CHAPTER 12

Linear Equations of Order n

A LINEAR DIFFERENTIAL EQUATION of order n has the form

$$1) \qquad P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + P_{n-1} \frac{dy}{dx} + P_n y = Q,$$

where $P_0 \neq 0$, P_1, P_2, $\cdots\cdots$, P_n, Q are functions of x or constants.

If $Q = 0$, 1) has the form

$$2) \qquad P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + P_{n-1} \frac{dy}{dx} + P_n y = 0$$

and is called *homogeneous* to indicate that all of the terms are of the same (first) degree in y and its derivatives.

Examples. *A)* $x^3 \dfrac{d^3 y}{dx^3} + 2x \dfrac{d^2 y}{dx^2} - 5 \dfrac{dy}{dx} - xy = \sin x$, of order 3.

B) $\dfrac{d^2 y}{dx^2} - 3 \dfrac{dy}{dx} + 2y = 0$, of order 2.

Equation *B)* is an example of a homogeneous linear equation.

SOLUTIONS. If $y = y_1(x)$ is a solution of 2), then $y = C_1 y_1(x)$, where C_1 is an arbitrary constant, is also a solution. If $y = y_1(x)$, $y = y_2(x)$, $y = y_3(x)$, $\cdots\cdots$ are solutions of 2), then $y = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x) + \cdots\cdots$ is also a solution.

A set of solutions $y = y_1(x)$, $y = y_2(x)$, $\cdots\cdots$, $y = y_n(x)$ of 2) is said to be *linearly independent* if the equality $c_1 y_1 + c_2 y_2 + c_3 y_3 \cdots\cdots + c_n y_n = 0$,

where the c's are constants, holds only when $c_1 = c_2 = c_3 = \cdots\cdots = c_n = 0$.

Example 1. The functions e^x and e^{-x} are linearly independent. To show this, form $c_1 e^x + c_2 e^{-x} = 0$, where c_1 and c_2 are constants, and differentiate to obtain $c_1 e^x - c_2 e^{-x} = 0$. When the two relations are solved simultaneously for c_1 and c_2, we find $c_1 = c_2 = 0$.

Example 2. The functions e^x, $2e^x$, and e^{-x} are linearly dependent, since

$$c_1 e^x + 2c_2 e^x + c_3 e^{-x} = 0 \quad \text{when} \quad c_1 = 2, \; c_2 = -1, \; c_3 = 0.$$

A necessary and sufficient condition that the set of n solutions be linearly independent is that:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0.$$

If $y = y_1(x)$, $y = y_2(x)$, $\cdots\cdots$, $y = y_n(x)$ are n linearly independent solutions of 2), then

3) $$y = C_1 y_1(x) + C_2 y_2(x) + \cdots\cdots + C_n y_n(x)$$

is the primitive of 2).

If $y = R(x)$ is a particular solution, also called *particular integral*, of 1) then

4) $$y = C_1 y_1(x) + C_2 y_2(x) + \cdots\cdots + C_n y_n(x) + R(x)$$

is the primitive of 1). Note that 4) contains all of 3). This part of 4) is called the *complementary function*. Thus the primitive of 1) consists of the sum of the complementary function and a particular integral.

Attention has been called to the fact that the primitive of a differential equation is not necessarily the complete solution of the equation. However, when the equation is linear, the primitive is its complete solution. Thus 3) and 4) may be called complete solutions of 2) and 1) respectively.

LINEAR DIFFERENTIAL EQUATIONS with constant coefficients (equations B) above) will be treated in Chapters 13 -16. Those with variable coefficients (equation A) above) will be considered in Chapters 17-19.

SOLVED PROBLEMS

1. Show that the equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$ has two distinct solutions of the form $y = e^{ax}$.

If $y = e^{ax}$, for some value of a, is a solution then the given equation is satisfied when the replacements $y = e^{ax}$, $\frac{dy}{dx} = ae^{ax}$, $\frac{d^2y}{dx^2} = a^2e^{ax}$ are made in it.

We obtain $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^{ax}(a^2 - a - 2) = 0$ which is satisfied when $a = -1, 2$.

Thus $y = e^{-x}$ and $y = e^{2x}$ are solutions.

2. Show that $y = C_1e^{-x} + C_2e^{2x}$ is the primitive of the equation of Problem 1.

Substituting for y and its derivatives in the differential equation, it is readily checked that $y = C_1e^{-x} + C_2e^{2x}$ is a solution. To show that it is the primitive, we note first that the number (2) of arbitrary constants and the order (2) of the equation agree and second that

since $W = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^{x} \neq 0$, $y = e^{-x}$ and $y = e^{2x}$ are linearly independent.

3. Show that the differential equation $x^3 \frac{d^3y}{dx^3} - 6x \frac{dy}{dx} + 12y = 0$ has three linearly independent solutions of the form $y = x^r$.

After making the replacements

$$y = x^r, \quad \frac{dy}{dx} = rx^{r-1}, \quad \frac{d^2y}{dx^2} = r(r-1)x^{r-2}, \quad \frac{d^3y}{dx^3} = r(r-1)(r-2)x^{r-3}$$

in the left member of the given equation, we have $x^r(r^3 - 3r^2 - 4r + 12) = 0$ which is satisfied when $r = 2, 3, -2$. The corresponding solutions $y = x^2$, $y = x^3$, $y = x^{-2}$ are linearly independent

since $W = \begin{vmatrix} x^2 & x^3 & x^{-2} \\ 2x & 3x^2 & -2x^{-3} \\ 2 & 6x & 6x^{-4} \end{vmatrix} = 20 \neq 0$. The primitive is $y = C_1x^2 + C_2x^3 + C_3x^{-2}$.

4. Verify that $y = -\sin x$ is a particular integral of $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \cos x + 3\sin x$ and write the primitive.

Substituting for y and its derivatives in the differential equation, it is found that the equation is satisfied. From Problem 2, the complementary function is $y = C_1e^{-x} + C_2e^{2x}$.

Hence, the primitive is $y = C_1e^{-x} + C_2e^{2x} - \sin x$.

5. Verify that $y = \ln x$ is a particular integral of $x^3 \frac{d^3y}{dx^3} - 6x \frac{dy}{dx} + 12y = 12\ln x - 4$ and write the primitive.

Substituting for y and its derivatives in the given equation it is found that the equation is satisfied. From Problem 3, the complementary function is $y = C_1x^2 + C_2x^3 + C_3x^{-2}$.

Hence, the primitive is $y = C_1x^2 + C_2x^3 + C_3x^{-2} + \ln x$.

6. Show that $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$ has only two linearly independent solutions of the form $y = e^{ax}$.

Substituting for y and its derivatives in the given equation, we have $e^{ax}(a^4 - a^3 - 3a^2 + 5a - 2) = 0$ which is satisfied when $a = 1, 1, 1, -2$.

Since $\begin{vmatrix} e^{x} & e^{-2x} \\ e^{x} & -2e^{-2x} \end{vmatrix} \neq 0$, but. $\begin{vmatrix} e^{x} & e^{x} & e^{x} & e^{-2x} \\ e^{x} & e^{x} & e^{x} & -2e^{-2x} \\ e^{x} & e^{x} & e^{x} & 4e^{-2x} \\ e^{x} & e^{x} & e^{x} & -8e^{-2x} \end{vmatrix} = 0$, the linearly independent solutions are $y = e^{x}$ and $y = e^{-2x}$.

7. Verify that $y = e^x$, $y = xe^x$, $y = x^2e^x$, and $y = e^{-2x}$ are four linearly independent solutions of the equation of Problem 6 and write the primitive.

By Problem 6, $y = e^x$ and $y = e^{-2x}$ are solutions. By direct substitution in the given equation it is found that the others are solutions.

Since $W = \begin{vmatrix} e^x & xe^x & x^2e^x & e^{-2x} \\ e^x & xe^x + e^x & x^2e^x + 2xe^x & -2e^{-2x} \\ e^x & xe^x + 2e^x & x^2e^x + 4xe^x + 2e^x & 4e^{-2x} \\ e^x & xe^x + 3e^x & x^2e^x + 6xe^x + 6e^x & -8e^{-2x} \end{vmatrix} = e^x \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 \\ 1 & 2 & 2 & 4 \\ 1 & 3 & 6 & -8 \end{vmatrix} = -54e^x \neq 0,$

these solutions are linearly independent and the primitive is

$$y = C_1e^x + C_2xe^x + C_3x^2e^x + C_4e^{-2x}.$$

8. Verify that $y = e^{-2x}\cos 3x$ and $y = e^{-2x}\sin 3x$ are solutions of $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13y = 0$ and write the primitive.

Substituting for y and its derivatives, it is found that the equation is satisfied.
Since $W = 3e^{-4x} \neq 0$, the solutions are linearly independent.
Hence, the primitive is $y = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x)$.

SUPPLEMENTARY PROBLEMS

9. Show that each of the following sets of functions are linearly independent.

a) $\sin ax$, $\cos ax$ *c)* 1, x, x^2 *e)* $\ln x$, $x \ln x$, $x^2 \ln x$

b) $e^{ax}\sin bx$, $e^{ax}\cos bx$ *d)* e^{ax}, e^{bx}, e^{cx} $(a \neq b \neq c)$

Form the differential equation having the given primitive.

10. $y = C_1e^{2x} + C_2e^{-3x}$ *Ans.* $y'' + y' - 6y = 0$

11. $y = C_1e^{2x} + C_2xe^{2x} + C_3x^2e^{2x}$ $y''' - 6y'' + 12y' - 8y = 0$

12. $y = C_1e^x + C_2e^{2x} + e^{3x}/12$ $y'' - 3y' + 2y = e^{3x}$

13. $y = C_1\cos 3x + C_2\sin 3x + (4x\cos x + \sin x)/32$ $y'' + 9y = x\cos x$

14. $y = C_1x^2 + C_2x^2 \ln x$ $x^2y'' - 3xy' + 4y = 0$

15. $y = C_1x + C_2x\ln x + C_3x\ln^2 x + x^4/9$ $x^3y''' + xy' - y = 3x^4$

16. $y = C_1\sin x^2 + C_2\cos x^2$ $xy'' - y' + 4x^3y = 0$

17. $y = \ln\sin(x - C_1) + C_2$ $y'' + (y')^2 + 1 = 0$

18. $y^2 = C_1x + C_2 + 2x^2$ $yy'' + (y')^2 = 2$

19. $x = C_1 + C_2y + y\ln y$ $yy'' + (y')^3 = 0$

CHAPTER 13

Homogeneous Linear Equations with Constant Coefficients

THE HOMOGENEOUS LINEAR EQUATION with constant coefficients has the form

$$1)\quad P_0\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots\cdots + P_{n-1}\frac{dy}{dx} + P_n y = 0$$

in which $P_0 \neq 0$, P_1, P_2, $\cdots\cdots$, P_n are constants.

By a convenient change of notation, writing $\frac{dy}{dx} = Dy$, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = D\cdot Dy = D^2y$, etc., 1) becomes

$$(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + \cdots\cdots\cdots + P_{n-1}D + P_n)y = 0.$$

Now $D = \frac{d}{dx}$ is an operator which acts on y and

$$3)\quad P_0D^n + P_1D^{n-1} + P_2D^{n-2} + \cdots\cdots\cdots + P_{n-1}D + P_n$$

is simply a much more complex operator. However, we shall find it very convenient to consider 3) at times as a polynomial in the variable D and to denote it by $F(D)$. Thus, 1) may be written briefly as

$$4)\qquad F(D)y = 0.$$

It can be shown in general and will be indicated by an example that when 3) is treated as a polynomial and factored as

$$5)\quad F(D) = P_0(D-m_1)(D-m_2)(D-m_3)\cdots\cdots\cdots(D-m_{n-1})(D-m_n),$$

then

$$6)\quad F(D)y = P_0(D-m_1)(D-m_2)(D-m_3)\cdots\cdots\cdots(D-m_{n-1})(D-m_n)y = 0$$

remains valid, i.e., is equivalent to 1) when D is treated as an operator.

