given in equations (9) and (10) obey superposition principle.

1.11. Simple Pendulum

A simple pendulum consists of a light string supporting a small sphere and fixed firmly at its upper end. An ideal simple pendulum should consist of a heavy particle suspended by means of a weightless, inextensible, flexible string from a rigid support.

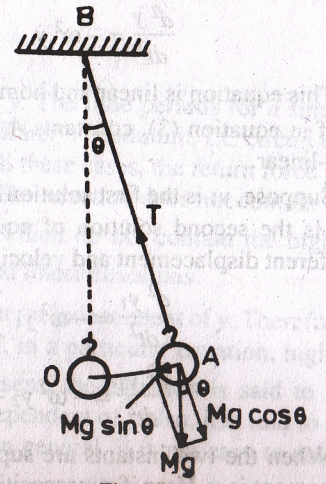


Fig. 1.7.

Let a pendulum be displaced from its mean position O and allowed to oscillate (Fig. 1.7). Suppose at any instant of time t, it is at A. The force acting upon the bob vertically downward = Mg. Resolve Mg into two rectangular components :

(1) Force along the string = $Mg \cos \theta$

(2) Force perpendicular to the string = $Mg \sin \theta$

Let the tension in the string be T. The component $Mg \cos \theta$ balances the tension T

$$Mg \cos \theta = T$$

Thus the only force acting on the oscillating particle is $-Mg \sin \theta$.

$$\therefore \quad F = -Mg \sin \theta$$

(–ve sign shows that the acceleration is directed towards the mean position)

According to Taylor's series of expansion

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots .$$

For small angular displacement θ,

$$\sin \theta = \theta$$

Tangential force $\quad F = -Mg\theta$

The linear displacement $y = l\theta$

Acceleration $\quad \dfrac{d^2y}{dt^2} = l\dfrac{d^2\theta}{dt^2}$

$$\therefore \quad \text{Force} = Ml\frac{d^2\theta}{dt^2}$$

From Newton's second law

$$Ml\frac{d^2\theta}{dt^2} = -Mg\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \qquad \ldots (1)$$

This equation is similar to the equation of simple harmonic motion

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \ldots (2)$$

From (1) and (2)

$$\omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

Time period, $\quad T = \dfrac{2\pi}{\omega}$

$$\therefore \quad T = 2\pi\sqrt{\frac{l}{g}} \qquad \ldots (3)$$

[**Bob of large size.** In the case of a simple pendulum, if the size of the bob is large, a correction has to be applied. In this case

$$t = 2\pi\sqrt{\frac{l + (\frac{2}{5}r^2)/l}{g}}$$

Here $l + \left(\dfrac{\frac{2}{5}r^2}{l}\right)$ represents the equivalent length of a simple pendulum.]

Example 1.13. *A disc of radius* 15 cm *is executing small oscillations about a point on its rim. Find the time period of oscillation.* (I.A.S.)

Moment of inertia of the disc about an axis passing through its centre of gravity and perpendicular to its plane,

$$I = \frac{MR^2}{2}$$

$$MK^2 = \frac{MR^2}{2}$$

$$K^2 = R^2/2$$

Time period of oscillation,

$$T = 2\pi\sqrt{\frac{K^2 + l^2}{lg}}$$

Here $l = R$

$$T = 2\pi\left[\frac{(R/2)+R^2}{Rg}\right]^{\frac{1}{2}}$$

$$T = 2\pi\left[\frac{3R}{2g}\right]^{\frac{1}{2}}$$

Here $R = 15 \text{ cm} = 0{\cdot}15 \text{ m}$

$g = 9{\cdot}8 \text{ m/s}^2$

$$T = 2\pi\left[\frac{3\times 0{\cdot}15}{2\times 9{\cdot}8}\right]^{\frac{1}{2}}$$

$T = 0{\cdot}95$ s.

1.12. Simple Harmonic Oscillations of a Mass between two Springs

Consider two springs S_1 and S_2 each having a length l in the relaxed position. Mass M is placed midway between the two springs on a frictionless surface [Fig. 1.8 (*i*)]. One end of the spring S_1 is attached to a rigid wall at A and the other end is attached to the mass M. Similarly one end of the spring S_2 is

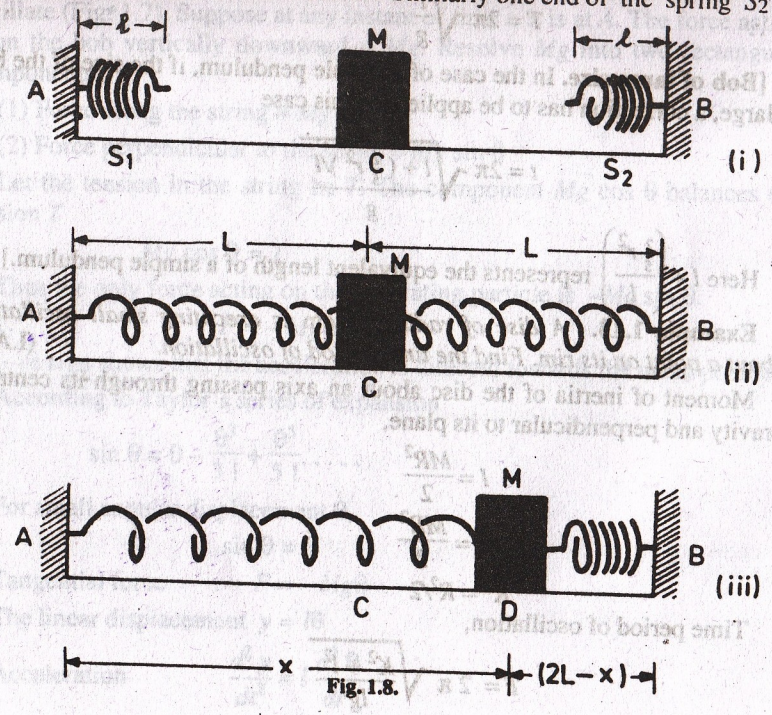


Fig. 1.8.

attached to a rigid wall at B and the other end is connected to the mass M. Here $AC = BC = L$ [Fig. 1.8 (*ii*)]. At C the mass is equally pulled by both the springs and it is the equilibrium position.

When the mass M is displaced from its equilibrium position and left, it executes simple harmonic oscillations. Let, at any instant, D be the displaced position of the mass M.

Here $AD = x$, and $BD = (2L - x)$

Let the tension per unit displacement in the spring be K. The displacement of the spring of S_1 is $(x - l)$ and it exerts a force $= K[x - l]$ in the direction DA. The displacement of the spring S_2 is $(2L-x-l)$ and it exerts a force $= K[2L-x-l]$ in the direction DB.

The resultant force on the mass M

$= K\{2L - x - l] - K[x - l]$ in the direction DB

$= -2K[x - L]$ in the direction DB

According to Newton's second law of motion

$$F = M\frac{d^2x}{dt^2} = -2K[x - L] \quad \ldots (1)$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{2K}{M}[x - L]$$

or $$\frac{d^2x}{dt^2} + \frac{2K}{M}[x - L] = 0 \quad \ldots (2)$$

Taking the displacement from the mean position

$$x - L = y$$

Differentiating twice,

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2}$$

Substituting these values in equation (2)

$$\frac{d^2y}{dt^2} + \frac{2K}{M}y = 0 \quad \ldots (3)$$

This equation is similar to the equation of simple harmonic motion

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \ldots (4)$$

From equations (3) and (4)

$$\omega^2 = \frac{2K}{M}$$

$$\omega = \sqrt{\frac{2K}{M}}$$

Time period $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{2K}} \quad \ldots (5)$$

Thus, the mass M executes simple harmonic oscillations and the time period is given by equation (5). Knowing the values of M and K, the time period can be calculated.

1.13. Simple Harmonic Oscillations of a Loaded Spring

Consider a spring S whose upper end is fixed to a rigid support and the lower end is attached to a mass M (Fig. 1.9). In the equilibrium position, the mass is at A. When the mass is displaced downwards and left, it oscillates simple harmonically in the vertical direction.

Suppose at any instant the mass is at B. The distance $AB=y$. Let the tension per unit displacement of the spring be K.

$\therefore$ Force exerted by the spring $=Ky$

According to Newton's second law

$$\text{Force} = M\frac{d^2y}{dt^2} = -Ky$$

[– ve sign shows that the force is directed upwards]

$$\therefore \quad M\frac{d^2y}{dt^2} + Ky = 0$$

$$\frac{d^2y}{dt^2} + \left(\frac{K}{M}\right)y = 0 \qquad \ldots (1)$$

This equation is similar to the equation of simple harmonic motion,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \ldots (2)$$

Comparing (1) and (2)

$$\omega^2 = \frac{K}{M}$$

$$\omega = \sqrt{\frac{K}{M}}$$

Time period $\quad T = \dfrac{2\pi}{\omega}$

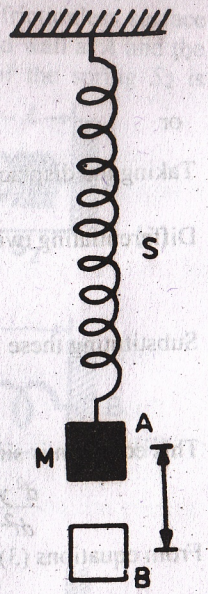


Fig. 1.9.

or $\quad T = 2\pi\sqrt{\dfrac{M}{K}} \qquad \ldots (3)$

Knowing the values of M and K, the value of T can be calculated.

Determination of K. To determine the value of tension per unit displacement of the spring, a small mass m is attached to the free end of the spring. The

increase in length of the spring is noted. Let it be x.

Then, $\quad K = \left(\dfrac{mg}{x}\right)$

Substituting the value of K in equation (3),

$$T = 2\pi\sqrt{\frac{Mx}{mg}} \qquad \ldots (4)$$

It is to be noted that $\dfrac{mg}{x}$ is constant for a given spring.

Example 1.14. *A spring is hung vertically and loaded with a mass of* 100 grams *and allowed to oscillate. Calculate (i) the time period and (ii) the frequency of oscillation. When the spring is loaded with* 200 grams *it extends by* 10 cm.

Here $\quad M = 100$ grams

$m = 200$ grams

$x = 10$ cm

$g = 980$ cm/s^2

(*i*) $\quad T = 2\pi\sqrt{\dfrac{Mx}{mg}}$

$$T = 2\pi\sqrt{\frac{100\times 10}{200\times 980}}$$

$$T = \frac{2\pi}{14} = \mathbf{0{\cdot}449\ s.}$$

(*ii*) $\quad \text{Frequency} = n = \dfrac{1}{T} = \dfrac{1}{0{\cdot}449}$,

$n = \mathbf{2{\cdot}22\ hertz.}$

Example 1.15. *The scale of a spring balance reading from* 0 – 10 kg *is* 0·25 m. *A body suspended from a balance oscillates with a frequency of* $\dfrac{10}{\pi}$ hertz.

Calculate the mass of the body attached to the spring.

Here $\quad m = 10$ kg

$x = 0{\cdot}25$ m

$M = ?$

$g = 9{\cdot}8$ m/s^2

$n = \dfrac{10}{\pi}$ hertz

or $$T = \frac{1}{n} = \frac{\pi}{10} \text{ s}$$

$$T = 2\pi\sqrt{\frac{Mx}{mg}}$$

or $$M = \frac{T^2 mg}{4\pi^2 x}$$

$$M = \frac{\pi^2 \times 10 \times 9{\cdot}8}{100 \times 4\pi^2 \times 0{\cdot}25}$$

$$M = \mathbf{0{\cdot}98 \text{ kg}}.$$

Example 1.16. *A body of mass* 0·5 kg *is suspended from a spring of negligible mass and it stretches the spring by* 0.7 m. *For a displacement of* 0·03 m *it has a downward velocity* 0·4 m/s. *Calculate (i) the time period; (ii) the frequency and (iii) amplitude of vibration of the spring.*

Here $m = 0{\cdot}5$ kg

$x = 0{\cdot}07$ m

$M = 0{\cdot}5$ kg

$g = 9{\cdot}8$ m/s^2

(*i*) Time period $$T = 2\pi\sqrt{\frac{Mx}{mg}}$$

$$T = 2\pi\sqrt{\frac{0{\cdot}5 \times 0{\cdot}07}{0{\cdot}5 \times 9{\cdot}8}}$$

$$T = \mathbf{0{\cdot}5311 \text{ s}}.$$

(*ii*) Frequency $$n = \frac{1}{T} = \frac{1}{0{\cdot}5311}$$

$$n = 1{\cdot}882 \text{ hertz}$$

Angular frequency $$\omega = \frac{2\pi}{T} = \frac{2\pi}{0{\cdot}5311} = \mathbf{11{\cdot}8 \text{ radians}}.$$

(*iii*) Amplitude

$$y = a \sin \omega t$$

$$\frac{dy}{dt} = a\omega \cos \omega t$$

$$\frac{dy}{dt} = a\omega\sqrt{1 - \frac{y^2}{a^2}}$$

$$\frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

Here $\frac{dy}{dt} = 0{\cdot}4$ m/s

$y = 0{\cdot}03$ m

and $\omega = 11{\cdot}8$ radians/s

$\therefore$ $0{\cdot}4 = 11{\cdot}8\sqrt{a^2 - (0{\cdot}03)^2}$

By simplification

$\therefore$ $a = \mathbf{0{\cdot}04526 \text{ m}}.$

Example 1.17. *A uniform spring of force constant K is cut into two pieces of equal length. What is the force constant of each piece?*

Suppose, Force $= F$

Increase in length $= l$

$$K = \frac{F}{l} \quad \ldots (i)$$

In the second case, the length is half and increase in length for the force $F = l/2$

Force constant for each price

$$K_1 = \frac{F}{l/2}$$

$$K_1 = 2\left[\frac{F}{l}\right]$$

$$K_1 = 2K \quad \ldots (ii)$$

Force constant for each piece = **2K.**

Example 1.18. *Two identical springs each of force constant K are connected as shown in Fig 1·10. Find the force constant of the system.*

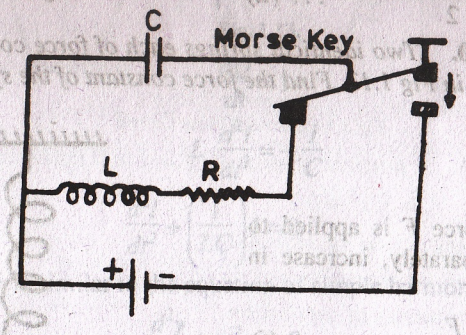


Fig. 1.10.

When the force F is applied to each spring separately, increase in length $= l$

When the force F is applied to the system,

$\therefore$ $$K = \frac{F}{l} \quad \ldots (i)$$

When the force F is applied to the system, force $F/2$ acts on each spring and the increase in length $= \frac{l}{2}$

Force constant of the system

$$K_1 = \frac{F}{l/2} = 2\left[\frac{F}{l}\right]$$

$$K_1 = \mathbf{2K}$$

Example 1.19. *Two identical springs each of force constant K are connected as shown in Fig 1.11. Calculate the force constant of the system.*

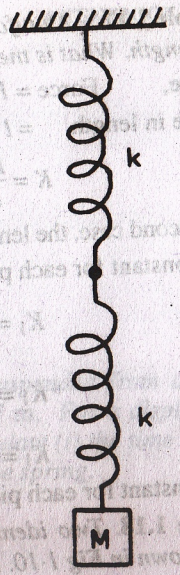


Fig. 1.11.

When the force F is applied to each spring separately, increase in length $= l$

$$\therefore \quad K = \frac{F}{l} \qquad \ldots (i)$$

When the force F is applied to the system, each spring extends by l and increase in length of the system $= 2l$

$$K_1 = \frac{F}{2l} = \frac{1}{2}\left[\frac{F}{l}\right]$$

$$K_1 = \frac{K}{2}. \qquad \ldots (ii)$$

Example 1.20. *Two identical springs each of force constant K are connected as shown in Fig 1.12. Find the force constant of the system.*

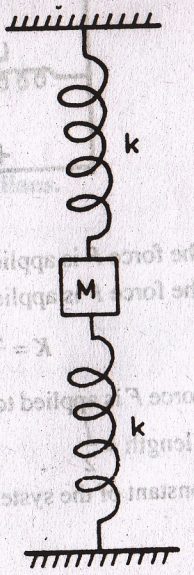


Fig. 1.12.