EXAMPLE. In the D notation $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ becomes $(D^3 - D^2 - 4D + 4)y = 0$ and, in factored form, $(D-1)(D-2)(D+2)y = 0$. Now

$$(D-1)(D-2)(D+2)y = (D-1)(D-2)(\frac{d}{dx}+2)y = (D-1)(D-2)(\frac{dy}{dx}+2y)$$

$$= (D-1)\{\frac{d}{dx}(\frac{dy}{dx}+2y) - 2(\frac{dy}{dx}+2y)\} = (D-1)(\frac{d^2y}{dx^2}-4y)$$

$$= \frac{d}{dx}(\frac{d^2y}{dx^2}-4y) - 1(\frac{d^2y}{dx^2}-4y)$$

$$= \frac{d^3y}{dx^3} - 4\frac{dy}{dx} - \frac{d^2y}{dx^2} + 4y = \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

In Problem 1 below, it will be indicated that the order of the factors here is immaterial.

THE EQUATION $F(D) = (D-m_1)(D-m_2)(D-m_3)\cdots\cdots(D-m_{n-1})(D-m_n) = 0$ is sometimes called the characteristic equation of 1) and the roots $m_1, m_2, m_3, \cdots, m_n$ are called the characteristic roots. Note that it is never necessary to write the characteristic equation since its roots can be read directly from 6).

TO OBTAIN THE PRIMITIVE of 1) we first write the equation in the form 6).

a) Suppose $m_1 \neq m_2 \neq m_3 \neq \cdots\cdots\cdots \neq m_{n-1} \neq m_n$. Then

$$y = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x} + \cdots\cdots\cdots + C_ne^{m_nx},$$

involving n linearly independent solutions of 1) with n arbitrary constants, is the primitive.

Thus in the example above, where $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$ or $(D-1)(D-2)(D+2)y = 0$, the characteristic roots are 1,2,-2 and the primitive is $y = C_1e^x + C_2e^{2x} + C_3e^{-2x}$.

See also Problems 5-7

b) Suppose $m_1 = m_2 \neq m_3 \neq \cdots\cdots\cdots \neq m_{n-1} \neq m_n$. Then

$$y = C_1e^{m_1x} + C_2xe^{m_1x} + C_3e^{m_3x} + \cdots\cdots\cdots + C_ne^{m_nx}$$

is the primitive.

In general, to a root m occuring r times there corresponds

$$C_1e^{mx} + C_2xe^{mx} + C_3x^2e^{mx} + \cdots\cdots + C_rx^{r-1}e^{mx}$$

in the primitive.

Thus to solve $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 8y = 0$, write the equation as

$(D^3 - 2D^2 - 4D + 8)y = (D-2)^2(D+2)y = 0$. The characteristic roots are 2,2,-2 and the primitive is

$y = C_1e^{2x} + C_2xe^{2x} + C_3e^{-2x}$.

See also Problems 8-10.

c) If the coefficients of 1) are real and if $a+bi$ is a complex root of 6), so also is $a-bi$. The corresponding terms in the primitive are

$$\begin{aligned} Ae^{(a+bi)x} + Be^{(a-bi)x} &= e^{ax}(Ae^{bix} + Be^{-bix}) \\ &= e^{ax}(C_1\cos bx + C_2\sin bx) \\ &= Pe^{ax}\sin(bx+Q) = Pe^{ax}\cos(bx+R), \end{aligned}$$

where A, B, C_1, C_2, P, Q, R are arbitrary constants.

Thus the characteristic roots of $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 5y = 0$ or $(D^2 - 4D + 5)y = 0$ are $2 \pm i$. Here $a = 2$, $b = 1$ and the primitive is $y = e^{2x}(C_1\cos x + C_2\sin x)$.

See also Problems 11 -15.

SOLVED PROBLEMS

1. Show that $(D-a)(D-b)(D-c)y = (D-b)(D-c)(D-a)y$.

$$(D-a)(D-b)(D-c)y = (D-a)(D-b)\left(\frac{dy}{dx} - cy\right) = (D-a)\left(\frac{d^2y}{dx^2} - (b+c)\frac{dy}{dx} + bcy\right)$$

$$= \frac{d^3y}{dx^3} - (a+b+c)\frac{d^2y}{dx^2} + (ab+bc+ac)\frac{dy}{dx} - abcy.$$

$$(D-b)(D-c)(D-a)y = (D-b)(D-c)\left(\frac{dy}{dx} - ay\right) = (D-b)\left(\frac{d^2y}{dx^2} - (a+c)\frac{dy}{dx} + acy\right)$$

$$= \frac{d^3y}{dx^3} - (a+b+c)\frac{d^2y}{dx^2} + (ab+ac+bc)\frac{dy}{dx} - abcy.$$

2. Verify that $y = C_1e^{ax} + C_2e^{bx} + C_3e^{cx}$ satisfies the differential equation $(D-a)(D-b)(D-c)y = 0$.

We are to show that $(D-a)(D-b)(D-c)(C_1e^{ax} + C_2e^{bx} + C_3e^{cx}) = 0$.

$(D-a)(D-b)(D-c)C_1e^{ax} = (D-b)(D-c)(D-a)C_1e^{ax} = (D-b)(D-c)0 = 0$, and similarly for the other two terms.

3. Verify that $y = C_1e^{mx} + C_2xe^{mx} + C_3x^2e^{mx}$ satisfies the differential equation $(D-m)^3y = 0$.

This follows since: *a)* $(D-m)^3C_1e^{mx} = (D-m)^2(D-m)C_1e^{mx} = (D-m)^20 = 0$,

b) $(D-m)^3C_2xe^{mx} = (D-m)^2C_2e^{mx} = (D-m)0 = 0$, and

c) $(D-m)^3C_3x^2e^{mx} = 2(D-m)^2C_3xe^{mx} = 2(D-m)0 = 0$.

4. Find the primitive of $(D-m)^2y = 0$ *(a)* by assuming a solution of the form $y = x^re^{mx}$ and *(b)* by solving the equivalent pair of equations $(D-m)y = v$, $(D-m)v = 0$.

a) $(D-m)^2y = (D-m)(D-m)x^re^{mx} = (D-m)rx^{r-1}e^{mx} = r(r-1)x^{r-2}e^{mx} = 0$ when $r = 0, 1$.

Thus the equation has two linearly independent solutions $y = e^{mx}$ and $y = xe^{mx}$.

The primitive is $y = C_1e^{mx} + C_2xe^{mx}$.

b) If we write $(D-m)y = v$, then $(D-m)^2y = (D-m)(D-m)y = (D-m)v = 0$.

Solving $(D-m)v = 0$, we obtain $v = C_2e^{mx}$. Since $(D-m)y = \frac{dy}{dx} - my = C_2e^{mx}$ is linear of the first order, its solution by the method of Chapter 6 is

$$ye^{-mx} = \int e^{-mx}(C_2e^{mx})dx = C_1 + C_2x \quad \text{or} \quad y = C_1e^{mx} + C_2xe^{mx}.$$

DISTINCT REAL ROOTS.

5. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

We write the equation as $(D^2 + D - 6)y = (D-2)(D+3)y = 0$.

The characteristic roots are 2,-3, and the primitive is $y = C_1e^{2x} + C_2e^{-3x}$.

6. Solve $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 12\frac{dy}{dx} = 0.$

We write the equation as $(D^3 - D^2 - 12D)y = 0$ or $D(D-4)(D+3)y = 0.$

The characteristic roots are 0,4,-3, and the primitive is $y = C_1 + C_2e^{4x} + C_3e^{-3x}.$

7. Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0.$

We write the equation as $(D^3 + 2D^2 - 5D - 6)y$ or $(D-2)(D+1)(D+3)y = 0.$

The characteristic roots are 2, -1, -3, and the primitive is $y = C_1e^{2x} + C_2e^{-x} + C_3e^{-3x}.$

REPEATED ROOTS.

8. Solve $(D^3 - 3D^2 + 3D - 1)y = 0$ or $(D-1)^3y = 0.$

The characteristic roots are 1,1,1, and the primitive is $y = C_1e^x + C_2xe^x + C_3x^2e^x.$

9. Solve $(D^4 + 6D^3 + 5D^2 - 24D - 36)y = 0$ or $(D-2)(D+2)(D+3)^2y = 0.$

The characteristic roots are 2, -2, -3, -3. The primitive is $y = C_1e^{2x} + C_2e^{-2x} + C_3e^{-3x} + C_4xe^{-3x}.$

10. Solve $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$ or $(D+1)^3(D-4)y = 0.$

The characteristic roots are -1,-1,-1,4. The primitive is $y = e^{-x}(C_1 + C_2x + C_3x^2) + C_4e^{4x}.$

COMPLEX ROOTS.

11. Solve $(D^2 - 2D + 10)y = 0.$

The characteristic roots are $1 \pm 3i$, and the primitive is

$$y = e^x(C_1\cos 3x + C_2\sin 3x) \quad \text{or} \quad C_3e^x\sin(3x + C_4) \quad \text{or} \quad C_3e^x\cos(3x + C_5).$$

12. Solve $(D^3 + 4D)y = 0$ or $D(D^2 + 4)y = 0.$

The characteristic roots are $0, \pm 2i$, and the primitive is $y = C_1 + C_2\cos 2x + C_3\sin 2x.$

13. Solve $(D^4 + D^3 + 2D^2 - D + 3)y = 0$ or $(D^2 + 2D + 3)(D^2 - D + 1)y = 0.$

The characteristic roots are $-1 \pm i\sqrt{2}$, $\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$, and the primitive is

$$y = e^{-x}(C_1\cos\sqrt{2}\,x + C_2\sin\sqrt{2}\,x) + e^{\frac{1}{2}x}(C_3\cos\tfrac{1}{2}\sqrt{3}\,x + C_4\sin\tfrac{1}{2}\sqrt{3}\,x).$$

14. Solve $(D^4 + 5D^2 - 36)y = 0$ or $(D^2 - 4)(D^2 + 9)y = 0.$

The characteristic roots are $\pm 2, \pm 3i$, and the primitive is

$$y = Ae^{2x} + Be^{-2x} + C_3\cos 3x + C_4\sin 3x$$
$$= C_1\cosh 2x + C_2\sinh 2x + C_3\cos 3x + C_4\sin 3x$$

since $\cosh 2x = \frac{1}{2}(e^{2x} + e^{-2x})$ and $\sinh 2x = \frac{1}{2}(e^{2x} - e^{-2x}).$

15. Solve $(D^2 - 2D + 5)^2y = 0.$ The characteristic roots are $1 \pm 2i$, $1 \pm 2i$, and the primitive is

$$y = e^x(C_1\cos 2x + C_2\sin 2x) + xe^x(C_3\cos 2x + C_4\sin 2x)$$
$$= e^x\{(C_1 + C_3x)\cos 2x + (C_2 + C_4x)\sin 2x\}.$$

SUPPLEMENTARY PROBLEMS

Solve.

16. $(D^2 + 2D - 15)y = 0$ *Ans.* $y = C_1e^{3x} + C_2e^{-5x}$

17. $(D^3 + D^2 - 2D)y = 0$ $y = C_1 + C_2e^{x} + C_3e^{-2x}$

18. $(D^2 + 6D + 9)y = 0$ $y = C_1e^{-3x} + C_2xe^{-3x}$

19. $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$ $y = C_1 + C_2e^{2x} + C_3xe^{2x} + C_4x^2e^{2x}$

20. $(D^2 - 4D + 13)y = 0$ $y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$

21. $(D^2 + 25)y = 0$ $y = C_1 \cos 5x + C_2 \sin 5x$

22. $(D^3 - D^2 + 9D - 9)y = 0$ $y = C_1e^{x} + C_2 \cos 3x + C_3 \sin 3x$

23. $(D^4 + 4D^2)y = 0$ $y = C_1 + C_2x + C_3 \cos 2x + C_4 \sin 2x$

24. $(D^4 - 6D^3 + 13D^2 - 12D + 4)y = 0$ $y = (C_1 + C_2x)e^{x} + (C_3 + C_4x)e^{2x}$

25. $(D^6 + 9D^4 + 24D^2 + 16)y = 0$ $y = C_1 \cos x + C_2 \sin x + (C_3 + C_4x)\cos 2x + (C_5 + C_6x)\sin 2x$

CHAPTER 14

Linear Equations with Constant Coefficients

THE PRIMITIVE OF

1) $$F(D)y = (P_0 D^n + P_1 D^{n-1} + \cdots\cdots + P_{n-1} D + P_n)y = Q(x),$$

where $P_0 \neq 0$, P_1, P_2, $\cdots\cdots$, P_n are constants and $Q = Q(x) \neq 0$, is the sum of the complementary function (primitive of $F(D)y = 0$ obtained in the preceding chapter) and any particular integral of 1). (See Chapter 12.)