When the force F is applied to each spring separately, increase in length $= l$

$$\therefore \quad K = \frac{F}{l} \qquad \ldots (i)$$

When the force F is applied to the system, change in length of each spring $= \frac{l}{2}$

Therefore, change in length of the system $= \frac{l}{2}$

$$\therefore \quad K_1 = \frac{F}{l/2} = 2\left[\frac{F}{l}\right]$$

$$K_1 = 2K.$$

1.14. LC CIRCUIT

A condenser is charged by pressing the Morse key and when the Morse key is released, the condenser gets discharged through the inductance L (Fig. 1.13).

Let the charge on the condenser at any instant of time be Q.

$$\therefore \quad \frac{Q}{C} + L\frac{dI}{dt} = 0$$

$$L\frac{dI}{dt} = -\frac{Q}{C} \qquad L\frac{d^2I}{dt^2} = -\frac{1}{C}\frac{dQ}{dt}$$

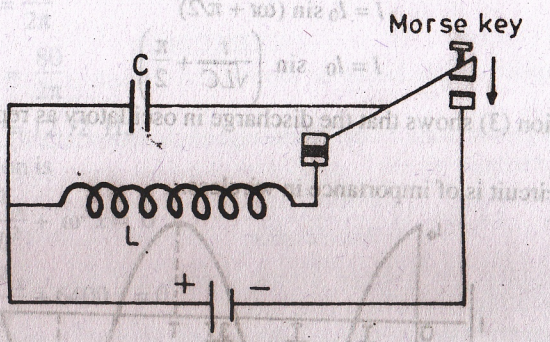


Fig. 1.13.

But $$\frac{dQ}{dt} = I$$

$$\therefore \quad L\frac{d^2I}{dt^2} = -\frac{I}{C}$$

$$\frac{d^2I}{dt^2} + \left(\frac{1}{LC}\right)I = 0 \qquad \ldots (1)$$

This equation is similar to the equation of simple harmonic motion,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \ldots (2)$$

From equations (1) and (2)

$$\omega^2 = \frac{1}{LC}$$

or $$\omega = \sqrt{\frac{1}{LC}}$$

Time period $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$$

Frequency $$n = \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}}$$

This shows that in a circuit containing an inductance and a capacitance, the discharge is oscillatory and undamped.

The general solution of equation (1) is given by

$$I = A \sin(\omega t + \alpha)$$

When $t = 0$

$I = I_0$

and when $\alpha = \pi/2,\ A = I_0$

$$I = I_0 \sin(\omega t + \pi/2)$$

or
$$I = I_0 \sin\left(\frac{t}{\sqrt{LC}} + \frac{\pi}{2}\right) \quad \ldots (3)$$

Equation (3) shows that the discharge in oscillatory as represented in Fig. 1.14

This circuit is of importance in wireless.

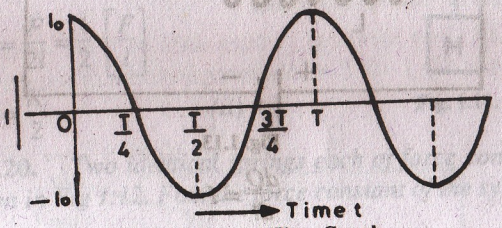


Fig.1.14 Current Time Graph

Example 1.21. *A particle of mass* 10 g *lies in a potential given by*

$$V_x = 32x^2 + 0{\cdot}2$$

Here x is in metres *and* V_x *is in* joules.

Write the equation of motion and solve it.

What is the frequency of oscillation? (IAS, 1991)

$$V_x = 32x^2 + 0{\cdot}2 \quad \ldots (i)$$

Here V_x is the potential energy in joules and x is in metres.

Also potential energy of oscillation,

$$V_x = \frac{1}{2} K x^2 + \text{constant} \quad \ldots (ii)$$

Comparing (*i*) and (*ii*)

$$\frac{1}{2} K = 32$$

$$K = 64 \text{ N/m}$$

Mass, $M = 10 \text{ g} = 10^{-2} \text{ kg}$

K is the force constant

$$\omega = \sqrt{\frac{K}{M}}$$

$$\omega = \left[\frac{64}{10^{-2}}\right]^{\frac{1}{2}}$$

$$\omega = 80 \text{ radians/s}$$

Frequency, $\nu = \dfrac{\omega}{2\pi}$

$$= \frac{80}{2\pi}$$

$$= 12.72 \text{ Hz}$$

Equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \ldots (iii)$$

$$\frac{d^2x}{dt^2} + 6400\, x = 0$$

$$x = A \sin \omega t$$

$$x = A \sin 80\, t \quad \ldots (iv)$$

Example 1.22. *Calculate the frequency of oscillation of hydrogen molecule if its force constant is* $4{\cdot}8 \times 10^2$ N/m *and mass of hydrogen atom is* $1{\cdot}67 \times 10^{-27}$ kg. (Bombay, 1990)

Here, $K = 4{\cdot}8 \times 10^2$ N/m

$m_1 = 1{\cdot}67 \times 10^{-27}$ kg

$m_2 = 1{\cdot}67 \times 10^{-27}$ kg

$m_1 = m_2 = m$

Reduced mass, $\mu = \left[\dfrac{m_1 \times m_2}{m_1 + m_2}\right]$

$$= \frac{m^2}{2m} = \frac{m}{2}$$

$$\mu = \frac{m}{2} = \frac{1{\cdot}67 \times 10^{-27}}{2}$$

$$= 0{\cdot}835 \times 10^{-27} \text{ kg.}$$

Frequency,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

$$= \frac{1}{2\pi}\left[\frac{4{\cdot}8 \times 10^{2}}{0{\cdot}835 \times 10^{-27}}\right]^{\frac{1}{2}}$$

$$1{\cdot}2 \times 10^{14}\ \text{Hz.}$$

EXERCISES

1. Explain simple harmonic motion and discuss its characteristics *(Delhi)*
2. Show that for a body vibrating simple harmonically, the time period is given by *(Lucknow)*

$$t = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

3. Calculate the average kinetic energy and the total energy of a body executing simple harmonic motion. *(Lucknow)*
4. Give examples of the systems that oscillate with one degree of freedom. Explain the term damped oscillations.
5. Show that the superposition principle is valid only in the case of homogeneous, linear vibrations.
6. Obtain the expression for the time period of a simple pendulum. Also derive the expression for the time period for a bob of large size.
7. Discuss the case of simple harmonic oscillations of a mass held between two linear springs.
8. Show that the time period of oscillations of a loaded spring is

$$t = 2\pi\sqrt{\frac{Mx}{mg}}.$$

9. Discuss the LC circuit and calculate the expression for the frequency of oscillations.
10. A particle vibrates simple harmonically with an amplitude of 13 cm. The time period of oscillation is 2π seconds. Calculate the velocity of the vibrating particle at the instant the displacement is 5 cm. Also calculate the frequency of oscillation. **[Ans.** *(i)* 12 cm/s *(ii)* $1/2\pi$ hertz]
11. A simple periodic wave disturbance with an amplitude of 8 units travels a line of particles in the positive x direction. At a given instant, the displacement of a particle 10 cm from the origin is 6 units, and that of a particle 25 cm from the origin is 4 units, both particles being in positive displacement. What is the wavelength of the disturbance. **[Ans.** 290 cm]

CHAPTER 2

Lissajous' Figures

2.1. Lissajous' Figures

When a particle is influenced simultaneously by two simple harmonic motions at right angles to each other, the resultant motion of the particle traces a curve. These curves are called a Lissajous' figures. The shape of the curve depends on the time period, phase difference and the amplitude of the two constituent vibrations. Lissajous' figures are helpful in determining the ratio of the time periods of two vibrations and to compare the frequencies of two tuning forks.

2.2. Composition of Two Simple Harmonic Motions in a Straight Line

Analytical method. Let the two simple harmonic vibrations be represented by the equations

$$y_1 = a_1 \sin(\omega t + \alpha_1) \quad \ldots(1)$$

and

$$y_2 = a_2 \sin(\omega t + \alpha_2) \quad \ldots(2)$$

where y_1 and y_2 are the displacements of a particle due to the two vibrations, a_1 and a_2 are the amplitudes of the two vibrations and α_1 and α_2 are the epoch angles. Here, the two vibrations are assumed to be of the same frequency and hence ω is the same for both. The resultant displacement y of the particle is given by

$$y = y_1 + y_2$$

$$= a_1 \sin(\omega t + \alpha_1) + a_2 \sin(\omega t + \alpha_2)$$

$$= a_1(\sin\omega t\cos\alpha_1 + \cos\omega t\sin\alpha_1) + a_2(\sin\omega t\cos\alpha_2 + \cos\omega t\sin\alpha_2)$$

$$y = (a_1\cos\alpha_1 + a_2\cos\alpha_2)\sin\omega t + (a_1\sin\alpha_1 + a_2\sin\alpha_2)\cos\omega t \quad \ldots(3)$$

Since the amplitudes a_1 and a_2 and the angles α_1 and α_2 are constant, the coefficients of $\sin\omega t$ and $\cos\omega t$ in equation (3) can be substituted by $A\cos\phi$ and $A\sin\phi$

$$A\cos\phi = a_1\cos\alpha_1 + a_2\cos\alpha_2 \quad \ldots(4)$$

and $$A \sin \phi = a_1 \sin \alpha_1 + a_2 \sin \alpha_2 \quad \ldots(5)$$

Squaring and adding equations (4) and (5)

$$A^2(\sin^2 \phi + \cos^2 \phi) = a_1^2 (\sin^2 \alpha_1 + \cos^2 \alpha_1) + a_2^2 (\sin^2 \alpha_2 + \cos^2 \alpha_2) + 2a_1 a_2 [\cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2]$$

$$\therefore \quad A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos (\alpha_1 - \alpha_2) \quad \ldots(6)$$

Dividing equation (5) by (4)

$$\tan \phi = \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2} \quad \ldots(7)$$

Equations (6) and (7) give the values of A and ϕ in terms of a_1, a_2, α_1 and α_2.

$$\therefore \quad y = A \cos \phi \sin \omega t + A \sin \phi \cos \omega t$$

$$y = A \sin (\omega t + \phi) \quad \ldots(8)$$

Equation (8) is similar to the original equations (1) and (2). The amplitude of the resultant vibration is A and epoch angle is ϕ. The time period of the resultant vibration is the same as the original vibrations. The values of A and ϕ are given by equations (6) and (7). Thus, the resultant of two simple harmonic vibrations of the same period and acting in the same line is also a simple harmonic vibration with a resultant amplitude A and epoch angle ϕ.

Special case. If $\alpha_1 = \alpha_2 = \alpha$, then

$$A = a_1 + a_2 \text{ and } \phi = \alpha$$

or $$y = (a_1 + a_2) \sin (\omega t + \alpha) \quad \ldots(9)$$

Graphical method. Let OP and OQ represent the radius vectors at any instant (Fig. 2.1).

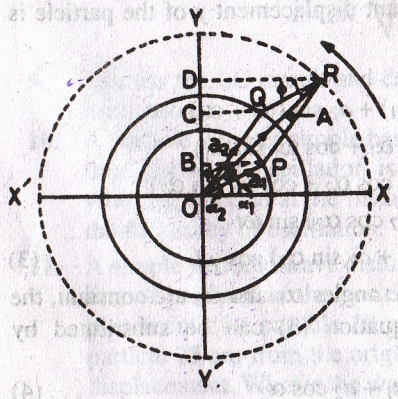


Fig. 2.1.

$$\angle POX = \alpha_1$$

and $$\angle QOX = \alpha_2$$

$$OP = a_1, \; OQ = a_2$$

and $$OR = A.$$

The displacements of a particle along the line YY' are given by

$OB = y_1 = a_1 \sin(\omega t + \alpha_1)$ and

$OC = y_2 = a_2 \sin(\omega t + \alpha_2)$

where a_1 and a_2 are the amplitudes and α_1 and α_2 are the epoch angles.

The resultant of the vectors OP and OQ is given by OR.

According to the parallelogram law of vectors

$$OR^2 = A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos \angle QOP$$

$$OR^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos (\alpha_2 - \alpha_1) \quad \ldots(1)$$

From the Δs OPB and OQC

$$y_1 = OB = a_1 \sin \alpha_1$$

and $$y_2 = OC = a_2 \sin \alpha_2$$

But the projection of OQ on the Y-axis is equal to the projection of PR on the y-axis.

$$\therefore \quad y_2 = OC = BD = a_2 \sin \alpha_2$$

Resultant displacement

$$y = OB + BD = y_1 + y_2$$

$$= a_1 \sin \alpha_1 + a_2 \sin \alpha_2 \quad \ldots(2)$$

Similarly the projections of OP and OQ on the X-axis will be $a_1 \cos \alpha_1$ and $a_2 \cos \alpha_2$.

$$\therefore \quad RD = a_1 \cos \alpha_1 + a_2 \cos \alpha_2$$

In the ΔORD, $\angle ORD = \phi$

and $$\tan \phi = \frac{OD}{RD} = \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2} \quad \ldots(3)$$

Thus the diagonal OR represents completely the resultant of two collinear simple harmonic motions. The resultant amplitude A and the epoch angle ϕ are given by the equations (1) and (3). The resultant displacement is represented by the equation

$$y = A \sin (\omega t + \phi) \quad \ldots(4)$$

If $\alpha_1 = \alpha_2$, then $\alpha_1 - \alpha_2 = 0$ and $A = a_1 + a_2$ and $\tan \alpha = \tan \phi$.

Example 2.1. *Two SHMs acting simultaneously on a particle are given by*

$$y_1 = \sin \left(\omega t + \frac{\pi}{3} \right)$$

$$y_2 = 2 \sin \omega t$$

Find the equation of the resultant vibration.

Here $$y_1 = \sin \left(\omega t + \frac{\pi}{3} \right) \quad \ldots(1)$$

and $$y_2 = 2 \sin \omega t \quad \ldots(2)$$

These equations are similar to

$$y_1 = a_1 \sin (\omega t + \alpha_1)$$

$$y_2 = a_2 \sin (\omega t + \alpha_2)$$

$$\therefore \quad a_1 = 1, a_2 = 2, \alpha_1 = \frac{\pi}{3}, \alpha_2 = 0$$

The resultant vibration is given by

$$y = A \sin(\omega t + \phi)$$

Here $$A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\alpha_1 - \alpha_2)}$$

$$A = \sqrt{1 + 4 + 2 \times 1 \times 2 \cos\left(\frac{\pi}{3}\right)}$$

$$A = \sqrt{7} = 2{\cdot}645$$

$$\tan\phi = \frac{a_1 \sin\alpha_1 + a_2 \sin\alpha_2}{a_1 \cos\alpha_1 + a_2 \cos\alpha_2}$$

$$\tan\phi = \frac{1 \times 0{\cdot}866 + 0}{1 \times 0{\cdot}5 + 2 \times 1} = \frac{0{\cdot}866}{2{\cdot}5}$$

$$\phi = 19{\cdot}1^0 = \frac{19{\cdot}1 \times \pi}{180} \text{ radian} = 0{\cdot}1061\,\pi \text{ radian}$$

$\therefore$ $$y = 2{\cdot}645 \sin(\omega t + 0{\cdot}1061\,\pi).$$

Example 2.2. *Two SHMs acting simultaneously on a particle are given by the equations*

$$y_1 = 2 \sin(\omega t + \pi/6)$$

$$y_2 = 3 \sin(\omega t + \pi/3)$$

Calculate (i) amplitude, (ii) phase constant and (iii) time period of the resultant vibration.

Here $$y_1 = 2 \sin(\omega t + \pi/6) \quad \ldots(1)$$

and $$y_2 = 3 \sin(\omega t + \pi/3) \quad \ldots(2)$$

These equations are similar to the equations,

$$y_1 = a_1 \sin(\omega t + \alpha_1) \quad \ldots(3)$$

and $$y_2 = a_2 \sin(\omega t + \alpha_2) \quad \ldots(4)$$

The resultant vibration is,

$$y = A \sin(\omega t + \phi) \quad \ldots(5)$$

Here $$A^2 = a_1^2 + \alpha_2^2 + 2\alpha_1 a_2 \cos(\alpha_1 - a_2) \quad \ldots(6)$$

and $$\tan\phi = \frac{a_1 \sin\alpha_1 + a_2 \sin\alpha_2}{a_1 \cos\alpha_1 + a_2 \cos\alpha_2} \quad \ldots(7)$$

Here $$a_1 = 2, \quad a_2 = 3$$

$$\alpha_1 = \frac{\pi}{6}, \alpha_2 = \frac{\pi}{3}$$

(*i*) Resultant amplitude

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\alpha_1 - \alpha_2)}$$

$$A = \sqrt{4 + 9 + 2 \times 2 \times 3 \cos\left(-\frac{\pi}{6}\right)}$$

$$A = 4{\cdot}939.$$

(*ii*) $$\phi = \tan^{-1}\left[\frac{2 \times 0{\cdot}5 + 3 \times 0{\cdot}866}{2 \times 0{\cdot}866 + 3 \times 0{\cdot}5}\right]$$

$$\phi = \tan^{-1}[1{\cdot}114]$$

$$\phi = 48{\cdot}1^\circ$$

$$\phi = \frac{48{\cdot}1 \times \pi}{180} = \frac{4\pi}{15} \text{ (app.)}$$

$\therefore$ Phase constant $= \left(\omega t + \frac{4\pi}{15}\right)$.

(*iii*) The resultant time period is the same as the individual time periods.

2.3. Composition of a Number of Simple Harmonic Motions Acting along a Line

When a particle is influenced by a number of collinear simple harmonic vibrations of different amplitudes and different epoch angles, the resultant vibration can be obtained by the polygon method. In Fig. 2.2, *OP*, *PQ* and *QR* are the vectors representing three simple harmonic vibrations. The amplitude of the individual vibrations are a_1, a_2 and a_3 and α_1, α_2 and α_3 are the corresponding epoch angles. The vector *OR* represents the resultant vibration. The amplitude and the epoch angle of the resultant vibration are A and ϕ respectively. Proceeding in the same way this method can be employed when a number of collinear simple harmonic vibrations influence the same particle of the medium.

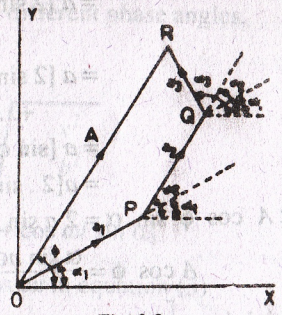


Fig. 2.2.

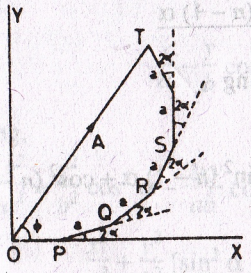


Fig. 2.3.

Let n simple harmonic vibrations of the same amplitude a and epoch angles $0, 2\alpha, 4\alpha \ldots 2(n-1)\alpha$ influence a vibrating particle (Fig. 2.3). If the displacements of the vibrating particle are considered along the y-axis, the individual dis-

placements are given by

$$y_1 = a \sin(\omega t - 0)$$

$$y_2 = a \sin(\omega t - 2\alpha)$$

Let A be the amplitude of the resultant vibration and ϕ the epoch angle. Then $Y = A \sin(\omega t - \phi)$.

The projections of the individual vectors OP, PQ, QR etc. on the y-axis are given by

$$0, a \sin 2\alpha, \alpha \sin 4\alpha \text{ etc.}$$

Similarly the projections on the X-axis are given by

$$a, a \cos 2\alpha, a \cos 4\alpha \text{ etc.}$$

If OT represents the resultant vector, then $A \sin \phi$ will give the projection along the Y- axis and $A \cos \phi$ gives the projection along the X-axis.

$$\therefore \quad A \sin \phi = 0 + a \sin 2\alpha + a \sin 4\alpha + \ldots + a \sin 2(n-1)\alpha$$

$$= a[\sin 2\alpha + \sin 4\alpha + \ldots + \sin 2(n-1)\alpha] \quad \ldots (1)$$

Similarly,

$$A \cos \phi = a + a \cos 2\alpha + a \cos 4\alpha + \ldots + a \cos 2(n-1)\alpha$$

$$= a[1 + \cos 2\alpha + \cos 4\alpha + \ldots + \cos 2(n-1)\alpha] \quad \ldots (2)$$

Multiplying equation (2) by $2 \sin \alpha$

$$2A \cos \phi \sin \alpha = 2a \sin \alpha [1 + \cos 2\alpha + \cos 4\alpha + \ldots + \cos 2(n-1)\alpha]$$

$$= a[2 \sin \alpha + 2 \cos 2\alpha \cdot \sin \alpha + 2 \cos 4\alpha \sin \alpha + \ldots + 2 \cos 2(n-1)\alpha \sin \alpha]$$

$$= a[2 \sin \alpha + (\sin 3\alpha - \sin \alpha) + (\sin 5\alpha - \sin 3\alpha) + \ldots + \{\sin (2n-1)\alpha - \sin (2n-3)\alpha\}]$$

$$= a[\sin \alpha + \sin (2n-1)\alpha]$$

$$= a[2 \cdot \sin n\alpha \cdot \cos (n-1)\alpha]$$

$$\therefore 2A \cos \phi \sin \alpha = 2a \sin n\alpha \cdot \cos (n-1)\alpha$$

$$\therefore \quad A \cos \phi = \frac{\alpha \sin n\alpha \cdot \cos (n-1)\alpha}{\sin \alpha} \quad \ldots (3)$$

Multiplying equation (1) by $2 \sin \alpha$ and proceeding in a similar way, it can be shown that

$$A \sin \phi = \frac{a \sin n\alpha \cdot \sin (n-1)\alpha}{\sin \alpha} \quad \ldots (4)$$

Squaring equations (3) and (4) and adding

$$A^2 (\sin^2 \phi + \cos^2 \phi) = A^2$$

$$= \frac{a^2 \sin^2 n\alpha}{\sin^2 \alpha} [\sin^2 (n-1)\alpha + \cos^2 (n-1)\alpha]$$

$$= \frac{a^2 \sin^2 n\alpha}{\sin^2 \alpha} \quad \ldots (5)$$

$$\therefore \quad A = \frac{a \sin n\alpha}{\sin \alpha} \quad \ldots (6)$$

Dividing equation (4) by (3)

$$\frac{A \sin \phi}{A \cos \phi} = \tan \phi = \frac{a \sin n\alpha \cdot \sin (n-1)\alpha \cdot \sin \alpha}{\sin \alpha \cdot a \sin n\alpha \cdot \cos (n-1)\alpha}$$

$$= \tan (n-1)\alpha$$

or $$\phi = (n-1)\alpha \quad \ldots (7)$$

where ϕ represents the epoch angle for the resultant vibration, n is the number of simple harmonic vibrations influencing a particle and α is half the increase in the epoch angle between successive vibrations.

2.4. Composition of Two Simple Harmonic Vibrations of Equal Time Periods Acting at Right Angles

Let $$x = a \sin (\omega t + \alpha) \quad \ldots (1)$$

and $$y = b \sin \omega t \quad \ldots (2)$$

represent the displacements of a particle along the X- and Y-axes due to the influence of two simple harmonic vibrations acting simultaneously on a particle in perpendicular directions. Here, the two vibrations are of the same time period but are of different amplitudes and different phase angles.

From equation (2), $$\sin \omega t = \frac{y}{b}$$

$$\therefore \quad \cos \omega t = \sqrt{1 - \frac{y^2}{b^2}}$$

From equation (1), $$\frac{x}{a} = [\sin \omega t \cos \alpha + \cos \omega t \sin \alpha] \quad \ldots (3)$$

Substituting the values of $\sin \omega t$ and $\cos \omega t$ in equation (3)

$$\frac{x}{a} = \left[\frac{y}{b} \cos \alpha + \sqrt{1 - \frac{y^2}{b^2}} \cdot \sin \alpha\right]$$

or $$\frac{x}{a} - \frac{y}{b} \cos \alpha = \sqrt{1 - \frac{y^2}{b^2}} \sin \alpha$$

Squaring,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \cos^2 \alpha - \frac{2xy}{ab} \cos \alpha = \left(1 - \frac{y^2}{b^2}\right) \sin^2 \alpha$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} [\sin^2 \alpha + \cos^2 \alpha] - \frac{2xy}{ab} \cos \alpha = \sin^2 \alpha$$

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha=\sin^2\alpha \qquad \ldots(4)$$

This represents the general equation of an ellipse. Thus, due to the superimposition of two simple harmonic vibrations, the displacement of the particle will be along a curve (Fig. 2.4) given by equation (4).

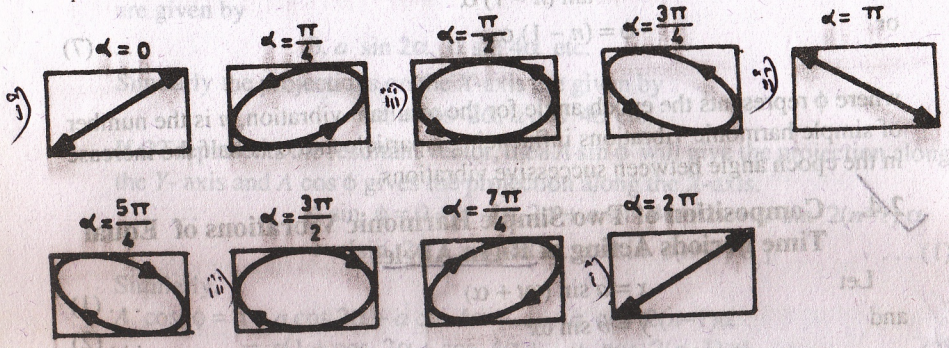


Fig. 2.4.

The resultant vibration of the particle will depend upon the value of α. Fig. 2.4 represents the resultant vibration for values of α changing from 0 to 2π.

Special cases

(*i*) If $\alpha=0$ or 2π; $\quad \cos\alpha=1; \quad \sin\alpha=0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}=0$$

or $$\frac{x}{a}-\frac{y}{b}=0$$

or $$y=\frac{b}{a}x.$$

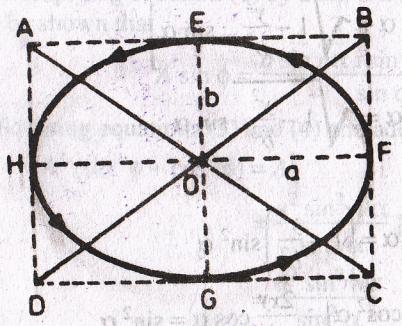


Fig. 2.5.

This represents the equation of the straight line *BD* (Fig. 2.5) *i.e.*, the particle vibrates simple harmonically along the line *DB*.

(*ii*) if $\alpha=\pi$; $\sin\alpha=0$; $\cos\alpha=-1$

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xy}{ab}=0$$

$$\left(\frac{x}{a}+\frac{y}{b}\right)=0$$

$$y=-\frac{b}{a}x. \qquad \ldots(5)$$

This represents the equation of the straight line *AC* (Fig. 2.5).

(*iii*) If $\quad \alpha=\frac{\pi}{2}$ or $\frac{3\pi}{2}$

$$\sin\alpha=1; \quad \cos\alpha=0$$

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This represents the equation of the ellipse *EHGF* (Fig. 2.5) with *a* and *b* as the semi-major and semi-minor axes.

(*iv*) If $\quad \alpha=\frac{\pi}{2}$ or $\frac{3\pi}{2}$

and $\quad a=b;$

then $$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$$

or $$x^2+y^2=a^2.$$

This represents the equation of a circle (Fig. 2.6).

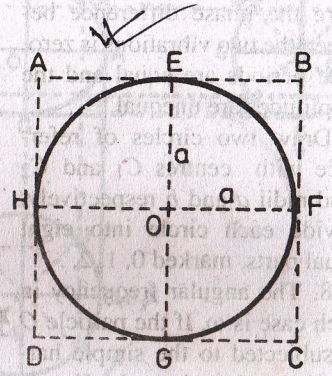


Fig. 2.6.

(*v*) If $\alpha=\frac{\pi}{4}$ or $\frac{7\pi}{4}$, the resultant vibration is an oblique ellipse *KLMN* as shown in Fig. 2.7 (*i*).

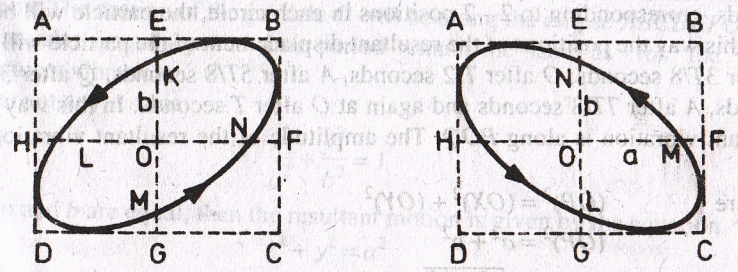


Fig. 2.7.

On the other hand if $\alpha=\frac{3\pi}{4}$ or $\frac{5\pi}{4}$, the resultant vibration is again an oblque ellipse *KLMN* as shown in Fig. 2.7 (*ii*). The cycle of changes is repeated after every time period.

2.5. Composition of Two SHMs of Equal Periods at Right Angles

Graphical method. (1) Let a particle be influenced simultaneously by two simple harmonic vibrations at right angles to each other. The two vibrations are represented by the equations

$$x = a \sin \omega t$$
$$y = b \sin \omega t$$

Here the phase difference between the two vibrations is zero, time periods are equal and the amplitudes are unequal.

Draw two circles of reference with centres C_1 and C_2 and radii a and b respectively. Divide each circle into eight equal parts, marked 0, 1, 2, ... 7, 8. The angular frequency in each case is ω. If the particle O is subjected to the simple harmonic motion along the **X-axis** only, the particle will vibrate along XX'. Similarly, if the particle O is subjected to the simple harmonic motion along the **Y-axis** only, the particle will vibrate along YY' (Fig. 2.8).

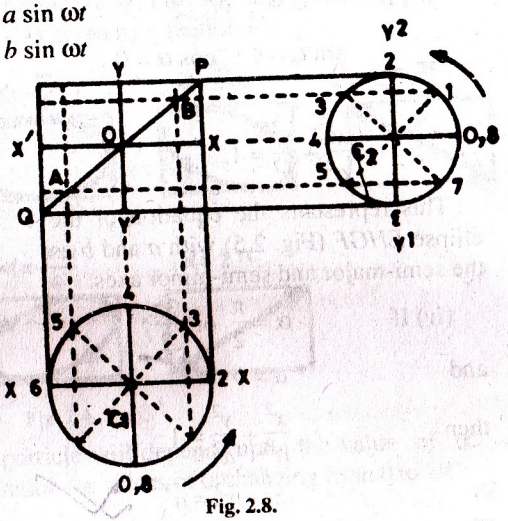


Fig. 2.8.

When the particle O is subjected to the two simple harmonic motions simultaneously ; the resultant vibration of O will be along the straight line PQ. At zero - zero position, the particle is at the mean position O. After $T/8$ seconds, corresponding to 1 - 1 position in each circle, the particle will be at B. After $T/4$ seconds, corresponding to 2 - 2 positions in each circle, the particle will be at P. In this way the positions of the resultant displacement of the particle will be, B after $3T/8$ seconds, O after $T/2$ seconds, A after $5T/8$ seconds, Q after $3T/4$ seconds, A after $7T/8$ seconds and again at O after T seconds. In this way the resultant vibration is along POQ. The amplitude of the resultant vibration is OP.

Here
$$(OP)^2 = (OX)^2 + (OY)^2$$
$$(OP)^2 = a^2 + b^2$$
$$OP = \sqrt{a^2 + b^2}$$

The angular frequency and time period remain the same as for the two constituent vibrations.

(2) Let a particle be influenced simultaneously by two simple harmonic vibrations at right angles to each other. The two vibrations are represented by

the equations

$$x = a \sin (\omega t + \alpha)$$
and
$$y = b \sin \omega t$$

Suppose the phase difference $\alpha = \pi/2$ and the time periods are equal. The amplitudes are a and b.

Draw two circles of reference with centres C_1 and C_2 and radii a and b respectively. Divide each circle into eight equal parts, marked 0, 1, 2, . . .7, 8. The angular frequency in each case is ω. If the particle O is subjected to the simple harmonic motion along the X-axis only, the particle will vibrate along XX'. Similarly if the particle O is subjected to the simple harmonic motion along the Y-axis only, the particle will vibrate along YY. Here, the initial position for vibration along XX' is at the extreme position at $t = 0$. The points on the circle of reference are marked showing that there is a phase difference of $\pi/2$ between the two vibrations (Fig. 2.9).

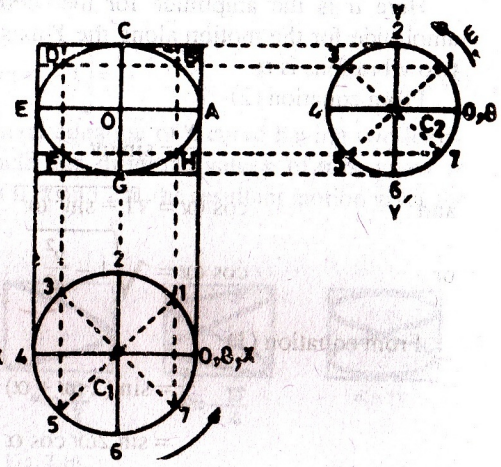


Fig. 2.9.