At times a particular integral may be found by inspection. For example, $y = \frac{1}{2}x$ is a particular integral of $(D^3 - 3D^2 + 2)y = x$, since $D^3 y = D^2 y = 0$. Such equations occur infrequently, however, and we proceed to consider in this chapter two general procedures for obtaining a *particular integral*. Other procedures will be given in the next two chapters.

In each of the procedures below, use will be made of an operator $\frac{1}{F(D)}$ defined by the relation $\frac{1}{F(D)} \cdot F(D)y = y$. When the operator is applied to 1) we obtain

$$\frac{1}{F(D)} \cdot F(D)y = y = \frac{1}{F(D)} Q$$

or

2) $$y = \frac{1}{D-m_1} \cdot \frac{1}{D-m_2} \cdot \frac{1}{D-m_3} \cdots\cdots\cdots\cdots \frac{1}{D-m_n} Q.$$

FIRST METHOD. This consists of solving a succession of linear differential equations of order one, as follows:

SET	SOLVE	TO OBTAIN
$u = \frac{1}{D-m_n} Q$	$\frac{du}{dx} - m_n u = Q$	$u = e^{m_n x} \int Q e^{-m_n x}\, dx$
$v = \frac{1}{D-m_{n-1}} u$	$\frac{dv}{dx} - m_{n-1} v = u$	$v = e^{m_{n-1} x} \int u e^{-m_{n-1} x}\, dx$
$\cdots\cdots\cdots\cdots$	$\cdots\cdots\cdots\cdots$	$\cdots\cdots\cdots\cdots$
$y = \frac{1}{D-m_1} w$	$\frac{dy}{dx} - m_1 y = w$	$y = e^{m_1 x} \int w e^{-m_1 x}\, dx.$

As is indicated in Problem 3 below, the following formula may be established:

A) $$y = e^{m_1 x} \int e^{(m_2 - m_1)x} \int e^{(m_3 - m_2)x} \int \cdots\cdots \int e^{(m_n - m_{n-1})x} \int Q e^{-m_n x} (dx)^n.$$

See Problems 1–6.

SECOND METHOD. This consists of expressing $\frac{1}{F(D)}$ as the sum of n partial fractions:

$\frac{N_1}{D-m_1} + \frac{N_2}{D-m_2} + \cdots\cdots\cdots + \frac{N_n}{D-m_n}$. Then

$$B) \quad y = N_1 e^{m_1 x} \int Q e^{-m_1 x}\,dx + N_2 e^{m_2 x} \int Q e^{-m_2 x}\,dx + \cdots + N_n e^{m_n x} \int Q e^{-m_n x}\,dx.$$

In evaluating both *A)* and *B)*, it is customary to discard the constants of integration as they appear: otherwise, one obtains the primitive rather than a particular integral of the differential equation. The complementary function is then obtained by inspection and added to the particular solution to form the primitive.

THE FOLLOWING FORMULAS will be found useful.

$$e^{ibx} = \cos bx + i \sin bx \qquad e^{-ibx} = \cos bx - i \sin bx$$

$$\sin bx = \frac{e^{ibx} - e^{-ibx}}{2i} \qquad \cos bx = \frac{e^{ibx} + e^{-ibx}}{2}$$

$$e^{bx} = \cosh bx + \sinh bx \qquad e^{-bx} = \cosh bx - \sinh bx$$

$$\sinh bx = \tfrac{1}{2}(e^{bx} - e^{-bx}) \qquad \cosh bx = \tfrac{1}{2}(e^{bx} + e^{-bx})$$

SOLVED PROBLEMS

1. Solve $(D^2 - 3D + 2)y = e^x$ or $(D-1)(D-2)y = e^x$.

The complementary function is $y = C_1 e^x + C_2 e^{2x}$, and a particular integral is $y = \frac{1}{D-1} \cdot \frac{1}{D-2} e^x$.

Let $u = \frac{1}{D-2} e^x$. Then $(D-2)u = e^x$ or $\frac{du}{dx} - 2u = e^x$, $ue^{-2x} = \int e^x e^{-2x}\,dx = \int e^{-x}dx = -e^{-x}$ and $u = -e^x$.

Now $y = \frac{1}{D-1} u$, $(D-1)y = u$ or $\frac{dy}{dx} - y = -e^{-x}$, and $y = e^x \int -e^x e^{-x}\,dx = -xe^x$.

The primitive is $y = C_1 e^x + C_2 e^{2x} - xe^x$.

2. Solve $(D^3 + 3D^2 - 4)y = xe^{-2x}$ or $(D-1)(D+2)^2 y = xe^{-2x}$.

The complementary function is $y = C_1 e^x + C_2 e^{-2x} + C_3 x e^{-2x}$, and a particular integral is

$$y = \frac{1}{D-1} \cdot \frac{1}{D+2} \cdot \frac{1}{D+2} xe^{-2x}.$$

Let $u = \frac{1}{D+2} xe^{-2x}$. Then $\frac{du}{dx} + 2u = xe^{-2x}$ and $u = e^{-2x} \int xe^{-2x} \cdot e^{2x}\,dx = \frac{1}{2} x^2 e^{-2x}$.

Let $v = \frac{1}{D+2} u$. Then $\frac{dv}{dx} + 2v = \frac{1}{2} x^2 e^{-2x}$ and $v = e^{-2x} \int \frac{1}{2} x^2 e^{-2x} \cdot e^{2x}\,dx = \frac{1}{6} x^3 e^{-2x}$.

Now $y = \frac{1}{D-1}v$. Then $\frac{dy}{dx} - y = \frac{1}{6}x^3e^{-2x}$ and $y = e^x\int\frac{1}{6}x^3e^{-2x}\cdot e^{-x}\,dx = \frac{1}{6}e^x\int x^3e^{-3x}\,dx$

$$= -\frac{1}{18}e^{-2x}(x^3 + x^2 + \frac{2}{3}x + \frac{2}{9}).$$

The primitive is $y = C_1e^x + C_2e^{-2x} + C_3xe^{-2x} - \frac{1}{18}(x^3 + x^2)e^{-2x}$, the remaining terms of the particular integral being absorbed by the complementary function.

3. Find a particular integral of $(D-a)(D-b)y = Q$.

A particular integral is given by $y = \frac{1}{D-a}\cdot\frac{1}{D-b}Q$.

Let $\frac{1}{D-b}Q = u$. Then $\frac{du}{dx} - bu = Q$ and $u = e^{bx}\int Qe^{-bx}\,dx$.

Now $y = \frac{1}{D-a}u$. Then $\frac{dy}{dx} - ay = u = e^{bx}\int Qe^{-bx}\,dx$ and

$$y = e^{ax}\int e^{bx}\,e^{-ax}\int Qe^{-bx}dx\,dx = e^{ax}\int e^{(b-a)x}\int Qe^{-bx}\,(dx)^2.$$

4. Solve $(D^2 - 3D + 2)y = e^{5x}$ or $(D-1)(D-2)y = e^{5x}$.

The complementary function is $y = C_1e^x + C_2e^{2x}$, and a particular integral is $y = \frac{1}{D-1}\cdot\frac{1}{D-2}e^{5x}$.

First Method. $y = \frac{1}{D-1}\cdot\frac{1}{D-2}e^{5x} = e^x\int e^{(2-1)x}\int e^{5x}\cdot e^{-2x}\,(dx)^2$

$$= e^x\int e^x\int e^{3x}\,(dx)^2 = e^x\int e^x\,\frac{1}{3}e^{3x}\,dx = \frac{1}{3}e^x\int e^{4x}\,dx = \frac{1}{12}e^{5x}.$$

Second Method. $y = \frac{1}{(D-1)(D-2)}e^{5x} = (-\frac{1}{D-1} + \frac{1}{D-2})e^{5x}$

$$= -e^x\int e^{5x}\cdot e^{-x}\,dx + e^{2x}\int e^{5x}\cdot e^{-2x}\,dx$$

$$= -\frac{1}{4}e^xe^{4x} + \frac{1}{3}e^{2x}e^{3x} = \frac{1}{12}e^{5x}.$$

The primitive is $y = C_1e^x + C_2e^{2x} + \frac{1}{12}e^{5x}$.

5. Solve $(D^2 + 5D + 4)y = 3 - 2x$ or $(D+1)(D+4)y = 3 - 2x$.

The complementary funtion is $y = C_1e^{-x} + C_2e^{-4x}$, and a particular integral is

$$y = \frac{1}{(D+1)(D+4)}(3 - 2x).$$

First Method. $y = \frac{1}{D+1}\cdot\frac{1}{D+4}(3-2x) = e^{-x}\int e^{(-4+1)x}\int(3-2x)e^{4x}\,(dx)^2$

$$= e^{-x}\int e^{-3x}(\frac{3}{4}e^{4x} - \frac{1}{2}xe^{4x} + \frac{1}{8}e^{4x})dx = e^{-x}\int(\frac{7}{8}e^x - \frac{1}{2}xe^x)dx = e^{-x}(\frac{7}{8}e^x - \frac{1}{2}xe^x + \frac{1}{2}e^x) = \frac{11}{8} - \frac{1}{2}x.$$

Second Method. $y = \frac{1}{D+1}\cdot\frac{1}{D+4}(3-2x) = (\frac{1/3}{D+1} - \frac{1/3}{D+4})(3-2x)$

$$= \frac{1}{3}e^{-x}\int(3-2x)e^{x}\,dx - \frac{1}{3}e^{-4x}\int(3-2x)e^{4x}\,dx$$

$$= \frac{1}{3}e^{-x}(3e^{x} - 2xe^{x} + 2e^{x}) - \frac{1}{3}e^{-4x}(\frac{3}{4}e^{4x} - \frac{1}{2}xe^{4x} + \frac{1}{8}e^{4x}) = \frac{11}{8} - \frac{1}{2}x.$$

The primitive is $y = C_1e^{-x} + C_2e^{-4x} - \frac{1}{2}x + \frac{11}{8}$.

6. Solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x}$ or $(D-1)(D-2)^2y = e^{2x}$.

The complementary function is $y = C_1e^{x} + C_2e^{2x} + C_3xe^{2x}$, and a particular integral is

$$y = \frac{1}{(D-1)(D-2)^2}e^{2x}.$$

$$y = \frac{1}{D-1}\cdot\frac{1}{D-2}\cdot\frac{1}{D-2}e^{2x}$$

$$= e^{x}\int e^{(2-1)x}\int e^{(2-2)x}\int e^{2x}e^{-2x}(dx)^3$$

$$= e^{x}\int e^{x}\iint(dx)^3 = e^{x}\int e^{x}\int x(dx)^2 = e^{x}\int e^{x}\tfrac{1}{2}x^2\,dx$$

$$= \tfrac{1}{2}e^{x}\int x^2e^{x}\,dx = \tfrac{1}{2}e^{x}(x^2e^{x} - 2xe^{x} + 2e^{x}) = \tfrac{1}{2}e^{2x}(x^2 - 2x + 2).$$

The primitive is $y = C_1e^{x} + C_2e^{2x} + C_3xe^{2x} + \frac{1}{2}x^2e^{2x}$, the remaining terms of the particular integral being absorbed in the complementary function.