When the particle O is subjected to the two SHMs simultaneously, the resultant vibration of O will be along an ellipse having a and b as semi-major and semi-minor axes. The resultant position after $T/8$ seconds corresponding to 1-1 position in each circle of reference will be the point B. Similarly C, D, E, F etc. will be the resultant positions after successive time intervals of $T/8$ seconds. The motion of the particle O will be along the ellipse $ABCDEFGHA$. The angular frequency and time period remain the same as for the two constituent vibrations.

Here at any instant

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If a and b are equal, then the resultant motion is given by the equation

$$x^2 + y^2 = a^2$$

which represents a circle with centre O and radius a.

[**Note.** If a particle is moving along a circular path of radius a with angular frequency ω, it can be considered to be under the influence of two rectangular SHMs having equal amplitude of value a and angular frequency ω and a phase difference of $\pi/2$.]

2.6. Composition of Two Simple Harmonic Motions at Right Angles to Each Other and Having Time Periods in the Ratio 1 : 2

Let two simple harmonic motions be given by the equations

$$x = a \sin (2\omega t + \alpha) \quad \ldots (1)$$

and

$$y = b \sin \omega t \quad \ldots (2)$$

Here a is the amplitude for the motion along the X-axis and b is the amplitude for the motion along the Y-axis. The phase difference between the two vibrations is α.

From equation (2)

$$\frac{y}{b} = \sin \omega t$$

and

$$\cos \omega t = \sqrt{1 - \sin^2 \omega t} \quad \ldots (3)$$

or

$$\cos \omega t = \sqrt{1 - \frac{y^2}{b^2}}$$

From equation (1)

$$\frac{x}{a} = \sin (2\omega t + \alpha)$$

$$= \sin 2\omega t \cos \alpha + \cos 2\omega t \sin \alpha$$

$$= 2 \sin \omega t \cos \omega t \cos \alpha + (1 - 2 \sin^2 \omega t) \sin \alpha$$

Substituting the values of $\sin \omega t$ and $\cos \omega t$,

$$\frac{x}{a} = 2 \cdot \frac{y}{b} \cdot \sqrt{1 - \frac{y^2}{b^2}} \cos \alpha + \left(1 - 2\frac{y^2}{b^2}\right) \sin \alpha$$

or

$$\left[\frac{x}{a} - \left(\frac{1 - 2y^2}{b^2}\right) \sin \alpha\right] = \frac{2y}{b} \cos \alpha \sqrt{1 - \frac{y^2}{b^2}}$$

$$\left[\left(\frac{x}{a} - \sin \alpha\right) + \frac{2y^2}{b^2} \sin \alpha\right] = \frac{2y \cos \alpha}{b} \sqrt{1 - \frac{y^2}{b^2}}$$

Squaring both sides

$$\left(\frac{x}{a} - \sin \alpha\right)^2 + \frac{4y^4}{b^4} \sin^2 \alpha + 2\left(\frac{x}{a} - \sin \alpha\right) \frac{2y^2}{b^2} \sin \alpha = \frac{4y^2 \cos^2 \alpha}{b^2} \left(1 - \frac{y^2}{b^2}\right)$$

$$\left(\frac{x}{a} - \sin \alpha\right)^2 + \frac{4y^4}{b^4} (\sin^2 \alpha + \cos^2 \alpha) - \frac{4y^2}{b^2} (\sin^2 \alpha + \cos^2 \alpha) + \frac{4y^2}{b^2} \cdot \frac{x}{a} \sin \alpha = 0$$

$$\left(\frac{x}{a} - \sin \alpha\right)^2 + \frac{4y^4}{b^4} - \frac{4y^2}{b^2} + \frac{4y^2}{b^2} \cdot \frac{x}{a} \sin \alpha = 0$$

$$\left(\frac{x}{a} - \sin \alpha\right)^2 + \frac{4y^2}{b^2}\left[\frac{y^2}{b^2} + \frac{x}{a} \sin \alpha - 1\right] = 0 \quad \ldots (5)$$

Equation (5) represents the general equation of a curve having two loops. The resultant motion of the particle for different values of α is given in Fig. 2.10. For a phase difference of 0, π and 2 π, the resultant motion gives the figure of eight.

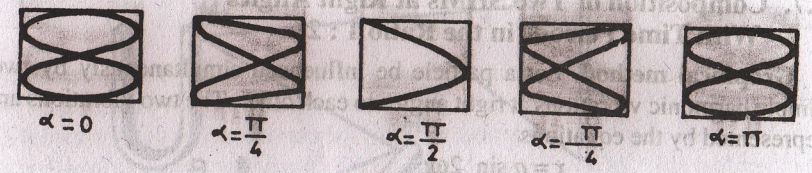


Fig. 2.10.

Special cases

(*i*) When $\alpha = 0, \pi, 2\pi$ etc.

$$\sin \alpha = 0$$

From equation (5)

$$\frac{x^2}{a^2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) = 0.$$

This equation represents the figure of eight and has two loops.

(*ii*) When $\alpha = \frac{\pi}{2}$

$$\sin \alpha = +1$$

From equation (5)

$$\left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a} - 1\right) = 0$$

or

$$\left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2}\left(\frac{x}{a} - 1\right) + \frac{4y^4}{b^4} = 0$$

or $$\left[\left(\frac{x}{a}-1\right)+\frac{2y^2}{b^2}\right]^2=0$$

or $$\left(\frac{x}{a}-1\right)+\frac{2y^2}{b^2}=0$$

or $$\frac{2y^2}{b^2}=-\left(\frac{x}{a}-1\right)$$

$$y^2=-\frac{b^2}{2}\left(\frac{x}{a}-1\right)$$

$$y^2=-\frac{b^2}{2a}(x-a)$$

This represents the equation of a parabola, with vertex at $(a, 0)$.

2.7. Composition of Two SHMs at Right Angles With Time Periods in the Ratio 1 : 2

Graphical method. Let a particle be influenced simultaneously by two simple harmonic vibrations at right angles to each other. The two vibrations are represented by the equations

$$x = a \sin 2\omega t$$

$$y = b \sin \omega t$$

Here the phase difference between the two vibrations is zero, amplitudes are unequal and the time periods are in the ratio of 1 : 2.

Draw two circles of reference with centres C_1 and C_2 and radii a and b respectively. Divide the circle with centre C_1 into 4 equal parts and the circle with centre C_2 into 8 equal parts. The angular frequencies are 2ω and ω. If the particle O is subjected to the SHM along the X-axis only, the particle will vibrate along XX'. (Fig. 2.11)

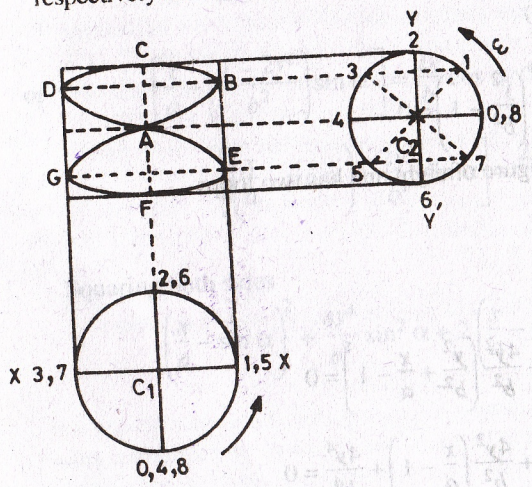


Fig. 2.11.

Similarly, if the particle O is subjected to SHM along the Y-axis only, the particle will vibrate along YY'.

When the particle O is subjected to two SHMs simultaneously, the resultant vibration of O will be along a curve $ABCDEFGA$ which represents the figure of eight. At zero-zero position the particle is

at the mean position A. After equal intervals of time the positions B, C, D etc. are obtained in the same order from the two circles of reference (Fig. 2.11). The angular frequency of the resultant vibration is ω.

2.8. Experimental Methods for Obtaining Lissajous' Figures

1. Optical method. A and B are two tuning forks with frequencies in the ratio of 2 : 1. The prongs of A vibrate in a horizontal plane and the prongs of B in a vertical plane. M_1 and M_2 are two small highly polished plane mirrors

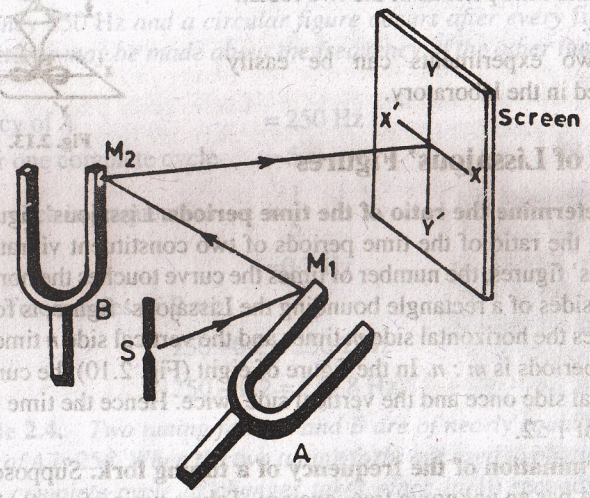


Fig. 2.12.

attached to the prongs of A and B respectively. It is adjusted that the spot of light after reflection from M_1 and M_2 is obtained at O, the centre of the screen. When only the tuning fork A vibrates the spot of light moves along XX'. When only the tuning fork B vibrates, the spot of light moves along YY'. When A and B vibrate simultaneously and are in phase, the spot of light traces the figure of eight (Fig. 2.12).

2. Blackburn's Pendulum. Blackburn's pendulum consists of a funnel having a fine nozzle. It is supported by three strings meeting at E (Fig. 2.13).

$EACBDA$ is a string passing over the hooks C and D. The length AE can be changed by moving the knot at A. When the pendulum is displaced about the point A along XX' it vibrates with a time period t_1.

Here $$t_1 = 2\pi\sqrt{\frac{l_1}{g}}$$

When the pendulum is displaced about the point B, along YY', it vibrates with a time period t_2.

Here $t_2 = 2\pi\sqrt{\frac{l_2}{g}}$

When both the motions are given simultaneously, the nozzle of the funnel traces the Lissajous' figure. If fine sand is put in the funnel, the sand falls through the nozzle and traces a fine curve. The shape and size of the curve will depend upon the relative amplitudes and time periods of the two rectangular SHMs.

These two experiments can be easily demonstrated in the laboratory.

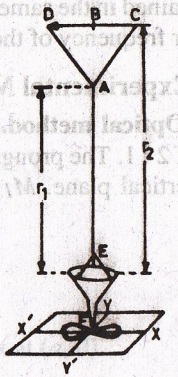


Fig. 2.13.

2.9. Uses of Lissajous' Figures

1. To determine the ratio of the time periods. Lissajous' figures help in determining the ratio of the time periods of two constituent vibrations. From the Lissajous' figures, the number of times the curve touches the horizontal and the vertical sides of a rectangle bounding the Lissajous' figure, is found. If the curve touches the horizontal side m times and the vertical side n times, the ratio of the time periods is $m : n$. In the figure of eight (Fig. 2.10) the curve touches the horizontal side once and the vertical side twice. Hence the time periods are in the ratio of 1 : 2.

2. Determination of the frequency of a tuning fork. Suppose the given tuning fork A has an unknown frequency n_1. Take a tuning fork B of nearly the same frequency as that of A. Suppose the frequency of $B = n$. Set the two tuning forks to vibrate in perpendicular planes as in the optical method for obtaining Lissajous' figures. Obtain the Lissajous' figure. As the tuning forks differ slightly in frequency, the phase difference between the two, changes with time. The Lissajous' figures obtained are as shown in Fig. 2.4. Due to the continuous change in shape, the pattern changes in shape continuously with the phase changing from 0 to 2π . Suppose the complete cycle of changes takes place in t seconds. Then, the difference in frequencies of A and $B = \frac{1}{t}$.

Therefore, frequency of $A = n \pm \left(\frac{1}{t}\right)$

Now attach a little wax to the tuning fork A. Repeat the experiment and note the time taken to complete one cycle of operation. Suppose time taken is t_1. If t_1 is less than t, then frequency of A

$$= \left(n + \frac{1}{t}\right)$$

If t_1 is greater than t, then the frequency of A

$$= \left(n - \frac{1}{t}\right)$$

Hence, the correct value of the frequency of the unknown tuning fork can be determined.

Example 2.3. *In an experiment to obtain Lissajous' figures, one tuning fork is of frequency* 250 Hz *and a circular figure occurs after every* five seconds. *What deductions may be made about the frequency of the other tuning fork.*

Frequency of A = 250 Hz

Time for one complete cycle = 5 s

$\therefore$ Difference in frequencies $= \frac{1}{5}$

= 0·2 Hz

Possible frequencies of B are

either 250 + 0·2 = 250·2 Hz

or 250 – 0.2 = 249·8 Hz.

Example 2.4. *Two tuning forks A and B are of nearly equal frequencies. Frequency of A is* 256. *When the two tuning forks are used to obtain Lissajous' figures the complete cycle of changes takes place in* 10 seconds. *When the tuning fork B is loaded with a little wax, the time taken is* 20 seconds. *Calculate the frequency of B before loading.*

Frequency of A = 256

Time for one complete cycle = 10 seconds

$\therefore$ Difference in frequencies $= \frac{1}{10}$

= 0·1

Possible frequencies of B are

either 256 + 0·1 = 256·1

or 256 – 0·1 = 255·9

When B is loaded with wax, the difference in frequencies

$= \frac{1}{20} = 0{\cdot}05$

Possible frequencies of B are

either $256 + 0{\cdot}05 = 256{\cdot}05$

or $256 - 0{\cdot}05 = 255{\cdot}9$

Suppose the frequency of B before loading is 256·1. After loading its frequency decreases and can be 256·05 or 255·95.

Therefore the original frequency of B = **256·1.**

If it is supposed that the original frequency of B before loading is 255.9, after loading its frequency should decrease. But it is given that possible frequencies after loading are 256.05 and 255.95 which are higher than 255.9. Therefore, this value is not possible.

Hence the original frequency of B **=256.1.**

Example 2.5. *Two tuning forks A and B are of nearly equal frequencies. The frequency of A is 288. When the two tuning forks are used to obtain Lissajous' figures, the complete cycle of changes takes place in* 20 seconds. *When the tuning fork B is loaded with a little wax, the time taken for one cycle of change is* 10 seconds. *Calculate the original frequency of B.*

Before loading

Frequency of A $= 288$

Time for one complete cycle $= 20$ s

Difference in frequencies $= \frac{1}{20} = 0{\cdot}05$

Possible frequencies of B are

either $288 + 0{\cdot}05 = 288{\cdot}05$

or $288 - 0{\cdot}05 = 287{\cdot}95$

After loading, the cycle of changes takes place after 10 seconds *i.e.*, time taken for the cycle decreases. Suppose the frequency of B is 288·05. After loading, the frequency of B will be lowered and its difference with the frequency of A becomes less. Therefore, the time taken for each cycle of changes will be more than 20 seconds. Hence it is not possible.

Suppose the frequency of B is 287.95. After loading its difference with the frequency of A is increased and the cycle of changes will take place in less time.

Hence the original frequency of B = **287.95.**

Example 2.6. *Two tuning forks produce Lissajous' figures. The figure changes in form from a parabola to a figure of 8 and again a parabola, the whole cycle takes place in* 6 seconds. *If the frequency of one is 100, find the possible frequencies of the other.* (Delhi, 1990)

Frequency of A $= 100$ Hz

Time for one complete cycle $= 6$ s

Difference in frequency $= \frac{1}{6}$ Hz

$= 0{\cdot}167$

Possible frequencies of B are

either $100 + 0{\cdot}167 =$ **100·67 Hz**

or $100 - 0{\cdot}167 =$ **99·833 Hz.**

EXERCISES

1. Explain how two simple harmonic vibrations acting simultaneously on a particle and in perpendicular directions, can be compounded. Deduce an expression for the resultant vibration. *(Lucknow)*
2. Deduce the resultant vibration of a particle influenced by two mutually perpendicular simple harmonic vibrations, having the same amplitude but differing in phase by *(i)* 0, *(ii)* $\pi/4$, *(iii)* $\pi/2$ and *(iv)* π.
3. Two simple harmonic vibrations acting at right angles to each other have the time periods in the ratio 1 : 2. The phase difference between the two vibrations is $\pi/2$. Show graphically that the resultant curve is a parabola.
4. Two mutually perpendicular simple harmonic vibrations of amplitudes a and b are acting simultaneously on a particle. Their time periods are in the ratio 1 : 2 and their phase difference is ϕ. Derive an expression for the resultant vibration of the particle.
5. What are Lissajous' figures? Show how they can be produced and demonstrated ? *(Calcutta ; Madras)*
6. What are Lissajous' figures? Discuss briefly their importance in acoustical measurements. *(Delhi)*
7. Describe an optical method for obtaining Lissajous' figures.
8. What are Lissajous' figures? Give the analytical treatment of the formation of these figures when the period of two simple harmonic motions are in the ratio of 1 : 1. Explain how these figures are useful in the laboratory. *(Delhi)*
9. What are Lissajous' figures? Discuss the formation of Lissajous' figures when the periods of the two vibrations are equal and the phase difference is $\pi/2$. *(Delhi)*
10. The lower of the two tuning forks has a frequency of 380 hertz and a complete cycle of changes takes place in 10 seconds. Calculate the frequency of the other fork. **[Ans.** 380·1]
11. What are Lissajous' figures? Calculate the resultant of two simple harmonic motions of equal time period when they act at right angles to one another and when the phase difference is *(i)* zero, and *(ii)* $\pi/2$.

12. Find the resultant of two simple harmonic motions of equal periods when they act at right angles to each other. Discuss different important cases analytically. How can these be demonstrated.
13. Find the resultant of two simple harmonic motions of periods in the ratio 1 : 2 and having (*a*) zero phase difference and (*b*) phase difference of 45°.
14. Find the resultant of two simple harmonic motions of equal period when they act at right angles to one another. Discuss the different important cases. *(Delhi)*
15. What are Lissajous' figures? Describe in detail two important experimental methods for producing them.
16. Discuss in detail the resulting motion when two simple harmonic motions having a frequency ratio 1 : 1 and acting at right angles to each other superimpose. *(Kanpur)*
17. Two tuning forks *A* and *B* are used to produce Lissajous' figures. It is found that the figure completes its cycle in 10 seconds. If the frequency of *A* (slightly larger than that of *B*) is 200, calculate the frequency of *B*. *[Behrampur (Hons), 1992]* [**Ans.** 199.9 hertz]
18. Define simple harmonic motion. Two simple harmonic oscillators of same frequency and same amplitude but in perpendicular directions are compounded. Find the resultant motion. *[Utkal, 1990]*

CHAPTER 3

Free, Forced And Resonant Vibrations

3.1. Free Vibrations

When the bob of a simple pendulum (in vacuum) is displaced from its mean position and left, it executes simple harmonic motion. The time period of oscillation depends only on the length of the pendulum and the acceleration due to gravity at the place. The pendulum will continue to oscillate with the same time period and amplitude for any length of time. In such cases there is no loss of energy by friction or otherwise. In all similar cases, the vibrations will be undamped free vibrations. The amplitude of swing remains constant.

3.2. Undamped Vibrations

For a simple harmonically vibrating particle, the kinetic energy for displacement y, is given by

$$\frac{1}{2} m \left(\frac{dy}{dt}\right)^2$$

At the same instant, the potential energy of the particle is $\frac{1}{2} Ky^2$ where K is the restoring force per unit displacement.

The total energy at any instant,

$$\therefore = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \tfrac{1}{2} Ky^2$$

For an undamped harmonic oscillator this total energy remains constant.

$$\therefore \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \tfrac{1}{2} Ky^2 = \text{constant} \quad \ldots (1)$$

Differentiating equation (1) with respect to time,

$$m \frac{d^2 y}{dt^2} + Ky = 0 \quad \ldots (2)$$

or $$\frac{d^2y}{dt^2} + \left(\frac{K}{m}\right)y = 0 \quad \ldots (3)$$

Equation (3) is similar to the equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \ldots (4)$$

Here $\omega^2 = \frac{K}{m}$

The solution for equation (4) is

$$y = a \sin(\omega t - \alpha)$$

$$\therefore \quad y = a \sin\left[\sqrt{\frac{K}{m}}\ t - \alpha\right]$$

The frequency of oscillation is

$$n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

Thus, in the case of undamped free vibrations, the differential equation is

$$\frac{d^2y}{dt^2} + \left(\frac{K}{m}\right)y = 0 \quad \ldots (5)$$

This is only an ideal case. In the first chapter, for the motion of a pendulum, loaded spring, LC circuit etc, it has been assumed that the vibrations are free and undamped.

3.3. Damped Vibrations

In actual practice, when the pendulum vibrates in air medium, there are frictional forces and consequently energy is dissipated in each vibration. The amplitude of swing decreases continuously with time and finally the oscillations die out. Such vibrations are called **free damped** vibrations. The dissipated energy appears as heat either within the system itself or in the surrounding medium. The dissipative force due to friction etc. (resistance in *LCR* circuit) is proportional to the velocity of the particle at that instant. Let $\mu \frac{dy}{dt}$ be the dissipative force due to friction etc.

This term is to be introduced in equation (2).

Therefore, the differential equation in the case of free-damped vibrations is,

$$m\frac{d^2y}{dt^2} + Ky + \mu\frac{dy}{dt} = 0 \quad \ldots (6)$$

or $$\frac{d^2y}{dt^2} + \left(\frac{\mu}{m}\right)\frac{dy}{dt} + \left(\frac{K}{m}\right)y = 0 \quad \ldots (7)$$

This equation is similar to a general differential equation,

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + k^2y = 0 \quad \ldots (8)$$

The solution of this equation is

$$y = ae^{-bt}\sin(\omega t - \alpha) \quad \ldots (9)$$

The general solution of equation (7) is also given by

$$y = A\,e^{(-b+\sqrt{b^2-k^2})\,t} + B\,e^{(-b-\sqrt{b^2-k^2})\,t} \quad \ldots (10)$$

Here $b = \frac{\mu}{2m}$ and $k^2 = \frac{K}{m}$

and $\omega = \sqrt{k^2 - b^2}$

or $$\omega = \sqrt{\frac{K}{m} - \frac{\mu^2}{4m^2}}$$

or $$n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{k^2 - b^2}.$$

3.4. Forced Vibration

The time period of a body executing simple harmonic motion depends on the dimensions of the body and its elastic properties. The vibrations of such a body die out with time due to dissipation of energy. If some external periodic force is constantly applied on the body, the body continues to oscillate under the influence of such external forces. Such vibrations of the body are called *forced vibrations.*

Initially, the amplitude of the swing increases, then decreases with time, becomes minimum and again increases. This will be repeated if the external periodic force is constantly applied on the system. In such cases the body will finally be forced to vibrate with the same frequency as that of the applied force. The frequency of the forced vibration is different from the natural frequency of vibration of the body. The amplitude of the forced vibration of the body depends on the difference between the natural frequency and the frequency of the applied force. The amplitude will be large if difference in frequencies is small and *vice versa.*

For forced vibrations, equation (6) is modified in the form,

$$m\frac{d^2y}{dt^2} + Ky + \mu\frac{dy}{dt} = F\sin pt \quad \ldots (11)$$

Here p is the angular frequency of the applied periodic force.

The particular solution of equation (11) representing the forced vibrations is

$$y = a\sin(pt - \alpha) \quad \ldots (12)$$

$$\therefore \quad \frac{dy}{dt} = ap\cos(pt - \alpha) \quad \ldots (13)$$

$$\frac{d^2y}{dt^2} = -ap^2 \sin(pt-\alpha) = -p^2 y \qquad \ldots(14)$$

Substituting these values in equation (11)

$$-mp^2a \sin(pt-\alpha) + Ka \sin(pt-\alpha) + \mu ap \cos(pt-\alpha) = F \sin pt$$

$$-mp^2a [\sin pt \cos\alpha - \cos pt \sin\alpha]$$
$$+ Ka [\sin pt \cos\alpha - \cos pt \sin\alpha]$$
$$+ \mu ap [\cos pt \cos\alpha + \sin pt \sin\alpha] - F \sin pt = 0 \qquad \ldots(15)$$

When $\sin pt = 1$; $\cos pt = 0$

$$-mp^2a \cos\alpha + Ka \cos\alpha + \mu ap \sin\alpha - F = 0 \qquad \ldots(16)$$

When $\cos pt = 1$; $\sin pt = 0$

$$+mp^2a \sin\alpha - Ka \sin\alpha + \mu ap \cos\alpha = 0 \qquad \ldots(17)$$

Dividing equation (17) by $\cos\alpha$ and simplifying

$$\tan\alpha = \frac{\mu p}{K - mp^2} = \frac{A}{B} \qquad \ldots(18)$$

From equation (18) $$\sin\alpha = \frac{B}{\sqrt{A^2+B^2}} \qquad \ldots(19)$$

$$\cos\alpha = \frac{B}{\sqrt{A^2+B^2}} \qquad \ldots(20)$$

Dividing equation (16) by $\cos\alpha$

$$-mp^2a + Ka + \mu ap \tan\alpha - \frac{F}{\cos\alpha} = 0$$

or $$a[(K-mp^2) + \mu p \tan\alpha] = \frac{F}{\cos\alpha}$$

But $(K - mp^2) = B$, and $\mu p = A$

Substituting the values of $\tan\alpha$ and $\cos\alpha$

$$a\left[B + \frac{A^2}{B}\right] = \frac{F\sqrt{A^2+B^2}}{B}$$

$$a = \frac{F}{\sqrt{A^2+B^2}}$$

Substituting the values of A and B

$$a = \frac{F}{\sqrt{\mu^2p^2 + (K-mp^2)^2}} \qquad \ldots(21)$$

$\therefore$ $$y = a \sin(pt-\alpha)$$

or $$y = \frac{F}{\sqrt{\mu^2p^2 + (K-mp^2)^2}} \sin 16\, pt - \alpha) \qquad \ldots(22)$$

Applying the boundary conditions, another solution is obtained when $F = 0$. This corresponds to free vibrations. In the case of free vibrations the solution is

$$y = a e^{-bt} \sin(\omega t - \alpha) \qquad \ldots(23)$$

The general solution will include both the particular solutions for free and forced vibrations.

$$\therefore \quad y = a e^{-bt} \sin(\omega t - \alpha) + \frac{F}{\sqrt{\mu^2p^2 + (K-mp^2)^2}} \sin(pt-\alpha) \qquad \ldots(24)$$

Here $$b = \frac{\mu}{2m}.$$

3.5. Resonance and Sharpness of Resonance

In the case of forced vibrations, the general solution for the displacement at any instant is given by

$$y = a e^{-bt} \sin(\omega t - \alpha) + \frac{F}{\sqrt{\mu^2p^2 + (K-mp^2)^2}} \sin(pt-\alpha).$$

If the effect of viscosity of the medium is small, the amplitude,

$$\frac{F}{\sqrt{\mu^2p^2 + (K-mp^2)^2}}$$

under the action of the driving force is maximum when the denominator is minimum. This is possible if $K - mp^2 = 0$ or $K = mp^2$

or $$p = \sqrt{\frac{K}{m}}.$$

Further, the amplitude will be infinite if μ is also zero. The oscillations will have maximum amplitude and this state of vibration of a system is called *resonance*. It means that, when the forced frequency is equal to the natural frequency of vibration of the body, resonance takes place. If friction is present, the amplitude at resonance

$$= \frac{F}{\mu p} = \frac{F}{\mu\sqrt{K/m}}$$

or amplitude at resonance $$= \frac{F}{\mu}\sqrt{\frac{m}{K}}$$

In the case of sound, the study of sharpness of resonance is of great importance. Sharpness of resonance refers to the fall in amplitude with change in frequency on each side of the maximum amplitude.

The particular solution for displacement in the case of forced vibrations is,

$$y = \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \sin(pt - \alpha) \quad \ldots (1)$$

Differentiating equation (1) with respect to time

$$\frac{dy}{dt} = \frac{Fp}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \cos(pt - \alpha) \quad \ldots (2)$$

The velocity *(dy/dt)* is maximum when $\cos(pt - \alpha)$ is maximum *i.e.*, the instant at which the particle crosses the mean position.

$$\left(\frac{dy}{dt}\right)_{max} = \frac{Fp}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \quad \ldots (3)$$

Kinetic energy of the vibrating particle at the instant of crossing the mean position is given by

$$\text{K.E.} = \tfrac{1}{2} m \left(\frac{dy}{dt}\right)^2_{max}$$

$$\text{K.E} = \frac{\frac{1}{2} mF^2 p^2}{\mu^2 p^2 + (k - mp^2)^2} \quad \ldots (4)$$

The mean square of the driving force per unit mass

$$= \frac{\left(\frac{O + F^2}{2}\right)}{m} = \frac{F^2}{2m} \quad \ldots (5)$$

Dividing equation (4) by $\frac{F^2}{2m}$ we get *kinetic energy per unit force* which is called the *response R.*

$$R = \frac{\frac{1}{2} mF^2 p^2}{\mu^2 p^2 + (K - mp^2)^2} \div \frac{F^2}{2m}$$

$$R = \frac{m^2 p^2}{\mu^2 p^2 + (K - mp^2)^2}$$

$$R = \frac{p^2}{\frac{\mu^2 p^2}{m^2} + \left(\frac{K}{m} - p^2\right)^2} \quad \ldots (6)$$

The natural frequency of the system in the absence of damping is $\sqrt{\frac{K}{m}}$.

Therefore, the term $\left(\frac{K}{m} - p^2\right)$ in equation (6) represents the extent to which the natural frequency of the system deviates from the forced frequency.

When $\frac{K}{m} = p^2$

the natural frequency coincides with the forced frequency and the value of R will be maximum. From equation (6)

$$R = \frac{p^2}{\mu^2 p^2 / m^2} = \frac{m^2}{\mu^2} = \left(\frac{m}{\mu}\right)^2 \quad \ldots (7)$$

The response $R \propto \frac{1}{\mu}$

It means that the response R is inversely proportional to the frictional force. In the absence of friction, the response is maximum.

The term $\left(\frac{K}{m} - p^2\right)$ in equation (6), refers to mistuning. The larger is its value, the greater is the system away from resonance.

The graph between p/ω along the X-axis and the response R along the Y-axis is shown in Fig. 3.1.

(*i*) When p/ω is equal to 1 the response is maximum. For curve A, μ is large and for curve C, μ is less. The response decreases for values of p/ω greater than 1 or less than 1.

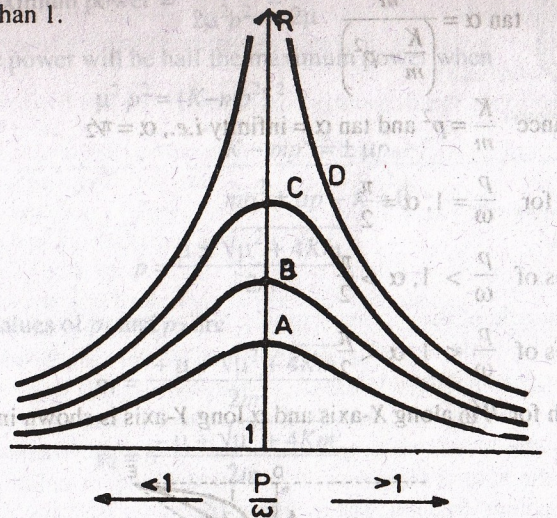


Fig. 3.1.

(*ii*) When the frictional forces are absent, *i.e.*, $\mu = 0$, R is infinite and the sharpness of resonance is maximum.

(*iii*) The sharpness of resonance decreases with increase in the value of μ.

(*iv*) The sharpness of resonance dies rapidly even for a very small change in the value of p/ω from 1, in the case, where μ is minimum.

In the case of the resonance tube, the damping force is large and the graph will be similar to the curve *A* in Fig. 3.1. The resonance persists over a wide range and it is difficult to exactly locate the position of maximum sharpness of resonance. Hence the results obtained with the resonance tube apparatus are not very accurate.

In the case of the sonometer wire, the damping forces are small and the graph will be similar to curve *C* in Fig. 3.1. In this case, the sharpness of resonance is maximum in a very narrow region. Even a slight variation in length or tension reduces the sharpness considerably. The vibrations die out rapidly. Thus, the results obtained with a sonometer are accurate.

3.6. Phase of Resonance

Considering the phase lead of the forced vibrations with reference to the driving force, in equation (18) of Art. 3.4,

$$\tan\alpha = \frac{\mu p}{K - mp^2}$$

$$\tan\alpha = \frac{\frac{\mu p}{m}}{\left(\frac{K}{m} - p^2\right)} \quad \ldots (8)$$

At resonance $\frac{K}{m} = p^2$ and $\tan\alpha =$ infinity *i.e.*, $\alpha = \pi/2$

It means, for $\frac{p}{\omega} = 1, \alpha = \frac{\pi}{2}$

For values of $\frac{p}{\omega} > 1, \alpha > \frac{\pi}{2}$

For values of $\frac{p}{\omega} < 1, \alpha < \frac{\pi}{2}$

The graph for p/ω along *X*-axis and α long *Y*-axis is shown in Fig. 3.2.