7. Solve $(D^2 + 9)y = x\cos x$.

The complementary function is $y = C_1\cos 3x + C_2\sin 3x$, and a particular integral is

$$y = \frac{1}{D^2+9}x\cos x = e^{-3ix}\int e^{(3i+3i)x}\int x\cos x\,e^{3ix}(dx)^2.$$

It will be simpler here to use $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, so that:

$$y = \tfrac{1}{2}e^{-3ix}\int e^{6ix}\int x(e^{-2ix} + e^{4ix})(dx)^2$$

$$= \frac{1}{2}e^{-3ix}\int e^{6ix}(\frac{1}{2}ixe^{-2ix} + \frac{1}{4}e^{-2ix} + \frac{1}{4}ixe^{4ix} + \frac{1}{16}e^{4ix})\,dx$$

$$= \frac{1}{2}e^{-3ix}\int(\frac{1}{2}ixe^{4ix} + \frac{1}{4}e^{4ix} + \frac{1}{4}ixe^{2ix} + \frac{1}{16}e^{2ix})\,dx$$

$$= \frac{1}{2}e^{-3ix}(\frac{1}{8}xe^{4ix} + \frac{1}{32}ie^{4ix} - \frac{1}{16}ie^{4ix} + \frac{1}{8}xe^{2ix} + \frac{1}{16}ie^{2ix} - \frac{1}{32}ie^{2ix})$$

$$= \frac{1}{16}x(e^{ix} + e^{-ix}) - \frac{1}{64}i(e^{ix} - e^{-ix}) = \frac{1}{8}x(\frac{e^{ix} + e^{-ix}}{2}) + \frac{1}{32}(\frac{e^{ix} - e^{-ix}}{2i})$$

$\frac{1}{8}x\cos x + \frac{1}{32}\sin x$. The primitive is $y = C_1\cos 3x + C_2\sin 3x + \frac{1}{8}x\cos x + \frac{1}{32}\sin x$.

8. Solve $(D^2+4)y = 2\cos x\cos 3x = \cos 2x + \cos 4x$.

The complementary function is $y = C_1\cos 2x + C_2\sin 2x$, and a particular integral is

$$y = \frac{1}{D^2+4}(\cos 2x + \cos 4x) = \frac{1}{4}i\left(\frac{1}{D+2i} - \frac{1}{D-2i}\right)(\cos 2x + \cos 4x)$$

$$= \tfrac{1}{4}i\{e^{-2ix}\int(\cos 2x + \cos 4x)e^{2ix}\,dx - e^{2ix}\int(\cos 2x + \cos 4x)e^{-2ix}\,dx\}$$

$$= \tfrac{1}{4}i\{e^{-2ix}\int e^{2ix}[\tfrac{1}{2}(e^{2ix}+e^{-2ix}) + \tfrac{1}{2}(e^{4ix}+e^{-4ix})]\,dx$$

$$- e^{2ix}\int e^{-2ix}[\tfrac{1}{2}(e^{2ix}+e^{-2ix}) + \tfrac{1}{2}(e^{4ix}+e^{-4ix})]\,dx\}$$

$$= \frac{1}{8}\{e^{-2ix}\int(e^{4ix}+1+e^{6ix}+e^{-2ix})i\,dx - e^{2ix}\int(1+e^{-4ix}+e^{2ix}+e^{-6ix})i\,dx\}$$

$$= \frac{1}{8}\{e^{-2ix}(\frac{1}{4}e^{4ix} + ix + \frac{1}{6}e^{6ix} - \frac{1}{2}e^{-2ix}) - e^{2ix}(ix - \frac{1}{4}e^{-4ix} + \frac{1}{2}e^{2ix} - \frac{1}{6}e^{-6ix})\}$$

$$= \frac{1}{8}\{-ix(e^{2ix}-e^{-2ix}) + \frac{1}{4}(e^{2ix}+e^{-2ix}) - \frac{1}{3}(e^{4ix}+e^{-4ix})\}$$

$$= \frac{1}{4}x\left(\frac{e^{2ix}-e^{-2ix}}{2i}\right) + \frac{1}{16}\left(\frac{e^{2ix}+e^{-2ix}}{2}\right) - \frac{1}{12}\left(\frac{e^{4ix}+e^{-4ix}}{2}\right)$$

$$= \frac{1}{4}x\sin 2x + \frac{1}{16}\cos 2x - \frac{1}{12}\cos 4x.$$

The primitive is $y = C_1\cos 2x + C_2\sin 2x + \frac{1}{4}x\sin 2x - \frac{1}{12}\cos 4x$.

9. Solve $(D^2-9D+18)y = e^{e^{-3x}}$.

The complementary function is $y = C_1e^{3x} + C_2e^{6x}$, and a particular integral is

$$y = \frac{1}{(D-6)(D-3)}e^{e^{-3x}} = e^{6x}\int e^{-3x}\int e^{e^{-3x}}\cdot e^{-3x}(dx)^2$$

$$= e^{6x}\int e^{-3x}(-\frac{1}{3}e^{e^{-3x}})dx = \frac{1}{3}e^{6x}\int e^{e^{-3x}}(-e^{-3x})\,dx = \frac{1}{9}e^{e^{-3x}}\cdot e^{6x}.$$

The complete solution is $y = C_1e^{3x} + (C_2 + \frac{1}{9}e^{e^{-3x}})e^{6x}$.

Note. When the factors are reversed, a particular integral is $y = e^{3x}\int e^{3x}\int e^{e^{-3x}}\cdot e^{-6x}(dx)^2$.

Using the substitution $e^{-3x} = v$, we obtain

$$y = \frac{1}{9v}\int\frac{1}{v^2}\int e^v v\,(dv)^2 = \frac{1}{9v}\int e^v(\frac{1}{v} - \frac{1}{v^2})dv = \frac{1}{9v^2}e^v$$

or $y = \frac{1}{9}e^{e^{-3x}}\cdot e^{6x}$, as before.

SUPPLEMENTARY PROBLEMS

10. Evaluate, omitting the arbitrary constant.

a) $\frac{1}{D+1}e^x$ *Ans.* $\frac{1}{2}e^x$

b) $\frac{1}{D-1}e^x$ *Ans.* xe^x

c) $\frac{1}{D+1}(x+1)$ *Ans.* x

d) $\frac{1}{D-1}(x^2+1)$ *Ans.* x^2-2x+3

e) $\frac{1}{D+2}\sin 3x$ *Ans.* $\frac{1}{13}(2\sin 3x - 3\cos 3x)$

f) $\frac{1}{D+2}e^{-2x}\sin 3x$ *Ans.* $-\frac{1}{3}e^{-2x}\cos 3x$

Solve.

11. $(D^2-4D+3)y = 1$ *Ans.* $y = C_1e^x + C_2e^{3x} + 1/3$

12. $(D^2-4D)y = 5$ $y = C_1 + C_2e^{4x} - 5x/4$

13. $(D^3-4D^2)y = 5$ $y = C_1 + C_2x + C_3e^{4x} - 5x^2/8$

14. $(D^5-4D^3)y = 5$ $y = C_1 + C_2x + C_3x^2 + C_4e^{2x} + C_5e^{-2x} - 5x^3/24$

15. $(D^3-4D)y = x$ $y = C_1 + C_2e^{2x} + C_3e^{-2x} - x^2/8$

16. $(D^2-6D+9)y = e^{2x}$ $y = C_1e^{3x} + C_2xe^{3x} + e^{2x}$

17. $(D^2+D-2)y = 2(1+x-x^2)$ $y = C_1e^x + C_2e^{-2x} + x^2$

18. $(D^2-1)y = 4xe^x$ $y = C_1e^x + C_2e^{-x} + e^x(x^2-x)$

19. $(D^2-1)y = \sin^2 x = \frac{1}{2}(1-\cos 2x)$ $y = C_1e^x + C_2e^{-x} - \frac{1}{2} + \frac{1}{10}\cos 2x$

20. $(D^2-1)y = (1+e^{-x})^{-2}$ $y = C_1e^x + C_2e^{-x} - 1 + e^{-x}\ln(1+e^x)$

21. $(D^2+1)y = \csc x$ $y = C_1\cos x + C_2\sin x + \sin x \ln\sin x - x\cos x$

22. $(D^2-3D+2)y = \sin e^{-x}$ $y = C_1e^x + C_2e^{2x} - e^{2x}\sin e^{-x}$

CHAPTER 15

Linear Equations with Constant Coefficients
VARIATION OF PARAMETERS, UNDETERMINED COEFFICIENTS

TWO OTHER METHODS for determining a *particular integral* of a linear differential equation with constant coefficients

1) $F(D)y = (D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots\cdots + P_{n-1} D + P_n)y = Q$

will be exhibited by means of examples.

VARIATION OF PARAMETERS. From the complementary function of 1),

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots\cdots + C_n y_n(x),$$

we obtain a basic relation

2) $y = L_1(x)\, y_1(x) + L_2(x)\, y_2(x) + \cdots\cdots + L_n(x)\, y_n(x)$

by replacing the C's by unknown functions of x, the L's. The method consists of a procedure for determining the L's so that 2) satisfies 1).

See Problems 1–4.

UNDETERMINED COEFFICIENTS. The basic relation here is

3) $y = A\, r_1(x) + B\, r_2(x) + C\, r_3(x) + \cdots\cdots + G\, r_t(x),$

where the functions $r_1(x), \cdots\cdots, r_t(x)$ are the terms of Q and those arising from these terms by differentiation, and $A, B, C, \cdots, G$ are constants.

For example, if the equation is $F(D)y = x^3$, we take for 3) $y = Ax^3 + Bx^2 + Cx + D$;

if the equation is $f(D)y = e^x + e^{3x}$, we take for 3) $y = Ae^x + Be^{3x}$,

since no new terms are obtained by differentiating e^x and e^{3x};

if the equation is $F(D)y = \sin ax$, we take for 3) $y = A \sin ax + B \cos ax$;

if the equation is $F(D)y = \sec x$, the method fails since the number of new terms obtained by differentiating $Q = \sec x$ is infinite.

Substitution 3) in 1), the coefficients $A, B, C, \cdots$ are found from the resulting identity.

See Problems 5-6.

The procedure must be modified in case:

a) A term of Q is also a term of the complementary function. If a term of Q, say u is also a term of the complementary function corresponding to an s-fold root m, then in 3) we introduce a term $x^s u$ plus terms arising from it by differentiation.

For example, in finding a particular integral of $(D-2)^2 (D+3)y = e^{2x} + x^2$,

the basic relation is $y = Ax^2e^{2x} + Bxe^{2x} + Ce^{2x} + Dx^2 + Ex + F$, the first three terms arising from the fact that the term e^{2x} of Q is also a term of the complementary function corresponding to a double root $m = 2$; hence, use is made of x^2e^{2x} and all terms arising by differentiation. See Problems 7–8.

b) A term of Q is $x^r u$ and u is a term of the complementary function. If u corresponds to an s-fold root a, 3) must contain the term $x^{r+s}u$ plus terms arising from it by differentiation.

For example, in finding a particular integral of $(D-2)^3(D+3)y = x^2e^{2x} + x^2$, the basic relation is

$$y = Ax^5e^{2x} + Bx^4e^{2x} + Cx^3e^{2x} + Dx^2e^{2x} + Exe^{2x} + Fe^{2x} + Gx^2 + Hx + J,$$

the first six terms arising from the fact that e^{2x} is a part of the complementary function corresponding to the triple root $m = 2$. See Problem 9.

SOLVED PROBLEMS

VARIATION OF PARAMETERS.

1. Show that if $y = C_1y_1 + C_2y_2 + C_3y_3$ is the complementary function of $F(D)y = (D^3 + P_1D^2 + P_2D + P_3)y = Q$ then

1) $$y = L_1y_1 + L_2y_2 + L_3y_3,$$

where L_1, L_2, L_3 satisfy the conditions

A)
$$\begin{aligned} L_1'y_1 + L_2'y_2 + L_3'y_3 &= 0 \\ L_1'y_1' + L_2'y_2' + L_3'y_3' &= 0 \\ L_1'y_1'' + L_2'y_2'' + L_3'y_3'' &= Q, \end{aligned}$$

is a particular solution of the differential equation.

We obtain, in view of *A)*, by successively differentiating $y = L_1y_1 + L_2y_2 + L_3y_3$:

B)
$$\begin{aligned} Dy &= L_1y_1' + L_2y_2' + L_3y_3' + (L_1'y_1 + L_2'y_2 + L_3'y_3) = L_1y_1' + L_2y_2' + L_3y_3' \\ D^2y &= L_1y_1'' + L_2y_2'' + L_3y_3'' + (L_1'y_1' + L_2'y_2' + L_3'y_3') = L_1y_1'' + L_2y_2'' + L_3y_3'' \\ D^3y &= L_1y_1''' + L_2y_2''' + L_3y_3''' + (L_1'y_1'' + L_2'y_2'' + L_3'y_3'') = L_1y_1''' + L_2y_2''' + L_3y_3''' + Q. \end{aligned}$$

Then $$\begin{aligned} F(D)y &= L_1(y_1''' + P_1y_1'' + P_2y_1' + P_3y_1) + L_2(y_2''' + P_1y_2'' + P_2y_2' + P_3y_2) \\ &\quad + L_3(y_3''' + P_1y_3'' + P_2y_3' + P_3y_3) + Q \\ &= L_1F(D)y_1 + L_2F(D)y_2 + L_3F(D)y_3 + Q = 0 + 0 + 0 + Q = Q, \end{aligned}$$

since y_1, y_2, y_3 are solutions of $F(D)y = 0$.