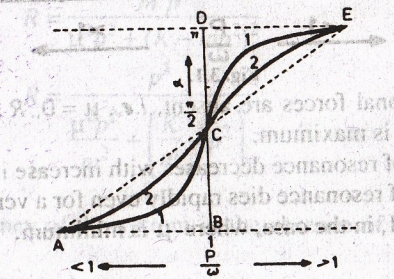


Fig. 3.2.

The shape of the curve will also depend on the value of μ, *i.e.*, the external frictional forces.

(1) For values of $\mu = 0$, the curve *ABCDE* represents the graph.

(2) For large values of μ, the curve will be similar to curve 2.

(3) For small values of μ the curve will be similar to curve 1.

3.7. Quality Factor

The sharpness of resonance is inversely proportional to μ and can be represented in terms of the quality factor. Suppose p_1 and p_2 are the values of the angular frequencies of the driving force on the two sides of the resonance angular frequency p_0 at which the power in the response is one half.

The power at any instant is the product of the force and the velocity at that instant.

$$\therefore \quad \text{Average value of power} = \frac{\mu F^2 p^2}{2\left[\mu^2 p^2 + (K - mp^2)^2\right]} \quad \ldots (1)$$

The power will be maximum when $(K - mp^2) = 0$

$$\therefore \quad \text{Maximum power} = \frac{\mu F^2 p^2}{2\mu^2 p^2} = \frac{F^2}{2\mu} \quad \ldots (2)$$

The average power will be half the maximum power when

$$\mu^2 p^2 = (K - mp^2)^2$$

or $$K - mp^2 = \pm \mu p$$

or $$mp^2 \pm \mu p - K = 0$$

or $$p = \frac{\mu \pm \sqrt{\mu^2 + 4Km}}{2m}$$

One set of values of p_1 and p_2 are

$$p_1 = \frac{+\mu + \sqrt{\mu^2 + 4Km}}{2m}$$

$$p_2 = \frac{-\mu + \sqrt{\mu^2 + 4Km}}{2m}$$

$$\therefore \quad p_1 - p_2 = \frac{2\mu}{2m} = \frac{\mu}{m}$$

The quality factor $$Q = \frac{p_0}{p_1 - p_2} = \frac{\sqrt{K/m}}{\mu/m} = \frac{\sqrt{Km}}{\mu} \quad \ldots (3)$$

3.8. Examples of Forced and Resonant Vibrations

1. An interesting example of forced vibrations and resonance is noticed in the case of troops marching over a suspension bridge. The marching troops

exert a certain **periodic force** on the bridge and the bridge is set into forced vibrations. If all the soldiers march in step, each will exert a periodic force in phase and this results in the increased swing of the bridge. If, however the natural frequency of the swing coincides with the forced frequency of the marching force, resonance occurs and the amplitude of the swing becomes very high. Due to this reason it is dangerous and the bridge may give way. Consequently the troops while crossing the bridge are usually ordered to march out of step.

2. A child exerts a periodic force to set the swing to move to and fro. The child also knows to give the push to the swing at periodic intervals. The amplitude of the swing gradually increases. When the natural frequency of the swing coincides with the forced frequency of the force exerted by the child, resonance occurs and the swing oscillates with a large amplitude.

3. When the prong of a tuning fork is struck against a rubber pad, the prongs vibrate with the natural frequency of the tuning fork. In this case the energy is dissipated only by the prongs of the tuning fork and the intensity of sound is

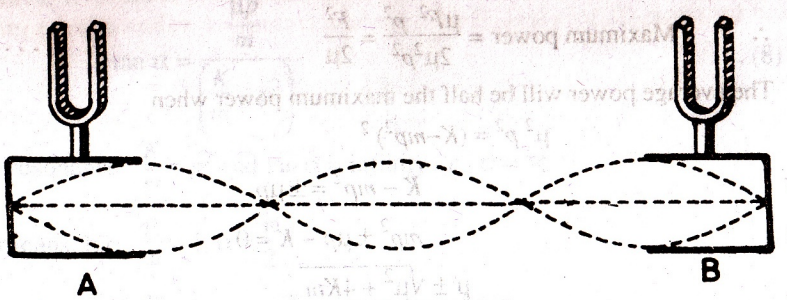


Fig. 3.3.

low. If the handle of the vibrating tuning fork is pressed against the table, the particles of the table are set into forced vibrations. A loud sound is heard because energy is dissipated through a large area.

Take two tuning forks A and B of the same frequency. Mount each tuning fork on a hollow wooden box open at one end (Fig. 3.3). The length of the box is such that the natural frequency of vibration of the air column in the box is equal to the natural frequency of the tuning fork.

When the prong of the tuning fork A is struck against a rubber pad, it is set into vibration. The air column in the box A resonates and maximum sound is heard. After some time it is found that the tuning fork on the box B also begins to vibrate although it has not been struck with the hammer. Here the tuning fork on the box B is forced to vibrate with its natural frequency due to resonance of the air column in B.

EXERCISES

1. Explain clearly the terms free vibration, forced vibration and resonance. Give examples.
2. Discuss the effect of a periodic force in producing vibrations in an elastic body. Give the necessary conditions for the production of forced vibrations.
3. Show that in the case of forced vibrations,

$$m\frac{d^2y}{dt^2} + Ky + \mu\frac{dy}{dt} = F\sin pt.$$

Hence explain the phenomenon of resonance.

4. Discuss sharpness of resonance and explain clearly the factors on which the sharpness depends.
5. What are forced vibrations? Discuss the phenomenon of resonance.
6. A system executing damped simple harmonic motion is subjected to an external periodic force. Derive the expression for the forced vibration and obtain the condition of resonance. Illustrate your answer with examples.
7. Distinguish between free and forced vibrations. Give the necessary conditions for the production of forced vibrations. *(Lucknow Osmania)*
8. Give the theory of forced vibrations and discuss sharpness of resonance.
9. Discuss analytically the phenomenon of forced vibrations and resonance. Give two examples of resonance.
10. Derive the equation of forced vibrations and discuss the conditions of resonance. Explain sharpness of resonance.
11. What are free and forced vibrations? Give the theory of forced vibrations and explain what you mean by sharpness of resonance. *(Delhi)*
12. Distinguish between free and forced vibrations. Give the theory of forced vibrations and discuss the conditions of resonance. *(Delhi)*
13. Give the theory of slightly damped forced vibrations. Prove that vibrating systems with very small damping and large damping have respectively sharp and flat resonance responses. Give two examples of each. *(Panjab)*
14. Write the differential equation for the motion of a particle executing forced vibrations in a resisting medium. Explain the physical meaning of each term and each constant in the equation. Solve the above equation and obtain the condition for amplitude resonance. Explain what is meant by sharpness of resonance. *(Nagpur)*
15. Discuss the theory of forced vibrations in the presence of a damping force proportional to the velocity of the vibrating particle. Deduce the condition of resonance for energy and displacement of the particle and discuss the dependence of the response on the damping factor. *[Behrampur (Hons.)]*

16. What are free, damped and forced vibrations? Give the theory of forced vibrations and discuss the condition of resonance.
17. What are damped vibrations? Obtain an expression for the displacement in the case of a damped oscillatory motion. Discuss the effect of damping on the natural frequency.
18. What is a forced vibration? Discuss mathematically, the vibration of a system executing damped simple harmonic motion when subjected to an external periodic force. What is sharpness of resonance?
19. Discuss the phenomenon of sharpness of resonance and show how it depends on the damping factor. *(Kanpur)*
20. Define quality factor and bandwidth of the sharpness of resonance. Obtain quality factor for a driven harmonic oscillator at resonance. *(Bhagalpur, 1990)*
21. (*a*) Derive the differential equation of damped oscillatory motion and give its general solution.
(*b*) What type of motion do you get when damping is small? *(Delhi, 1991)*
22. Discuss the phenomenon of sharpness of resonance and show how it depends upon the damping factor? *(Delhi, 1992)*

CHAPTER 4

Wave Motion

4.1. Wave Motion

Wave motion is a form of disturbance which travels through the medium due to the repeated periodic motion of the particles of the medium about their mean positions, the disturbance being handed over from one particle to the next. When a stone is dropped into a pond containing water, waves are produced at the point where the stone strikes the water in the pond. The waves travel outward, the particles of water vibrate only up and down about their mean positions. Water particles do not travel along with the wave. Similarly when a tuning fork is set into vibration, it produces waves in air. The wave travels from one particle to the next but the particles of air vibrate about their mean positions.

It is essential to understand the concept of wave motion in the study of various branches in Physics. Wave motion, in general, refers to the transfer of energy from one point to another point of the medium. Transference of various forms of energy like sound, heat, light, X-rays, γ-rays, radio-waves etc. takes place in the form of wave motion. For the transference of energy through a medium, the medium must possess the properties of *elasticity, inertia* and *negligible fricitional resistance.*

4.2. What Propagates in Wave Motion ?

Before studying the characteristics of the different forms of wave motion, it is essential to clearly understand,—*what is propagated in a wave motion* ? The answer to this question is that the physical condition due to a disturbance generated at some point in the medium is propagated to other points in the medium. In all the waves, the particles of the medium vibrate about their mean positions. Hence, in the case of wave motion, it is not matter that is propagated but it is only state of motion of the matter that is propagated. It is a form of *dynamic condition* that is propagated from one point to the other point in the medium.

According to the laws of Physics, any dynamic condition is related to momentum and energy. To conclude, it may be said that in wave motion *momentum and energy* are transferred or propagated. It is not a case of propagation of matter as a whole.

4.3. Characteristics of Wave Motion

1. Wave motion is a disturbance produced in the medium by the repeated periodic motion of the perticles of the medium.

2. Only the wave travels forward whereas the particles of the medium vibrate about their mean positions.

3. There is a regular phase change between the various particles of the medium. The particle ahead starts vibrating a little later than a particle just preceding it.

4. The velocity of the wave is different from the velocity with which the particles of the medium are vibrating about their mean positions. The wave travels with a uniform velocity whereas the velocity of the particles is different at different positions. It is maximum at the mean position and zero at the extreme position of the particles.

There are two types of wave motions :

(*i*) Transverse and (*ii*) Longitudinal.

Sound waves are longitudinal waves and light waves are transverse waves.

4.4. Transverse Wave Motion

In this type of wave motion, the particles of the medium vibrate at right angles to the direction of propagation of the wave.

To understand the propagation of transverse waves in a medium consider nine particles of the medium and the circle of reference (Fig. 4.1). The particles are vibrating about their mean positions up and down and the wave is travelling from left to right. The disturbance takes *T*/8 seconds to travel from one particle to the next.

(1) At $t = 0$ all the particles are at their mean position.

(2) After *T*/8 seconds, the particle 1 travels a certain distance upward and the disturbance reaches particle 2.

(3) After 2*T*/8 seconds, particle 1 has reached its extreme position and the disturbance has reached particle 3.

(4) After 3*T*/8 seconds, particle 1 has completed 3/8 of its vibration and the disturbance has reached particle 4. The positions of particles 2 and 3 are also shown in the diagram.

(5) In this way after *T*/2 seconds, particle 1 has come back to its mean position and the particles 2, 3 and 4 are at the positions shown in the diagram. The disturbance has reached particle 5.

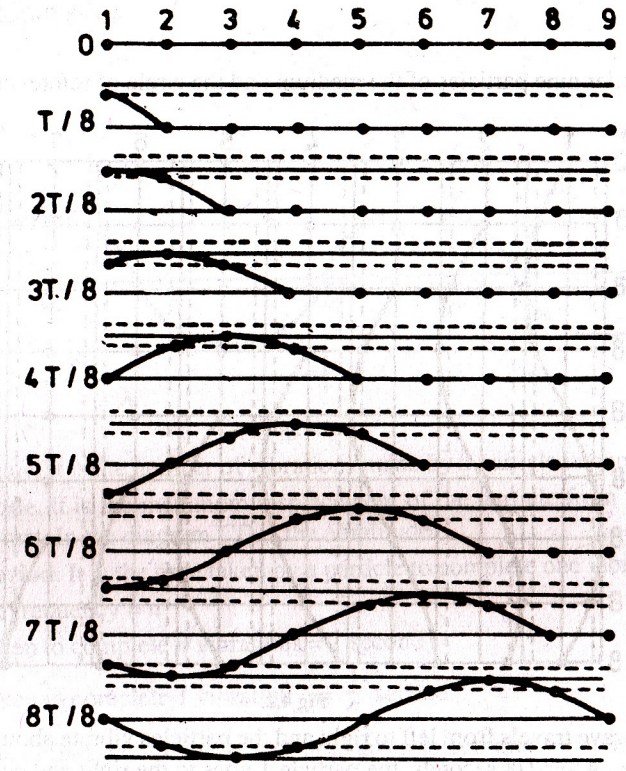


Fig. 4.1.

In this way the process continues and the positions of the particles after 5*T*/8, 6*T*/8, 7*T*/8 and *T* seconds are shown in the diagram.

After *T* seconds, the particles one, five and nine are at their mean positions. The wave has reached particle 9. Particles 1 and 9 are in the same phase. The wave has travelled a distance between particles 1 and 9 in the time in which the particle 1 has completed one vibration.

The top point on the wave at the maximum distance from the mean position is called *crest*, while the point at the maximum distance below the mean position is called *trough*. Thus in a transverse wave, crests and troughs are alternately formed. The contour of the displaced particles of the medium represents the wave. In the case of transverse (or longitudinal) progressive waves, this contour continuously changes position in space and the wave seems to advance in the direction of propagation.

4.5. Longitudinal Wave Motion

In this type of wave motion, particles of the medium vibrate along the direction of propagation of the wave. Fig. 4.2.

Consider nine particles of the medium and the circle of reference (Fig. 4.2).

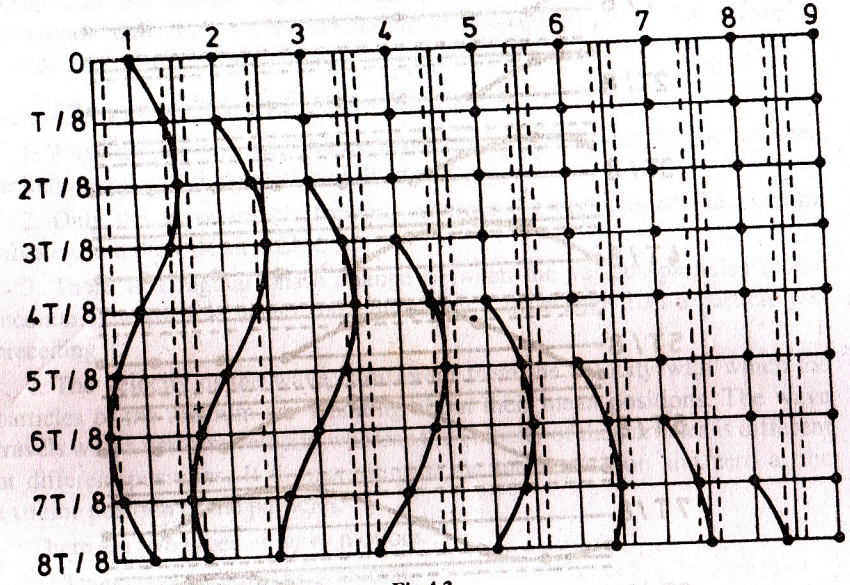


Fig.4.2.

The wave travels from left to right and the particles vibrate about their mean positions. After $T/8$ seconds, the particle 1 goes to the right and completes 1/8 of its vibration. The disturbance reaches the particle 2. After $T/4$ seconds the particle 1 has reached its extreme right position and completes 1/4 of tis vibration and the particle 2 completes 1/8 of its vibration. The disturbance reaches the particle 3. The process continues.

After one complete time period, the positions of the various particles is as shown in the diagram. The wave has reached particle 9. Here 1 and 9 are again in the same phase. Here particles 1, 5 and 9 are at their mean positions. The particles 1 and 3 are close to the particle 2. This is the position of condensation. Similarly particles 9 and 8 are close to the particle 7. This is also the position of condensation or compression. On the other hand, particles 4 and 6 are far away from the particle 5. This is the position of rarefaction. Hence in a longitudinal wave motion, condensations and rarefactions are alternately formed.

4.6. Definitions

Wavelength. It is the distance travelled by the wave in the time in which the particle of the medium completes one vibration. It is also defined as the distance between two nearest particles in the same phase.

The distance AB (Fig. 4.3) is equal to the wavelength λ.

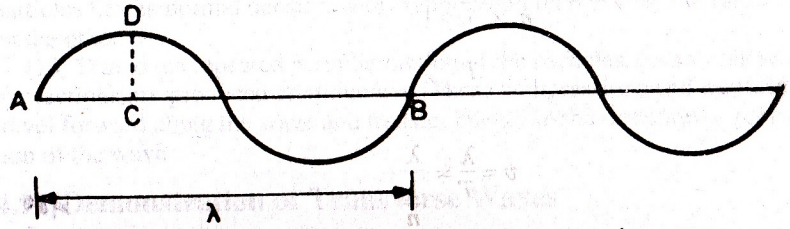


Fig. 4.3.

Frequency. It is the number of vibrations made by a particle in one second.

Amplitude. It is the maximum displacement of the particle from its mean position of rest. In the diagram CD is the amplitude.

Time period. It is the time taken by a particle to complete one vibration.

Suppose frequency = n

Time taken to complete n vibrations = 1 second

Time taken to complete 1 vibration $= \frac{1}{n}$ second

From the definition of time period, time taken to complete one vibration is the time period (T)

$$\therefore \qquad T = \frac{1}{n} \quad \text{or} \quad nT = 1$$

$\therefore$ Frequency × Time period = 1.

Vibration. It is the to- and -fro motion of a particle from one extreme position to the other and back again. It is also equal to the motion of a particle from the mean position to one extreme position, then to the other extreme position and finally back to the mean position.

Phase. It is defined as the ratio of the displacement of the vibrating particle at any instant to the amplitude of the vibrating particle or it is defined as the fraction of the time interval that has elapsed since the particle crossed the mean position of rest in the positive direction or it is also equal to the angle swept by the radius vector since the vibrating particle last crossed its mean position of rest.

4.7. Relation between Frequency and Wavelength

Velocity of the wave is the distance travelled by the wave in one second.

$$\text{Velocity} = \frac{\text{Distance}}{\text{Time}}$$

Wavelength (λ) is the distance travelled by the wave in one time period (T).

$$\text{Velocity} = \frac{\text{Wavelength}}{\text{Time period}} = \frac{\lambda}{T}$$

But, frequency × time period

$$n \times T = 1$$

$$T = \frac{1}{n}$$

$$v = \frac{\lambda}{T} = \frac{\lambda}{\frac{1}{n}}$$

$$v = n\lambda.$$

Example 4.1. *If the frequency of a tuning fork is 400* Hz *and the velocity of sound in air is* 320 metres/s, *find how far sound travels while the fork completes* 30 vibrations.

Here, $n = 400$, $v = 320$ metres/second, $\lambda = ?$

$$v = n\lambda$$

or $$\lambda = \frac{v}{n} = \frac{320}{400} = \mathbf{0 \cdot 8\ metre}$$

∴ Distance travelled by the wave when the fork completes 1 vibration

$= 0{\cdot}8$ metre

Distance travelled by the wave when the fork completes 30 vibrations

$= 0{\cdot}8 \times 30 = \mathbf{24\ metres.}$

4.8. Properties of Longitudinal Progressive Waves

1. The particles of the medium vibrate simple harmonically along the direction of propagation of the wave.

2. All the particles have the same amplitude, frequency and time period.

3. There is a gradual phase difference between the successive particles.

4. All the particles vibrating in phase will be at a distance equal to $n\lambda$. Here $n = 1, 2, 3$ etc. It means the minimum distance between two particles vibrating in phase is equal to the wavelength.

5. The velocity of the particle is maximum at their mean position and it is zero at their extreme positions.

6. When the particle moves in the same direction as the propagation of the wave, it is in a region of compression.

7. When the particle moves in a direction opposite to the direction of propagation of the wave, it is in a region of rarefaction.

8. When the particle is at the mean position, it is a region of maximum compression or rarefaction.

9. When the particle is at the extreme position, the medium around the particles has its normal density, with compression on one side and rarefaction on the other.

(10) Due to the repeated periodic motion of the particles, compressions and rarefactions are produced continuously. These compressions and rarefactions travel forward along the wave and transfer energy in the direction of propagation of the wave.

4.9. Demonstration of Transverse Waves

(*a*) **Wave apparatus.** The formation of transverse waves can be demonstrated in the laboratory with the help of the wave motion apparatus. The apparatus consists of a vertical rectangular frame fixed on a horizontal base. The axle passes eccentrically through a number of circular discs with grooved edges. The circular discs are equidistant. Vertical rods carrying spherical balls at their upper ends rest on the circumference of their respective discs. When the axle is rotated, the rods are displaced vertically through different distances. It is adjusted that there is a gradual phase difference between the successive rods. Each ball will execute simple harmonic motion and will complete one vibration in one rotation of the discs. When the axle is continuously rotated with a uniform speed, the balls show a wave pattern and the transverse wave appears to progress in the forward direction with the formation of alternate crests and troughs. It is observed that the balls move in the vertical direction and the wave advances in the horizontal direction. Therefore, transverse waves are produced.

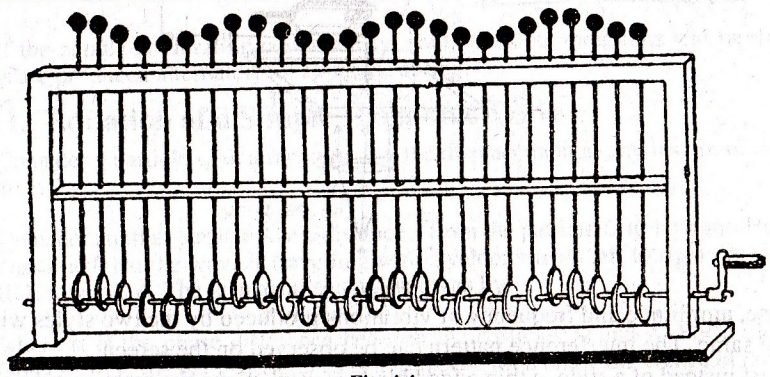


Fig. 4.4.

(*b*) **Ripple tank.** The formation of transverse waves on the surface of water can be demonstrated in the laboratory with the help of ripple tank apparatus.

The ripple tank apparatus consists of a shallow rectangular dish with a glass

bottom. The edges of the dish are sloping outwards so as to avoid interference between the direct and the reflected waves. An electrically maintained tuning fork having a fine style fixed to one of its prongs is adjusted so that the tip of the style just dips in water. The field of view is illuminated from below with the help of a strong source of light S and a condensing lens system C (Fig. 4.5).

The tuning fork is set into vibration. The style oscillates vertically up and down and transverse waves are produced on the surface of water. These waves originate at the tip of the style and travel radially outwards. The image of the water surface is obtained on the screen by reflection from the mirror M. The wave pattern is observed on the screen.

If a stroboscope is used so that the dish is illuminated intermittently with the same frequency as that of the tuning fork a stationary pattern is obtained on the screen.

Using the ripple tank apparatus, interference phenomenon can be demonstrated fixing two styles to the same prong of the tuning fork. In this

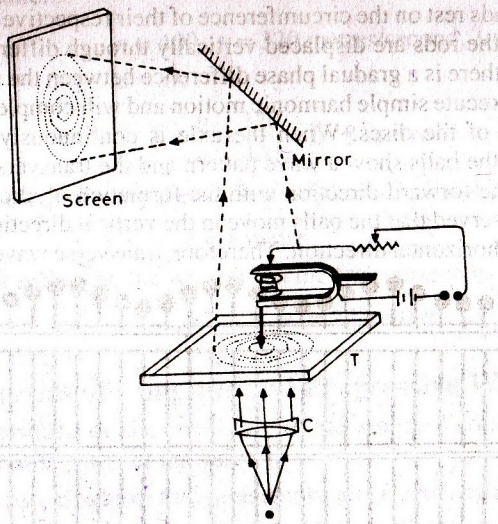


Fig. 4.5.

case, amplitude and frequency of vibrations produced by the two styles will be the same. The interference pattern can be observed on the screen.

If instead of a style, a thin edged blade is used, plane progressive transverse waves are produced on the surface of water.

4.10. Demonstration of Longitudinal Waves

The formation of longitudinal waves can be demonstrated with the help of a spring. One end of the spring is fixed to a handle H and the other end is free. A small push is given to the handle. The first three turns are compressed and the

rest of the spring is in the relaxed position (Fig. 4.6). When the handle is brought back to its original position, the compression travels forward and there is rarefaction between the handle and the compression. If the handle is kept fixed at the initial position it is seen that with time, the compression and rarefaction travel forward as shown in Fig. 4.6. This demonstrates the formation of longitudinal waves in which any particle on the spring vibrates simple harmonically along the direction of propagation of the wave.

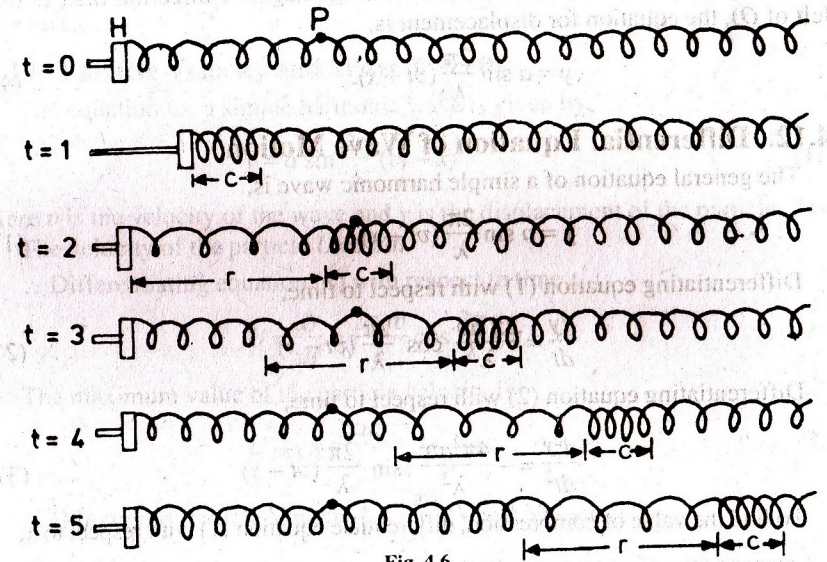


Fig. 4.6.

If the handle is vibrating continuously, continuous compressions and rarefactions are produced alternately all along the spring.

4.11. Equation of a Simple Harmonic Wave

Consider a particle O in a medium. Let the displacement at any instant of time be given by

$$y = a \sin \omega t \quad \ldots (1)$$

Consider another particle A at a distance x from the particle O to its right. Here, it is assumed that the wave is travelling with a velocity from left to right *i.e.* from particle towards A. The displacement at A is given by

$$y = a \sin (\omega t - \alpha) \quad \ldots (2)$$

where α is the phase difference between the particles O and A. For a phase difference of 2π, the pathg difference is λ. Suppose for a phase difference of α, the path difference is x.

$$\therefore \quad \frac{\alpha}{x} = \frac{2\pi}{\lambda}$$

$$\therefore \quad \alpha = \frac{2\pi x}{\lambda}$$

Also

$$\omega = \frac{2\pi}{T} = \frac{2\pi v}{\lambda}$$

Substituting the values of α and ω in equation (2),

$$y = a \sin\left(\frac{2\pi\, vt}{\lambda} - \frac{2\pi x}{\lambda}\right)$$

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (3)$$

Equation (3) represents the equation for a simple harmonic wave.

Similarly for a particle at a distance x in the negative direction (*i.e.*, to the left of O), the equation for displacement is,

$$y = a \sin \frac{2\pi}{\lambda}(vt + x). \qquad \ldots (4)$$

4.12. Differential Equation of Wave Motion

The general equation of a simple harmonic wave is,

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (1)$$

Differentiating equation (1) with respect to time,

$$\frac{dy}{dt} = \frac{2\pi\, av}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (2)$$

Differentiating equation (2) with respect to time,

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (3)$$

To find the value of compression, differentiate equation (1) with respect to x,

$$\therefore \quad \frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (4)$$

To find the rate of change of compression with respect to distance, differentiate equation (4) with respect to x,

$$\frac{d^2 y}{dx^2} = -\frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (5)$$

From equations (2) and (4)

$$\frac{dy}{dt} = -v\frac{dy}{dx} \qquad \ldots (6)$$

From equations (3) and (5)

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2 y}{dx^2} \qquad \ldots (7)$$

Equation (7) represents the differential equation of wave motion.

The general differential equation of wave motion can be written as

$$\frac{d^2 y}{dt^2} = K\frac{d^2 y}{dx^2} \qquad \ldots (8)$$

Here $K = v^2$

or $v = \sqrt{K}$

Thus, knowing the value of K, the value of the wave velocity can be calculated.

4.13. Particle Velocity and Wave Velocity

The equation for a simple harmonic wave is given by

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (1)$$

Here v is the velocity of the wave and y is the displacement of the particle.

The velocity of the particle $U = dy/dt$.

$\therefore$ Differentiating equation (1) with respect to time t,

$$U = \frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (2)$$

The maximum value of the particle velocity is

$$U_{max} = \frac{2\pi av}{\lambda} \qquad \ldots (3)$$

$\therefore$ [Maximum Particle Velocity] $= \frac{2\pi a}{\lambda}$ [Wave Velocity]

To find the particle acceleration, differentiate equation (2) with respect to time

$$f = \frac{d^2 y}{dt^2} = -\frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (4)$$

$$f = -\frac{4\pi^2 v^2}{\lambda^2}\left[a \sin \frac{2\pi}{\lambda}(vt - x)\right]$$

$$\therefore \quad f = -\left(\frac{4\pi^2 v^2}{\lambda^2}\right) y$$

The acceleration is maximum when $y = a$

$$\therefore \quad f_{max} = -\left(\frac{4\pi^2 v^2}{\lambda^2}\right) a \qquad \ldots (5)$$

The negative sign shows that the acceleration of the particle is directed towards its mean position.

Differentiating equation (1) with respect to x

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \quad \ldots (6)$$

dy/dx represents the slope of the displacement curve.

From equations (2) and (6)

$$U = \frac{dy}{dt} = -v\left(\frac{dy}{dx}\right) \quad \ldots (7)$$

$$\therefore \begin{bmatrix}\text{Particle velocity}\\ \text{at any instant}\end{bmatrix} = \begin{bmatrix}\text{wave velocity}\end{bmatrix}\begin{bmatrix}\text{slope of the displacement}\\ \text{curve at that instant}\end{bmatrix}$$

4.14. Distribution of Velocity and Pressure in a Plane Progressive Wave

For a plane progressive wave

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \ldots (1)$$

The particle velocity,

$$U = \frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \quad \ldots (2)$$

The strain in the medium is dy/dx. If dy/dx is positive, it represents a region of rarefaction. If dy/dx is negative, it represents a region of compression. The bulk modulus of elasticity of the medium

$$K = \frac{\text{Change in pressure}}{\text{Volume strain}}$$

$$\therefore \quad K = \frac{-dp}{(dy/dx)}$$

$$\therefore \quad dP = -K \cdot \left(\frac{dy}{dx}\right)$$

$$\therefore \quad dP = K \cdot \left(\frac{-dy}{dx}\right) \quad \ldots (3)$$

It means if dy/dx is negative, dP is +ve *i.e.* it is a region of compression. If dy/dx is positive, dP is –ve, *i.e.* it is a region of rarefaction.

Differentiating equation (1) with respect to x

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x)$$

or

$$dP = K\left(-\frac{dy}{dx}\right)$$

$$dP = \frac{2\pi Ka}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \quad \ldots (4)$$

The graphs for displacement, velocity and change in pressure are shown in Fig. 4.7.