In using this method:

a) Write the complementary function.

b) From the L function 1), which is to be a particular integral, by replacing the C's of the complementary function with L's.

c) Obtain equations *B)* by differentiating 1) as many times as the degree of the differential equation. After each differentiation, set the sum of all terms containing derivatives of the L's equal to zero, except in the case of the last differentiation when the sum is set equal to Q. The equations obtained by setting the sums equal to zero and Q are the equations *A)*.

d) Solve these equations for L_1', L_2', ·····.

e) Obtain L_1, L_2, ····· by integration.

2. Solve $(D^2 - 2D)y = e^x \sin x$.

The complementary function is $y = C_1 + C_2 e^{2x}$.

We form the relation $y = L_1 + L_2 e^{2x}$,

obtain, by differentiation $Dy = 2L_2 e^{2x} + (L_1' + L_2' e^{2x})$,

and set 1) $L_1' + L_2' e^{2x} = 0$.

Since now $Dy = 2L_2 e^{2x}$, $D^2y = 4L_2 e^{2x} + 2L_2' e^{2x}$ and we set $2L_2' e^{2x} = Q = e^x \sin x$.

Thus, $L_2' = \frac{1}{2} e^{-x} \sin x$ and $L_2 = -\frac{1}{4} e^{-x}(\sin x + \cos x)$.

From 1), $L_1' = -L_2' e^{2x} = -\frac{1}{2} e^x \sin x$ and $L_1 = -\frac{1}{4} e^x (\sin x - \cos x)$.

A particular integral of the given equation is

$$y = L_1 + L_2 e^{2x} = -\tfrac{1}{4} e^x (\sin x - \cos x) - \tfrac{1}{4} e^x (\sin x + \cos x) = -\tfrac{1}{2} e^x \sin x,$$

and the primitive is $y = C_1 + C_2 e^{2x} - \frac{1}{2} e^x \sin x$.

3. Solve $(D^3 + D)y = \csc x$.

The complementary function is $y = C_1 + C_2 \cos x + C_3 \sin x$.

From the relation $y = L_1 + L_2 \cos x + L_3 \sin x$

we obtain $Dy = (-L_2 \sin x + L_3 \cos x) + (L_1' + L_2' \cos x + L_3' \sin x)$

and set 1) $L_1' + L_2' \cos x + L_3' \sin x = 0$

Then $Dy = -L_2 \sin x + L_3 \cos x$,

$D^2y = (-L_2 \cos x - L_3 \sin x) + (-L_2' \sin x + L_3' \cos x)$,

and we set 2) $-L_2' \sin x + L_3' \cos x = 0$.

Then $D^2y = -L_2 \cos x - L_3 \sin x$,

$D^3y = (L_2 \sin x - L_3 \cos x) + (-L_2' \cos x - L_3' \sin x)$,

and we set 3) $-L_2' \cos x - L_3' \sin x = Q = \csc x$.

Adding 1) and 3), $L_1' = \csc x$ and $L_1 = -\ln(\csc x + \cot x)$.

Solving 2) and 3), $L_3' = -1$ and $L_2' = -\cot x$, so that $L_3 = -x$ and $L_2 = -\ln \sin x$.

Thus, a particular integral of the differential equation is

$$y = L_1 + L_2 \cos x + L_3 \sin x = -\ln(\csc x + \cot x) - \cos x \ln \sin x - x \sin x,$$

and the primitive is

$$y = C_1 + C_2 \cos x + C_3 \sin x - \ln(\csc x + \cot x) - \cos x \ln \sin x - x \sin x.$$

4. Solve $(D^2 - 6D + 9)y = e^{3x}/x^2$.

The complementary function is $y = C_1 e^{3x} + C_2 x e^{3x}$.

From the relation $y = L_1 e^{3x} + L_2 x e^{3x}$

we obtain $Dy = (3L_1 + L_2)e^{3x} + 3L_2 x e^{3x} + (L_1' e^{3x} + L_2' x e^{3x})$

and set 1) $L_1' e^{3x} + L_2' x e^{3x} = 0$.

Then $D^2y = (9L_1 + 6L_2)e^{3x} + 9L_2xe^{3x} + (3L_1' + L_2')e^{3x} + 3L_2'xe^{3x}$,

and we set 2) $(3L_1' + L_2')e^{3x} + 3L_2'xe^{3x} = e^{3x}/x^2$.

Solving 1) and 2) $L_1' = -1/x$ and $L_2' = 1/x^2$, so that $L_1 = -\ln x$ and $L_2 = -1/x$.
Thus, a particular integral of the differential equation is

$$y = L_1e^{3x} + L_2xe^{3x} = -e^{3x}\ln x - e^{3x},$$

and the primitive is $y = C_1e^{3x} + C_2xe^{3x} - e^{3x}\ln x$.

UNDETERMINED COEFFICIENTS.

5. Solve $(D^2 - 2D)y = e^x \sin x$.

The complementary function is $y = C_1 + C_2e^x$. As a particular integral, we take

$$y = Ae^x \sin x + Be^x \cos x.$$

Then
$$Dy = (A-B)e^x \sin x + (A+B)e^x \cos x,$$
$$D^2y = -2Be^x \sin x + 2Ae^x \cos x,$$

and $(D^2 - 2D)y = -2Ae^x \sin x - 2Be^x \cos x = e^x \sin x = Q.$

Equating coefficients of like terms, $-2A = 1$ and $-2B = 0$, so that $A = -\frac{1}{2}$ and $B = 0$.

Hence, a particular integral of the differential equation is

$$y = Ae^x \sin x + Be^x \cos x = -\tfrac{1}{2}e^x \sin x,$$

and the primitive is $y = C_1 + C_2e^{2x} - \frac{1}{2}e^x \sin x$.

This was solved above as Problem 2.

6. Solve $(D^2 - 2D + 3)y = x^3 + \sin x$.

The complementary function is $y = e^x(C_1 \cos \sqrt{2}\,x + C_2 \sin \sqrt{2}\,x)$. As a particular integral, we take

$$y = Ax^3 + Bx^2 + Cx + E + F \sin x + G \cos x$$

Then
$$Dy = 3Ax^2 + 2Bx + C - G \sin x + F \cos x,$$
$$D^2y = 6Ax + 2B - F \sin x - G \cos x,$$

and $(D^2 - 2D + 3)y = 3Ax^3 + 3(B-2A)x^2 + (3C-4B+6A)x + (3E-2C+2B) + 2(F+G)\sin x + 2(G-F)\cos x$
$= x^3 + \sin x.$

Equating coefficients of like terms, $3A = 1$ and $A = 1/3$; $B - 2A = 0$ and $B = 2/3$; $3C - 4B + 6A = 0$ and $C = 2/9$; $3E - 2C + 2B = 0$ and $E = -8/27$; $2(F+G) = 1$, $G - F = 0$ and $F = G = \frac{1}{4}$.

Thus, a particular integral of the differential equation is

$$y = \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x),$$

and the primitive is

$$y = e^x(C_1 \cos \sqrt{2}\,x + C_2 \sin \sqrt{2}\,x) + \frac{1}{27}(9x^3 + 18x^2 + 6x - 8) + \frac{1}{4}(\sin x + \cos x).$$

7. Solve $(D^3 + 2D^2 - D - 2)y = e^x + x^2$.

The complementary function is $y = C_1e^x + C_2e^{-x} + C_3e^{-2x}$. Since e^x occurs in Q and also in the complementary function corresponding to a root of multiplicity one, we take as a particular integral

1) $$y = Ax^2 + Bx + C + Exe^x + Fe^x$$

Then
$$Dy = 2Ax + B + Exe^x + (E+F)e^x,$$
$$D^2y = 2A + Exe^x + (2E+F)e^x,$$
$$D^3y = Exe^x + (3E+F)e^x,$$
and
$$(D^3 + 2D^2 - D - 2)y = -2Ax^2 - 2(B+A)x + (4A - B - 2C) + 6Ee^x = e^x + x^2.$$

Equating coefficients of like terms, $-2A = 1$, $B + A = 0$, $4A - B - 2C = 0$, $6E = 1$; hence, $A = -\frac{1}{2}$, $B = \frac{1}{2}$, $C = -\frac{5}{4}$, $E = \frac{1}{6}$, and F is arbitrary. Now F should be arbitrary here, since C_1e^x is a term of the complementary function. Thus, in writing 1), the inclusion of Fe^x was unnecessary.

Hence, a particular integral is $y = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{5}{4} + \frac{1}{6}xe^x$,

and the primitive is $y = C_1e^x + C_2e^{-x} + C_3e^{-2x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{5}{4} + \frac{1}{6}xe^x$.

8. Solve $(D^2 - 4D + 4)y = x^3e^{2x} + xe^{2x}$.

The complementary function is $y = C_1e^{2x} + C_2xe^{2x}$ Now e^{2x} is a part of Q and also occurs in the complementary function corresponding to a root of multiplicity two. As a particular integral, we take

$$y = Ax^5e^{2x} + Bx^4e^{2x} + Cx^3e^{2x} + Ex^2e^{2x}.$$

Note that terms involving xe^{2x} and e^{2x} are not included, since they appear in the complementary function with arbitrary coefficients. Then

$$Dy = 2Ax^5e^{2x} + (5A + 2B)x^4e^{2x} + (4B + 2C)x^3e^{2x} + (3C + 2E)x^2e^{2x} + 2Exe^{2x},$$
$$D^2y = 4Ax^5e^{2x} + (20A + 4B)x^4e^{2x} + (20A + 16B + 4C)x^3e^{2x} + (12B + 12C + 4E)x^2e^{2x} + (6C + 8E)xe^{2x} + 2Ee^{2x},$$
and
$$(D^2 - 4D + 4)y = 20Ax^5e^{2x} + 12Bx^4e^{2x} + 6Cxe^{2x} + 2Ee^{2x} = x^3e^{2x} + xe^{2x}.$$

Equating coefficients of like terms, $20A = 1$, $12B = 0$, $6C = 1$, $2E = 0$; hence $A = 1/20$, $B = 0$, $C = 1/6$, $E = 0$.

Thus, a particular integral is $y = \frac{1}{20}x^5e^{2x} + \frac{1}{6}x^3e^{2x}$,

and the primitive is $y = C_1e^{2x} + C_2xe^x + \frac{1}{20}x^5e^{2x} + \frac{1}{6}x^3e^{2x}$.

9. Solve $(D^2 + 4)y = x^2 \sin 2x$.

The complementary function is $y = C_1\cos 2x + C_2\sin 2x$.

Since $x^2 \sin 2x$ occurs in Q and $\sin 2x$ is a part of the complementary function corresponding to a root of multiplicity one, we take as a particular integral

$$y = Ax^3 \cos 2x + Bx^3 \sin 2x + Cx^2 \cos 2x + Ex^2 \sin 2x + Fx \cos 2x + Gx \sin 2x.$$

Note that $H\cos 2x + K\sin 2x$ is not included, since these terms are in the complementary function. Then

$$Dy = 2Bx^3 \cos 2x - 2Ax^3 \sin 2x + (3A + 2E)x^2 \cos 2x + (3B - 2C)x^2 \sin 2x$$
$$+ (2C + 2G)x \cos 2x + (2E - 2F)x \sin 2x + F \cos 2x + G \sin 2x,$$

$$D^2y = -4Ax^3\cos 2x - 4Bx^3\sin 2x + (12B-4C)x^2\cos 2x + (-12A-4E)x^2\sin 2x$$
$$+ (6A+8E-4F)x\cos 2x + (6B-8C-4G)x\sin 2x + (2C+4G)\cos 2x + (2E-4F)\sin 2x,$$

and

$$(D^2+4)y = 12Bx^2\cos 2x - 12Ax^2\sin 2x + (6A+8E)x\cos 2x + (6B-8C)x\sin 2x$$
$$+ (2C+4G)\cos 2x + (2E-4F)\sin 2x = x^2\sin 2x.$$

Equating coefficients of like terms, $-12A = 1$, $12B = 0$, $6A + 8E = 0$, $6B - 8C = 0$, $2C + 4G = 0$, $2E - 4F = 0$; hence, $A = -1/12$, $B = 0$, $C = 0$, $E = 1/16$, $F = 1/32$, $G = 0$.

A particular integral is $y = -\frac{1}{12}x^3\cos 2x + \frac{1}{16}x^2\sin 2x + \frac{1}{32}x\cos 2x$,

and the primitive is $y = C_1\cos 2x + C_2\sin 2x - \frac{1}{12}x^3\cos 2x + \frac{1}{16}x^2\sin 2x + \frac{1}{32}x\cos 2x$.

SUPPLEMENTARY PROBLEMS

Solve, using the method of variation of parameters.

10. $(D^2+1)y = \csc x$ *Ans.* $y = C_1\cos x + C_2\sin x + \sin x\ \ln\sin x - x\cos x$

11. $(D^2+4)y = 4\sec^2 2x$ *Ans.* $y = C_1\cos 2x + C_2\sin 2x - 1 + \sin 2x\ \ln(\sec 2x + \tan 2x)$

12. $(D^2-4D+3)y = (1+e^{-x})^{-1}$ *Ans.* $y = C_1e^x + C_2e^{3x} + \frac{1}{2}e^{2x} + \frac{1}{2}(e^x - e^{3x})\ln(1+e^{-x})$

13. $(D^2-1)y = e^{-x}\sin e^{-x} + \cos e^{-x}$ *Ans.* $y = C_1e^x + C_2e^{-x} - e^x\sin e^{-x}$

14. $(D^2-1)y = (1+e^{-x})^{-2}$ *Ans.* $y = C_1e^x + C_2e^{-x} - 1 + e^{-x}\ln(1+e^x)$

Solve, using the method of undetermined coefficients.

15. $(D^2+2)y = e^x + 2$ *Ans.* $y = C_1\cos\sqrt{2}x + C_2\sin\sqrt{2}x + e^x/3 + 1$

16. $(D^2-1)y = e^x\sin 2x$ *Ans.* $y = C_1e^x + C_2e^{-x} - e^x(\sin 2x + \cos 2x)/8$

17. $(D^2+2D+2)y = x^2 + \sin x$ *Ans.* $y = e^{-x}(C_1\cos\ \ C_2\sin x) + \frac{1}{2}(x-1)^2 + \frac{1}{5}(\sin x - 2\cos x)$

18. $(D^2-9)y = x + e^{2x} - \sin 2x$ *Ans.* $y =\ \ + C_2e^{-3x} - x/9 - e^{2x}/5 + \frac{1}{13}\sin 2x$

19. $(D^3+3D^2+2D)y = x^2 + 4x + 8$ (Use $Ax^3 + Bx^2 + Cx$.)

Ans $y = C_1 + C_2e^{-x} + C_3e^{-2x} + \frac{1}{6}x^3 + \frac{1}{4}x^2 + \frac{11}{4}x$

20. $(D^2+1)y = -2\sin x + 4x\cos x$ *Ans.* $y = C_1\cos x + C_2\sin x + 2x\cos x + x^2\sin x$

21. $(D^3-D^2-4D+4)y = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$

Ans. $y = C_1e^x + C_2e^{2x} + C_3e^{-2x} + \frac{1}{2}x^2 + \frac{1}{6}x^3e^{2x}$

CHAPTER 16

Linear Equations with Constant Coefficients

SHORT METHODS

A PARTICULAR INTEGRAL of a linear differential equation $F(D)y = Q$ with constant coefficients is given by $y = \frac{1}{F(D)}Q$. For certain forms of Q the labour involved in evaluating this symbol may be considerably shortened, as follows:

a) If Q is of the form e^{ax},

$$y = \frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}, \qquad F(a) \neq 0.$$

See Problems 2-3 when $F(a) \neq 0$, and Problems 4-5 when $F(a) = 0$.

b) If Q is of the form $\sin(ax+b)$ or $\cos(ax+b)$,

$$y = \frac{1}{F(D^2)}\sin(ax+b) = \frac{1}{F(-a^2)}\sin(ax+b), \qquad F(-a^2) \neq 0,$$

$$y = \frac{1}{F(D^2)}\cos(ax+b) = \frac{1}{F(-a^2)}\cos(ax+b), \qquad F(-a^2) \neq 0.$$

See Problems 7-11 when $F(-a^2) \neq 0$, and Problem 12 when $F(-a^2) = 0$.

c) If Q is of the form x^n,

$$y = \frac{1}{F(D)}x^n = (a_0 + a_1D + a_2D^2 + \cdots\cdots + a_nD^n)x^n, \qquad a_0 \neq 0,$$

obtained by expanding $\frac{1}{F(D)}$ in ascending powers of D and suppressing all terms beyond D^n, since $D^nx^n = 0$ when $n > m$. See Problems 13-15.

d) If Q is of the form $e^{ax}V(x)$, $y = \frac{1}{F(D)}e^{ax}V = e^{ax}\frac{1}{F(D+a)}V$.

See Problems 17-20.

e) If Q is of the form $xV(x)$, $y = \frac{1}{F(D)}xV = x\frac{1}{F(D)}V - \frac{F'(D)}{(F(D))^2}V$.

See Problems 21-23.

SOLVED PROBLEMS

1. Establish the rule in *a)* above.

Since when $y = e^{ax}$, $Dy = ae^{ax}$, $D^2y = a^2e^{ax}$, ..., $D^re^{ax} = a^re^{ax}$,

$F(D)e^{ax} = \sum P_rD^re^{ax} = \sum P_ra^re^{ax} = F(a)e^{ax}$. Hence, $\frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}$.

2. Solve $(D^3 - 2D^2 - 5D + 6)y = e^{4x}$ or $(D-1)(D-3)(D+2)y = e^{4x}$.

The complementary function is $y = C_1e^x + C_2e^{3x} + C_3e^{-2x}$.

A particular integral is $y = \dfrac{1}{(D-1)(D-3)(D+2)}e^{4x}$

$$= \frac{1}{(4-1)(4-3)(4+2)}e^{4x} = \frac{1}{3\cdot 1\cdot 6}e^{4x} = \frac{1}{18}e^{4x}.$$

Hence, the primitive is $y = C_1e^x + C_2e^{3x} + C_3e^{-2x} + \dfrac{1}{18}e^{4x}$.

3. Solve $(D^3 - 2D^2 - 5D + 6)y = (e^{2x} + 3)^2$.

The complementary function is, from Problem 2, $y = C_1e^x + C_2e^{3x} + C_3e^{-2x}$.

A particular integral is $y = \dfrac{1}{(D-1)(D-3)(D+2)}(e^{2x} + 3)^2$

$$= \frac{1}{(D-1)(D-3)(D+2)}e^{4x} + \frac{6}{(D-1)(D-3)(D+2)}e^{2x} + \frac{9}{(D-1)(D-3)(D+2)}e^{0x}$$

$$= \frac{1}{3(1)6}e^{4x} + \frac{6}{1(-1)4}e^{2x} + \frac{9}{(-1)(-3)2} = \frac{e^{4x}}{18} - \frac{3e^{2x}}{2} + \frac{3}{2}.$$

The primitive is $y = C_1e^x + C_2e^{3x} + C_3e^{-2x} + \dfrac{e^{4x}}{18} - \dfrac{3e^{2x}}{2} + \dfrac{3}{2}$.

4. Solve $(D^3 - 2D^2 - 5D + 6)y = e^{3x}$.

The complementary function is $y = C_1e^x + C_2e^{3x} + C_3e^{-2x}$.

A particular integral is $y = \dfrac{1}{(D-1)(D-3)(D+2)}e^{3x}$. Now $F(a) = F(3) = 0$, and the short method does not apply. However, we may write

$$y = \frac{1}{(D-1)(D-3)(D+2)}e^{3x} = \frac{1}{D-3}\left(\frac{1}{(D-1)(D+2)}e^{3x}\right) = \frac{1}{D-3}\left(\frac{1}{2\cdot 5}e^{3x}\right) = \frac{1}{10}\frac{1}{D-3}e^{3x}$$

$$= \frac{1}{10}e^{3x}\int e^{3x}e^{-3x}\,dx = \frac{1}{10}e^{3x}\int dx = \frac{1}{10}xe^{3x}.$$

The primitive is $y = C_1e^x + C_2e^{3x} + C_3e^{-2x} + xe^{3x}/10$.

5. Solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^x + 3e^{-x}$.

The complementary function is $y = C_1e^x + C_2e^{2x} + C_3xe^{2x}$, and a particular integral is

$$y = \frac{1}{(D-1)(D-2)^2}e^{2x} + \frac{2}{(D-1)(D-2)^2}e^x + \frac{3}{(D-1)(D-2)^2}e^{-x}$$

$$= \frac{1}{(D-2)^2}\left(\frac{1}{D-1}e^{2x}\right) + \frac{2}{D-1}\left(\frac{1}{(D-2)^2}e^x\right) + \frac{3}{(D-1)(D-2)^2}e^{-x}$$

$$= \frac{1}{(D-2)^2}e^{2x} + \frac{2}{D-1}e^x + \frac{3}{(-2)(-3)^2}e^{-x}$$

$$= e^{2x}\iint (dx)^2 + 2e^x\int dx - \frac{1}{6}e^{-x} = \frac{1}{2}x^2e^{2x} + 2xe^x - \frac{1}{6}e^{-x}.$$

The primitive is $y = C_1e^x + C_2e^{2x} + C_3xe^{2x} + \frac{1}{2}x^2e^{2x} + 2xe^x - \frac{1}{6}e^{-x}$.

6. Establish the rule in *b)* above for $\cos(ax+b)$.

Since, when $y = \cos(ax+b)$, $D^2y = -a^2\cos(ax+b)$, $D^4y = (-a^2)^2\cos(ax+b)$, ..., $D^{2r}y = (-a^2)^r\cos(ax+b)$, then

$$F(D^2)\cos(ax+b) = \sum_r P_rD^{2r}\cos(ax+b) = \sum_r P_r(-a^2)^r\cos(ax+b) = F(-a^2)\cos(ax+b).$$

Hence, $\dfrac{1}{F(D^2)}\cos(ax+b) = \dfrac{1}{F(-a^2)}\cos(ax+b)$.

7. Solve $(D^2+4)y = \sin 3x$.

The complementary function is $y = C_1\cos 2x + C_2\sin 2x$, and a particular solution is

$$y = \frac{1}{D^2+4}\sin 3x = \frac{1}{-(3)^2+4}\sin 3x = -\frac{1}{5}\sin 3x.$$

The primitive is $y = C_1\cos 2x + C_2\sin 2x - \frac{1}{5}\sin 3x$.

8. Solve $(D^4+10D^2+9)y = \cos(2x+3)$.

The complementary function is $y = C_1\cos x + C_2\sin x + C_3\cos 3x + C_4\sin 3x$, and a particular integral is

$$y = \frac{1}{(D^2+1)(D^2+9)}\cos(2x+3) = \frac{1}{(-3)(5)}\cos(2x+3) = -\frac{1}{15}\cos(2x+3).$$

The primitive is $y = C_1\cos x + C_2\sin x + C_3\cos 3x + C_4\sin 3x - \frac{1}{15}\cos(2x+3)$.

9. Solve $(D^2+3D-4)y = \sin 2x$.

The complementary function is $y = C_1e^x + C_2e^{-4x}$, and a particular integral is

$$y = \frac{1}{D^2+3D-4}\sin 2x = \frac{1}{(D-1)(D+4)}\sin 2x.$$

The operator here is not of the form $\dfrac{1}{F(D^2)}$, and the short method does not apply. However, we may use either of the following procedures to shorten the work.

a) $y = \dfrac{1}{(D-1)(D+4)}\sin 2x = \dfrac{(D+1)(D-4)}{(D^2-1)(D^2-16)}\sin 2x = \dfrac{1}{100}(D^2-3D-4)\sin 2x$

$= \dfrac{1}{100}(-4\sin 2x - 6\cos 2x - 4\sin 2x) = -\dfrac{1}{50}(4\sin 2x + 3\cos 2x)$.

b) $y = \dfrac{1}{D^2+3D-4}\sin 2x = \dfrac{1}{(-4)+3D-4}\sin 2x = \dfrac{1}{3D-8}\sin 2x = \dfrac{3D+8}{9D^2-64}\sin 2x$

$= -\dfrac{1}{100}(3D+8)\sin 2x = -\dfrac{1}{100}(6\cos 2x + 8\sin 2x) = -\dfrac{1}{50}(4\sin 2x + 3\cos 2x)$.