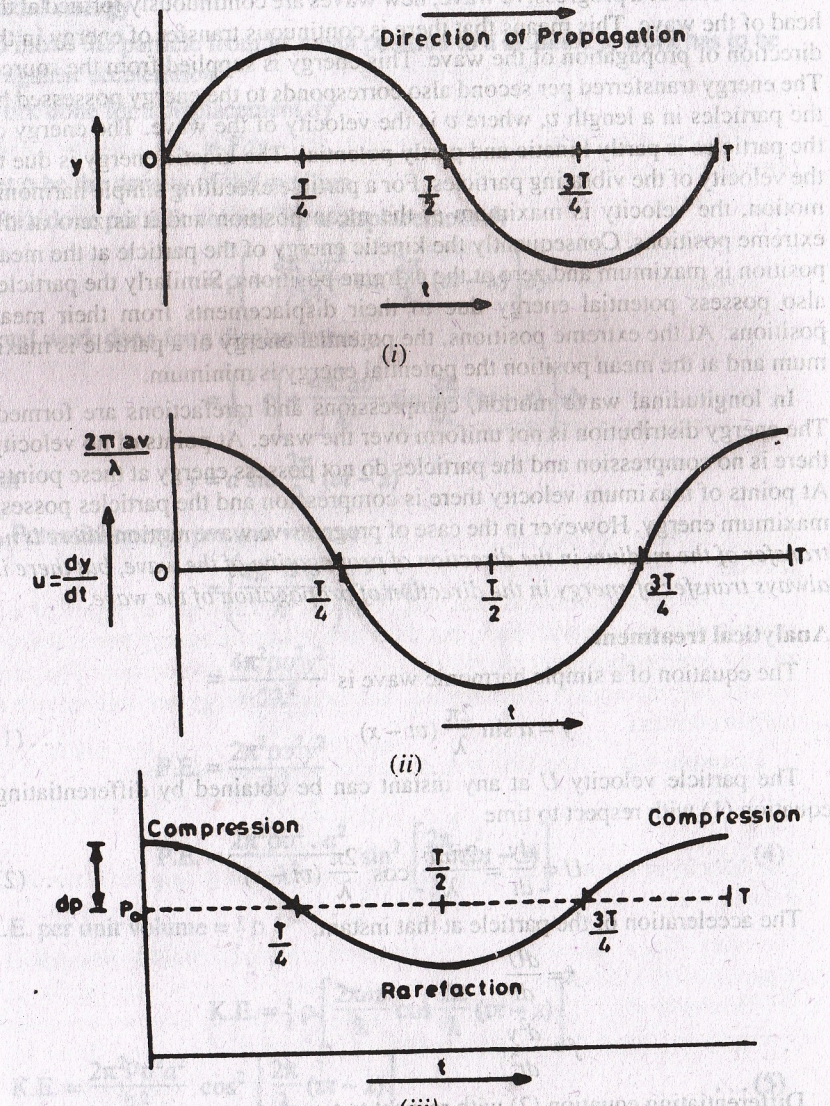


Fig. 4.7.

dP represents the change in pressure. P_0 is the normal pressure which the medium possesses in the absence of the propagation of the wave.

4.15. Energy of a Progressive Wave

In the case of a progressive wave, new waves are continuously formed at the head of the wave. This means that there is continuous transfer of energy in the direction of propagation of the wave. This energy is supplied from the source. The energy transferred per second also corresponds to the energy possessed by the particles in a length v, where v is the velocity of the wave. The energy of the particles is partly kinetic and partly potential. The kinetic energy is due to the velocity of the vibrating particles. For a particle executing simple harmonic motion, the velocity is maximum at the mean position and it is zero at the extreme positions. Consequently the kinetic energy of the particle at the mean position is maximum and zero at the extreme positions. Similarly the particles also possess potential energy due to their displacements from their mean positions. At the extreme positions, the potential energy of a particle is maximum and at the mean position the potential energy is minimum.

In longitudinal wave motion, compressions and rarefactions are formed. The energy distribution is not uniform over the wave. At points of no velocity there is no compression and the particles do not possess energy at these points. At points of maximum velocity there is compression and the particles possess maximum energy. However in the case of progressive wave motion, *there is no transfer of the medium in the direction of propagation of the wave, but there is always transfer of energy in the direction of propagation of the wave.*

Analytical treatment

The equation of a simple harmonic wave is

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (1)$$

The particle velocity U at any instant can be obtained by differentiating equation (1) with respect to time

$$U = \frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (2)$$

The acceleration of the particle at that instant,

$$f = \frac{dU}{dt}$$

$$\therefore \qquad f = \frac{d^2y}{dt^2}$$

Differentiating equation (2) with respect to time,

$$f = \frac{d^2y}{dt^2} = -\frac{4\pi^2 a v^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x) \qquad \ldots (3)$$

The –ve sign shows that acceleration is directed towards the mean position.

Potential energy

To move the particle from its mean position to a distance y, work has to be done against acceleration.

Work done for a displacement dy

$$= F dy$$

Let ρ be the density of the medium

Work done per unit volume for a displacement dy

$$= \rho\left(+\frac{4\pi^2 a v^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x)\right) dy$$

Total work done for a displacement y

$$= \int_0^y \rho\left(+\frac{4\pi^2 a v^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x)\right) dy$$

But $$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

∴ Potential energy per unit volume

$$= \left(\frac{4\pi^2 \rho v^2}{\lambda^2}\right) \int_0^y y\, dy$$

$$= \frac{4\pi^2 \rho v^2 y^2}{2\lambda^2}$$

$$\text{P.E.} = \frac{2\pi^2 \rho v^2 y^2}{\lambda^2}$$

$$\text{P.E.} = \frac{2\pi^2 \rho v^2 . a^2}{\lambda^2} \sin^2 \left[\frac{2\pi}{\lambda}(vt - x)\right] \qquad \ldots (4)$$

K.E. per unit volume $= \frac{1}{2} \rho U^2$

$$\text{K.E.} = \frac{1}{2} \rho \left[\frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x)\right]^2$$

$$\text{K.E.} = \frac{2\pi^2 \rho v^2 a^2}{\lambda^2} \cos^2 \left[\frac{2\pi}{\lambda}(vt - x)\right] \qquad \ldots (5)$$

Total energy per unit volume,

$$E = PE + KE$$

$$= \frac{2\pi^2 \rho v^2 a^2}{\lambda^2}\left[\sin^2\frac{2\pi}{\lambda}(vt - x) + \cos^2\frac{2\pi}{\lambda}(vt - x)\right]$$

$$\therefore \quad E = \frac{2\pi^2 \rho v^2 a^2}{\lambda^2} \quad \ldots (6)$$

But $v = n\lambda$

$$\therefore \quad E = 2\pi^2 \rho n^2 q^2 \quad \ldots (7)$$

The average kinetic energy per unit volume and the average potential energy per unit volume are equal and each is equal to half the total energy per unit volume.

Average K.E. per unit volume $= \pi^2 \rho n^2 a^2$ ... (8)

Average P.E. per unit volume $= \pi^2 \rho n^2 a^2$ ... (9)

Suppose the area of cross-section of a parallel beam of radiation is 1 unit and the velocity of the wave is v.

$$\text{Volume} = 1 \times v = v$$

$\therefore$ Energy transfer per unit area per second

$$= E \times v$$

$$= 2\pi^2 \rho n^2 a^2 v \quad \ldots (10)$$

Energy transfer per second is also called energy current per unit area of cross section.

It is to be remembered that the potential and kinetic energies of every particle will change with time, but the average kinetic energy per unit volume and the average potential energy per unit volume remain constant. The current density per unit area of cross-section and the total energy per unit volume also remain constant.

Example 4.2. *At a time t=0, a train of waves has the form*

$$y = 4 \sin 2\pi\left(\frac{x}{100}\right)$$

The velocity of the wave is 30 cm/s. Find the equation giving the wave form at a time t = 2 s.

$$y = a \sin 2\pi\left[\frac{t}{T} - \frac{x}{\lambda}\right]$$

When

$$y = a \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \quad \ldots (1)$$

$$y = 4 \sin 2\pi\left(\frac{x}{100}\right) \quad \ldots (2)$$

From equations (1) and (2),

$$a = 4$$

and

$$\lambda = 100 \text{ cm}$$

At time $t = 2$ s,

$$x = 2 \times 30 = 60 \text{ cm}$$

$$y = a \sin 2\pi\left[\frac{x}{100} - \frac{60}{100}\right]$$

$$y = 4 \sin 2\pi\left[\frac{x}{100} - 0{\cdot}6\right] \quad (3)$$

Example 4.3. *When a simple harmonic wave is propagated through a medium, the displacement of a particle in cm at any instant of time is given by*

$$y = 10 \sin \frac{2\pi}{100}(36000t - 20).$$

Calculate, the amplitude of the vibrating particle, wave velocity, wavelength, frequency and time period.

The general equation of a simple harmonic wave is,

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \ldots (1)$$

Here

$$y = 10 \sin \frac{2\pi}{100}(36000t - 20) \quad \ldots (2)$$

Comparing equations (1) and (2)

$$a = 10 \text{ cm}$$

$$\lambda = 100 \text{ cm}$$

Wave velocity, $v = 36000$ cm/s

Frequency $n = \frac{v}{\lambda} = \frac{36000}{100}$

or $n = 360$ hertz

Time period $T = \frac{1}{n} = \frac{1}{360}$ second.

Example 4.4. *A simple harmonic wave of amplitude 8* units *traverses a line of particles in the direction of the positive X-axis. At any given instant of time, for a particle at a distance of 10* cm *from the origin, the displacement is +6* units and for a particle at a distance of 25 cm *from the origin, the displacement is + 4 units. Calculate the wavelength.*

$$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

or
$$\frac{y}{a} = \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

(1) In the first case

$$\frac{y_1}{a} = \sin 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right)$$

Here $y_1 = +6,\ a = 8,\ x_1 = 10$ cm

$$\therefore \quad \frac{6}{8} = \sin 2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right) \qquad \ldots (1)$$

(2) In the second case,

$$\frac{y_2}{a} = \sin 2\pi\left(\frac{t}{T} - \frac{x_2}{\lambda}\right)$$

Here $y_2 = +4,\ a = 8,\ x_2 = 25$

$$\therefore \quad \frac{4}{8} = \sin 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right) \qquad \ldots (2)$$

From equation (1)

$$0{\cdot}75 = \sin 2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right)$$

But
$$\sin\left(\frac{48{\cdot}6\,\pi}{180}\right) = 0{\cdot}75$$

$$\therefore \quad 2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right) = \frac{48{\cdot}6\,\pi}{180} \qquad \ldots (3)$$

or
$$\frac{t}{T} - \frac{10}{\lambda} = \frac{48{\cdot}6}{360} \qquad \ldots (4)$$

From equation (2)

$$0{\cdot}5 = \sin 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right)$$

But
$$\sin\frac{\pi}{6} = 0{\cdot}5$$

$$\therefore \quad 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right) = \frac{\pi}{6} \qquad \ldots (5)$$

$$\frac{t}{T} - \frac{25}{\lambda} = \frac{1}{12} \qquad \ldots (6)$$

Subtracting (6) from (4)

$$\frac{25}{\lambda} - \frac{10}{\lambda} = \frac{48{\cdot}6}{360} - \frac{1}{12}$$

or
$$\lambda = 290{\cdot}8 \text{ cm.}$$

Example 4.5. *The velocity of a simple harmonic wave is 30 cm/s. At time $t = 0$ the displacement of a particle is given by*

$$y = 4 \sin 2\pi\left(\frac{x}{100}\right)$$

Find the equation for the displacement at time $t = 2$ s.

The general equation of a simple harmonic wave is

$$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

or
$$y = a \sin\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$$

When $t = 0$

$$y = a \sin\left(-\frac{2\pi x}{\lambda}\right)$$

or
$$y = -a \sin \frac{2\pi x}{\lambda} \qquad \ldots (1)$$

At $t = 0$, the given equation is

$$y = 4 \sin 2\pi\left(\frac{x}{100}\right) \qquad \ldots (2)$$

Comparing equations (1) and (2)

$$a = -4 \qquad \text{and} \qquad \lambda = 100 \text{ cm}$$

At time $t = 2$ s

$$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

Here $a = -4$ cm, $\lambda = 100$ cm, $t = 2$ s

$$y = -4 \sin \frac{2\pi}{100}(30 \times 2 - x)$$

$$y = -4 \sin\left[\frac{6\pi}{5} - 2\pi\left(\frac{x}{100}\right)\right]$$

or
$$y = 4 \sin\left[2\pi\left(\frac{x}{100}\right) - \frac{6\pi}{5}\right]$$

Example 4.6. *A simple harmonic wave is represented by*

$$y = 10 \sin\left(\frac{2\pi t}{T} + \alpha\right)$$

The time period is 30 s. At time $t = 0$, the displacement is 5 cm. Calculate (i) phase angle at $t = 7{\cdot}5$ s and (ii) phase difference between two positions at a time interval of 6 s.

Here $y = 10 \sin\left(\frac{2\pi t}{T} + \alpha\right)$

$T = 30$ s.

(*i*) At $t = 0$

$y = 5$ cm

$\therefore \quad 5 = 10 \sin \alpha$

$\sin \alpha = 0{\cdot}5$

$\alpha = \frac{\pi}{6}$ **radian**

or

(*ii*) At $t = 7{\cdot}5$ s,

the phase angle $\phi = \left(\frac{2\pi t}{T} + \alpha\right)$

$\phi = \frac{2\pi \times 7{\cdot}5}{30} + \frac{\pi}{6}$

$\phi = \frac{2\pi}{3}$ **radian**

The phase difference for a time interval of 6 s

$\theta = \frac{2\pi t}{T}$

Here $t = 6$ s

$\theta = \frac{2\pi \times 6}{30} = \frac{2\pi}{5}$ **radian.**

Example 4.7. *A source of sound has a frequency of 512 Hz and an amplitude of 0·25 cm. What is the flow of energy across a square cm per second, if the velocity of sound in air is 340 m/s and the density of air is 0·00129 g/cm^3 ?* (Delhi)

Here $n = 512$ Hz

$a = 0{\cdot}25$ cm

$\rho = 0{\cdot}00129$ g/cm^3

$v = 340$ m/s $= 34000$ cm/s

Total energy per unit volume

$= 2\pi^2 \rho n^2 a^2$.

Energy transferred across a sq cm per second

$= 2\pi^2 \rho n^2 a^2 v$

$= 2 \times (3{\cdot}14)^2 \times 0{\cdot}00129 \times (512)^2 (0{\cdot}25)^2\, 34000$

$= 1{\cdot}417 \times 10^7$ **ergs/cm^2-s.**

(Delhi)

EXERCISES

1. Explain the term wave motion. Distinguish clearly between transverse and longitudinal wave motion.
2. Describe briefly the properties of longitudinal progressive waves.
3. Describe experiments to demonstrate the formation of (*i*) transverse waves and (*ii*) longitudinal waves.
4. Obtain the equation for a plane simple harmonic wave.
5. Derive the relation

$$\frac{d^2y}{dt^2} = v^2 \cdot \frac{d^2y}{dx^2}.$$

6. Distinguish between particle velocity and wave velocity and obtain the relation between the two.
7. Show that

$$U = -v \frac{dy}{dx}$$

where U is the particle velocity and v is the wave velocity. *(Panjab, 1993)*

8. Discuss the variation of velocity and pressure in a plane progressive wave.
9. Show that the energy of a plane progressive wave is given by

$$E = 2\pi^2 \rho n^2 a^2.$$

10. Discuss the distribution of energy in a plane progressive wave.
11. Show how a transverse wave may be represented by a curve and explain how the velocity of any particle can be determined from the curve.
12. Derive the equation of wave motion in the form

$$y = a \sin \frac{2\pi}{\lambda}(vt - x).$$

13. Show that for a plane progressive wave, on the average, half the energy is kinetic and half potential. Hence derive the expression for intensity of sound in terms of pressure amplitude. *(Delhi, 1992)*

14. Show that in the case of progressive longitudinal waves
Particle velocity=Wave velocity × Compression. *(Panjab)*

15. Deduce the equation of a simple harmonic wave travelling in the positive x direction in the form

$$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

and explain how energy is distributed in such a progressive wave. *(Delhi)*

CHAPTER 5

Velocity of Sound

5.1. Origin of Sound

Sound is produced by a vibrating body. When the gong of a bell is struck with a hammer, sound is produced. The bell is set into vibration and sound is propagated through air. These vibrations reach the ear and the ear drum is set into vibration. These vibrations are communicated to the brain. By touching the gong with the hand, one can feel the vibration of the gong. Similarly the cycle bell produces sound due to the vibrations produced by the gong. When the cycle bell is touched with hand, the vibrations are stopped and the bell does not produce sound. If a pith ball pendulum is held in contact with the edges of a vibrating gong, the pith ball moves to and fro. This shows that the gong vibrates as long as the sound is produced.

A tuning fork is set into vibration by striking one of its prongs against a rubber pad. The vibration of the prongs of the tuning fork can be seen. Similarly, vibrating strings, air columns, vibrating plates etc. produce sound.

5.2. Material Medium is a Necessity

It can be proved by means of an experiment that a material medium is a necessity for the propagation of sound waves. In the absence of a medium, no sound waves can travel.

Experiment. Take a jar and fix a bell inside it as shown. The connecting wires pass through an air tight rubber cork fixed to the neck of the jar. The electric bell is connected to a battery and a key outside the jar (Fig. 5.1). The jar is placed on the platform of an air pump. Insert the key. The bell rings and the sound is heard. When the air in the jar is removed gradually, the sound becomes fainter although the same current is passing

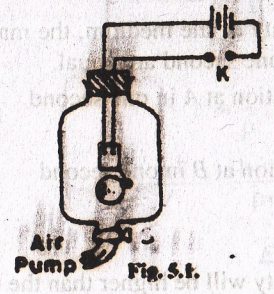


Fig. 5.1.