The primitive is $y = C_1e^x + C_2e^{-4x} - \frac{1}{50}(4\sin 2x + 3\cos 2x)$.

10. Solve $(D^3+D^2+D+1)y = \sin 2x + \cos 3x$.

The complementary function is $y = C_1\cos x + C_2\sin x + C_3e^{-x}$, and a particular integral is

$$y = \frac{1}{(D^2+1)(D+1)}(\sin 2x + \cos 3x) = \frac{1}{(D^2+1)(D+1)}\sin 2x + \frac{1}{(D^2+1)(D+1)}\cos 3x$$

$$= -\frac{1}{3}\frac{1}{D+1}\sin 2x - \frac{1}{8}\frac{1}{D+1}\cos 3x = -\frac{1}{3}\frac{D-1}{D^2-1}\sin 2x - \frac{1}{8}\frac{D-1}{D^2-1}\cos 3x$$

$$= \frac{1}{15}(D-1)\sin 2x + \frac{1}{80}(D-1)\cos 3x = \frac{1}{15}(2\cos 2x - \sin 2x) - \frac{1}{80}(3\sin 3x + \cos 3x).$$

The primitive is

$$y = C_1\cos x + C_2\sin x + C_3e^{-x} + \frac{1}{15}(2\cos 2x - \sin 2x) - \frac{1}{80}(3\sin 3x + \cos 3x).$$

11. Solve $(D^2-D+1)y = \sin 2x$.

The complementary function is $y = e^{\frac{1}{2}x}(C_1\cos \frac{1}{2}\sqrt{3}\,x + C_2\sin \frac{1}{2}\sqrt{3}\,x)$, and a particular integral is

$$y = \frac{1}{D^2-D+1}\sin 2x = \frac{1}{(-4)-D+1}\sin 2x = -\frac{1}{D+3}\sin 2x = -\frac{D-3}{D^2-9}\sin 2x$$

$$= \frac{1}{13}(D-3)\sin 2x = \frac{1}{13}(2\cos 2x - 3\sin 2x).$$

The primitive is $y = e^{\frac{1}{2}x}(C_1\cos \frac{1}{2}\sqrt{3}\,x + C_2\sin \frac{1}{2}\sqrt{3}\,x) + \frac{1}{13}(2\cos 2x - 3\sin 2x)$.

12. Solve $(D^2+4)y = \cos 2x + \cos 4x$.

The complementary function is $y = C_1\cos 2x + C_2\sin 2x$, and a particular integral is

$$y = \frac{1}{D^2+4}(\cos 2x + \cos 4x) = \frac{1}{D^2+4}\cos 2x + \frac{1}{D^2+4}\cos 4x.$$

The method of this chapter cannot be used to evaluate $\frac{1}{D^2+4}\cos 2x$ since, when D^2 is replaced by -4, $D^2+4 = 0$. However, the following procedure may be used.

Consider $\frac{1}{D^2+4}\cos(2+h)x = \frac{1}{-(2+h)^2+4}\cos(2+h)x = -\frac{1}{4h+h^2}\cos(2+h)x$

$$= -\frac{1}{h(4+h)}(\cos 2x - hx\sin 2x - \tfrac{1}{2}(hx)^2\cos 2x + \cdots\cdots\cdots)$$

by Taylor's theorem. The first term, $\cos 2x$, is part of the complementary function and need not be considered here. Hence, a particular integral is

$$\frac{1}{D^2+4}\cos(2+h)x = \frac{1}{h(4+h)}(hx\sin 2x + \tfrac{1}{2}(hx)^2\cos 2x - \cdots\cdots\cdots)$$

$$= \frac{1}{4+h}(x\sin 2x + \tfrac{1}{2}hx^2\cos 2x - \cdots\cdots\cdots).$$

Letting $h \to 0$, we obtain $\frac{1}{D^2+4}\cos 2x = \frac{1}{4}x\sin 2x$. Since $\frac{1}{D^2+4}\cos 4x = -\frac{1}{12}\cos 4x$, the primitive is $y = C_1\cos 2x + C_2\sin 2x + \frac{1}{4}x\sin 2x - \frac{1}{12}\cos 4x$. (Compare this solution with that given in Problem 8, Chapter 14.)

13. Solve $(2D^2 + 2D + 3)y = x^2 + 2x - 1$.

The complementary function is $y = e^{-\frac{1}{2}x}(C_1 \cos \frac{1}{2}\sqrt{5}\,x + C_2 \sin \frac{1}{2}\sqrt{5}\,x)$, and a particular integral is

$$y = \frac{1}{2D^2 + 2D + 3}(x^2 + 2x - 1) = (\frac{1}{3} - \frac{2}{9}D - \frac{2}{27}D^2)(x^2 + 2x - 1)$$

$$= \frac{1}{3}(x^2 + 2x - 1) - \frac{2}{9}(2x + 2) - \frac{2}{27}(2) = \frac{1}{3}x^2 + \frac{2}{9}x - \frac{25}{27}.$$

Note: $\dfrac{1}{2D^2 + 2D + 3} = (\frac{1}{3} - \frac{2}{9}D - \frac{2}{27}D^2 + \cdots\cdots\cdots)$ by direct division.

The primitive is $y = e^{-\frac{1}{2}x}(C_1 \cos \frac{1}{2}\sqrt{5}\,x + C_2 \sin \frac{1}{2}\sqrt{5}\,x) + \frac{1}{3}x^2 + \frac{2}{9}x - \frac{25}{27}$.

14. Solve $(D^3 - 2D + 4)y = x^4 + 3x^2 - 5x + 2$.

The complementary function is $y = C_1 e^{-2x} + e^x(C_2 \cos x + C_3 \sin x)$, and a particular integral is

$$y = \frac{1}{D^3 - 2D + 4}(x^4 + 3x^2 - 5x + 2) = (\frac{1}{4} + \frac{1}{8}D + \frac{1}{16}D^2 - \frac{1}{32}D^3 - \frac{3}{64}D^4)(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{3}{2}x^2 - \frac{5}{4}x - \frac{7}{8}.$$

The primitive is $y = C_1 e^{-2x} + e^x(C_2 \cos x + C_3 \sin x) + \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{3}{2}x^2 - \frac{5}{4}x - \frac{7}{8}$.

15. Solve $(D^3 - 4D^2 + 3D)y = x^2$.

The complementary function is $y = C_1 + C_2 e^x + C_3 e^{3x}$, and a particular integral is

$$y = \frac{1}{D(D^2 - 4D + 3)}x^2 = \frac{1}{D}(\frac{1}{D^2 - 4D + 3})x^2 = \frac{1}{D}(\frac{1}{3} + \frac{4}{9}D + \frac{13}{27}D^2)x^2$$

$$= \frac{1}{D}(\frac{1}{3}x^2 + \frac{8}{9}x + \frac{26}{27}) = \frac{1}{9}x^3 + \frac{4}{9}x^2 + \frac{26}{27}x, \quad \text{since } \frac{1}{D}\{f(x)\} = \int f(x)\,dx.$$

The primitive is $y = C_1 + C_2 e^x + C_3 e^{3x} + \frac{1}{9}x^3 + \frac{4}{9}x^2 + \frac{26}{27}x$.

16. Solve $(D^4 + 2D^3 - 3D^2)y = x^2 + 3e^{2x} + 4 \sin x$.

The complementary function is $y = C_1 + C_2 x + C_3 e^x + C_4 e^{-3x}$, and a particular integral is

$$y = \frac{1}{D^2(D^2 + 2D - 3)}(x^2 + 3e^{2x} + 4 \sin x)$$

$$= \frac{1}{D^2(D^2 + 2D - 3)}x^2 + 3\frac{1}{D^2(D^2 + 2D - 3)}e^{2x} + 4\frac{1}{D^2(D^2 + 2D - 3)}\sin x$$

$$= \frac{1}{D^2}\{\frac{1}{D^2 + 2D - 3}x^2\} + \frac{3}{4(4 + 4 - 3)}e^{2x} + \frac{4}{(-1)(-1 + 2D - 3)}\sin x$$

$$= \frac{1}{D^2}\left(-\frac{1}{3} - \frac{2}{9}D - \frac{7}{27}D^2\right)x^2 + \frac{3}{20}e^{2x} - \frac{2}{D-2}\sin x$$

$$= -\frac{1}{27}\frac{1}{D^2}(9x^2 + 12x + 14) + \frac{3}{20}e^{2x} - 2\frac{D+2}{D^2-4}\sin x$$

$$= -\frac{1}{27}\left(\frac{3}{4}x^4 + 2x^3 + 7x^2\right) + \frac{3}{20}e^{2x} + \frac{2}{5}(\cos x + 2\sin x).$$

The primitive is

$$y = C_1 + C_2x + C_3e^x + C_4e^{-3x} - \frac{x^2}{108}(3x^2 + 8x + 28) + \frac{3}{20}e^{2x} + \frac{2}{5}(\cos x + 2\sin x).$$

17. Establish the rule in *d)* above by first showing that $F(D)e^{ax}U = e^{ax}F(D+a)U$.

Since when $y = e^{ax}U$, $Dy = ae^{ax}U + e^{ax}DU = e^{ax}(D+a)U$,

$D^2y = ae^{ax}(D+a)U + e^{ax}D(D+a)U = e^{ax}(D^2 + 2aD + a^2)U = e^{ax}(D+a)^2U$, ...,
$D^ry = e^{ax}(D+a)^rU$, and

1) $F(D)e^{ax}U = \sum P_rD^r(e^{ax}U) = \sum P_re^{ax}(D+a)^rU = e^{ax}\sum P_r(D+a)^rU = e^{ax}F(D+a)U$.

Let $V = F(D+a)U$ so that $U = \frac{1}{F(D+a)}V$. Then, from 1),

$$F(D)e^{ax}\frac{1}{F(D+a)}V = e^{ax}V \quad \text{and} \quad \frac{1}{F(D)}e^{ax}V = \frac{1}{F(D)}\left\{F(D)e^{ax}\frac{1}{F(D+a)}V\right\} = e^{ax}\frac{1}{F(D+a)}V.$$

18. Solve $(D^2-4)y = x^2e^{3x}$.

The complementary function is $y = C_1e^{2x} + C_2e^{-2x}$, and a particular integral is

$$y = \frac{1}{D^2-4}x^2e^{3x} = e^{3x}\frac{1}{(D+3)^2-4}x^2 = e^{3x}\frac{1}{D^2+6D+5}x^2$$

$$= e^{3x}\left(\frac{1}{5} - \frac{6}{25}D + \frac{31}{125}D^2\right)x^2 = e^{3x}\left(\frac{x^2}{5} - \frac{12}{25}x + \frac{62}{125}\right).$$

The primitive is $y = C_1e^{2x} + C_2e^{-2x} + \frac{1}{125}e^{3x}(25x^2 - 60x + 62)$.

19. Solve $(D^2 + 2D + 4)y = e^x\sin 2x$.

The complementary function is $y = e^{-x}(C_1\cos\sqrt{3}\,x + C_2\sin\sqrt{3}\,x)$, and a particular integral is

$$y = \frac{1}{D^2+2D+4}e^x\sin 2x = e^x\frac{1}{(D+1)^2 + 2(D+1) + 4}\sin 2x = e^x\frac{1}{D^2+4D+7}\sin 2x$$

$$= e^x\frac{1}{4D+3}\sin 2x = e^x\frac{4D-3}{16D^2-9}\sin 2x = -\frac{e^x}{73}(4D-3)\sin 2x = -\frac{e^x}{73}(8\cos 2x - 3\sin 2x).$$

The primitive is $y = e^{-x}(C_1\cos\sqrt{3}\,x + C_2\sin\sqrt{3}\,x) - \frac{e^x}{73}(8\cos 2x - 3\sin 2x)$.

20. Solve $(D^2-4D+3)y = 2xe^{3x} + 3e^x \cos 2x$.

The complementary function is $y = C_1e^x + C_2e^{3x}$, and a particular integral is

$$y = \frac{1}{D^2-4D+3}(2xe^{3x} + 3e^x \cos 2x) = 2\frac{1}{D^2-4D+3}xe^{3x} + 3\frac{1}{D^2-4D+3}e^x \cos 2x$$

$$= 2e^{3x}\frac{1}{D^2+2D}x + 3e^x\frac{1}{D^2-2D}\cos 2x = 2e^{3x}\frac{1}{D}\cdot\frac{1}{D+2}x + 3e^x\frac{1}{-4-2D}\cos 2x$$

$$= 2e^{3x}\frac{1}{D}\left(\frac{1}{2} - \frac{1}{4}D\right)x - \frac{3}{2}e^x\frac{D-2}{D^2-4}\cos 2x = \frac{1}{2}e^{3x}\frac{1}{D}(2x-1) + \frac{3}{16}e^x(D-2)\cos 2x$$

$$= \frac{1}{2}e^{3x}(x^2-x) - \frac{3}{8}e^x(\cos 2x + \sin 2x).$$

The primitive is $y = C_1e^x + C_2e^{3x} + \frac{1}{2}e^{3x}(x^2-x) - \frac{3}{8}e^x(\cos 2x + \sin 2x)$.

21. Establish the rule in *e)* above by first showing that $F(D)xU = xF(D)U + F'(D)U$.

Since when $y = xU$, $Dy = xDU + U$, $D^2y = xD^2U + 2DU$, ..., $D^ry = xD^rU + rD^{r-1}U = xD^rU + (\frac{d}{dD}D^r)U$, then

1) $F(D)xU = \sum_r P_r D^r(xU) = \sum_r P_r xD^rU + \sum_r P_r(\frac{d}{dD}D^r)U = xF(D)U + F'(D)U$.

Let $V = F(D)U$ so that $U = \frac{1}{F(D)}V$. Then, substituting in 1),

$$F(D)x\frac{1}{F(D)}V = xF(D)\frac{1}{F(D)}V + F'(D)\frac{1}{F(D)}V, \quad xV = F(D)x\frac{1}{F(D)}V - F'(D)\frac{1}{F(D)}V,$$

and $$\frac{1}{F(D)}xV = x\frac{1}{F(D)}V - \frac{1}{F(D)}F'(D)\frac{1}{F(D)}V = x\frac{1}{F(D)}V - \frac{F'(D)}{\{F(D)\}^2}V.$$

22. Solve $(D^2+3D+2)y = x \sin 2x$.

The complementary function is $y = C_1e^{-x} + C_2e^{-2x}$, and a particular integral is

$$y = \frac{1}{D^2+3D+2}x\sin 2x = x\frac{1}{D^2+3D+2}\sin 2x - \frac{2D+3}{(D^2+3D+2)^2}\sin 2x$$

$$= x\frac{1}{3D-2}\sin 2x - \frac{2D+3}{D^4+6D^3+13D^2+12D+4}\sin 2x$$

$$= x\frac{1}{3D-2}\sin 2x - \frac{2D+3}{(-4)^2+6(-4)D+13(-4)+12D+4}\sin 2x, \text{ replacing } D^2 \text{ by } -4,$$

$$= x\frac{3D+2}{9D^2-4}\sin 2x + \frac{1}{4}\frac{(2D+3)(3D-8)}{9D^2-64}\sin 2x$$

$$= \frac{-x(3\cos 2x + \sin 2x)}{20} + \frac{24\sin 2x + 7\cos 2x}{200}.$$

The primitive is $y = C_1e^{-x} + C_2e^{-2x} - \frac{30x-7}{200}\cos 2x - \frac{5x-12}{100}\sin 2x$.

23. Solve $(D^2-1)y = x^2 \sin 3x$.

The complementary function is $y = C_1e^x + C_2e^{-x}$, and a particular integral is

$$y = \frac{1}{D^2-1}x^2\sin 3x = x\frac{1}{D^2-1}x\sin 3x - \frac{2D}{(D^2-1)^2}x\sin 3x$$

$$= x^2\frac{1}{D^2-1}\sin 3x - x\frac{2D}{(D^2-1)^2}\sin 3x - 2D\left\{x\frac{1}{D^4-2D^2+1}\sin 3x - \frac{4D^3-4D}{(D^4-2D^2+1)^2}\sin 3x\right\}$$

$$= x^2\frac{1}{D^2-1}\sin 3x - x\frac{2D}{(D^2-1)^2}\sin 3x - 2D\left\{x\frac{1}{(D^2-1)^2}\sin 3x\right\} + \frac{8D^2}{(D^2-1)^3}\sin 3x$$

$$= -\frac{1}{10}x^2\sin 3x - \frac{3}{50}x\cos 3x - \frac{1}{50}D(x\sin 3x) + \frac{9}{125}\sin 3x$$

$$= -\frac{1}{10}x^2\sin 3x - \frac{3}{25}x\cos 3x + \frac{13}{250}\sin 3x.$$

The primitive is $y = C_1e^x + C_2e^{-x} - \frac{25x^2-13}{250}\sin 3x - \frac{3}{25}x\cos 3x$.

24. Solve $(D^3-3D^2-6D+8)y = xe^{-3x}$.

The complementary function is $y = C_1e^x + C_2e^{4x} + C_3e^{-2x}$, and a particular integral is

$$y = \frac{1}{D^3-3D^2-6D+8}xe^{-3x}.$$

By *a)*: $$y = e^{-3x}\frac{1}{(D-3)^3-3(D-3)^2-6(D-3)+8}x = e^{-3x}\frac{1}{D^3-12D^2+39D-28}x$$

$$= e^{-3x}\left(-\frac{1}{28} - \frac{39}{784}D\right)x = e^{-3x}\left(-\frac{1}{28}x - \frac{39}{784}\right).$$

By *e)*: $$y = x\frac{1}{D^3-3D^2-6D+8}e^{-3x} - \frac{3D^2-6D-6}{(D^3-3D^2-6D+8)^2}e^{-3x}$$

$$= -\frac{1}{28}xe^{-3x} - \frac{3D^2-6D-6}{(-28)^2}e^{-3x} = -\frac{1}{28}xe^{-3x} - \frac{39}{784}e^{-3x}.$$

The primitive is $y = C_1e^x + C_2e^{4x} + C_3e^{-2x} - \frac{e^{-3x}}{784}(28x+39)$.

25. Denote the real and imaginary parts of a complex number z by $\mathrm{Re}[z]$ and $\mathrm{Im}[z]$ respectively. An alternate short method for Problems 9-11 makes use of $\sin bx = \mathrm{Im}[e^{ibx}]$ and $\cos bx = \mathrm{Re}[e^{ibx}]$.

Consider, for example, $F(D)z = (D^3+D^2+D+1)z = e^{2ix} + e^{3ix}$ for which

$$z = \frac{e^{2ix}}{F(2i)} + \frac{e^{3ix}}{F(3i)} = -\frac{e^{2ix}}{3+6i} - \frac{e^{3ix}}{8+24i}$$

$$= \frac{2i-1}{15}(\cos 2x + i\sin 2x) + \frac{3i-1}{80}(\cos 3x + i\sin 3x) = z_1 + z_2$$

is a particular integral. Then

$$\mathrm{Re}[z_1] + \mathrm{Re}[z_2] = -\frac{1}{15}(2\sin 2x + \cos 2x) - \frac{1}{80}(3\sin 3x + \cos 3x)$$

is particular integral of $F(D)y = \cos 2x + \cos 3x$,

$$\mathrm{Im}[z_1] + \mathrm{Im}[z_2] = \frac{1}{15}(2\cos 2x - \sin 2x) + \frac{1}{80}(3\cos 3x - \sin 3x)$$

is a particular integral of $F(D)y = \sin 2x + \sin 3x$,

$$\mathrm{Re}[z_1] + \mathrm{Im}[z_2]$$

is a particular integral of $F(D)y = \cos 2x + \sin 3x$, and

$$\mathrm{Im}[z_1] + \mathrm{Re}[z_2]$$

is the particular integral in Problem 10.

SUPPLEMENTARY PROBLEMS

Find a particular integral.

26.	$(D^2 + D + 1)y = e^{3x} + 6e^x - 3e^{-2x} + 5$	*Ans.*	$y = e^{3x}/13 + 2e^x - e^{-2x} + 5$
27.	$(D^2 - 1)y = e^x$		$y = xe^x/2$
28.	$(D - 2)^2 y = e^x + xe^{2x}$		$y = e^x + x^3 e^{2x}/6$
29.	$(D^4 - 1)y = \sin 2x$		$y = \frac{1}{15}\sin 2x$
30.	$(D^3 + 1)y = \cos x$		$y = \frac{1}{2}(\cos x - \sin x)$
31.	$(D^2 + 4)y = \sin 2x$		$y = -\frac{1}{4}x\cos 2x$
32.	$(D^2 + 5)y = \cos\sqrt{5}\,x$		$y = \frac{\sqrt{5}}{10}x\sin\sqrt{5}\,x$
33.	$(D^3 + D^2 + D + 1)y = e^x + e^{-x} + \sin x$		$y = \frac{1}{4}(e^x + 2xe^{-x}) - \frac{1}{4}x(\sin x + \cos x)$
34.	$(D^2 - 1)y = x^2$		$y = -x^2 - 2$
35.	$D^4(D^2 - 1)y = x^2$		$y = -\frac{1}{360}(x^6 + 30x^4)$
36.	$(D^2 + 2)y = x^3 + x^2 + e^{-2x} + \cos 3x$		$y = \frac{1}{2}(x^3 + x^2 - 3x - 1) + \frac{1}{6}e^{-2x} - \frac{1}{7}\cos 3x$
37.	$(D^2 - 2D - 1)y = e^x \cos x$		$y = -\frac{1}{3}e^x \cos x$
38.	$(D - 2)^2 y = e^{2x}/x^2$		$y = -e^{2x}\ln x$
39.	$(D^2 - 1)y = xe^{3x}$		$y = \frac{1}{32}e^{3x}(4x - 3)$
40.	$(D^2 + 5D + 6)y = e^{-2x}(\sec^2 x)(1 + 2\tan x)$		$y = e^{-2x}\tan x$

CHAPTER 17

Linear Equations with Variable Coefficients

THE CAUCHY AND LEGENDRE LINEAR EQUATIONS

THE CAUCHY LINEAR EQUATION

1) $$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_{n-1} x \frac{dy}{dx} + P_n y = Q(x),$$

in which $P_0, P_1, \cdots\cdots, P_n$ are constants, and the *Legendre linear equation*

2) $$P_0 (ax+b)^n \frac{d^n y}{dx^n} + P_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_{n-1} (ax+b) \frac{dy}{dx} + P_n y = Q(x),$$

of which 1) is the special case $(a = 1, b = 0)$, may be reduced to linear equations with constant coefficients by properly chosen transformations of the independent variable.

THE CAUCHY LINEAR EQUATION. Let $x = e^z$; then if $\mathfrak{D}$ is defined by $\mathfrak{D} = \frac{d}{dz}$,

$$Dy = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \qquad \text{and} \qquad xDy = \frac{dy}{dz} = \mathfrak{D}y,$$

$$D^2 y = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x^2}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) \qquad \text{and} \qquad x^2 D^2 y = \mathfrak{D}(\mathfrak{D}-1)y,$$

$$D^3 y = -\frac{2}{x^3}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) + \frac{1}{x^3}\left(\frac{d^3 y}{dz^3} - \frac{d^2 y}{dz^2}\right)$$

$$= \frac{1}{x^3}\left(\frac{d^3 y}{dz^3} - 3\frac{d^2 y}{dz^2} + 2\frac{dy}{dz}\right) \qquad \text{and} \qquad x^3 D^3 y = \mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)y,$$

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$$x^r D^r y = \mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)\cdots\cdots(\mathfrak{D}-r+1)y.$$

After making these replacements, 1) becomes

$$\{P_0 \mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)\cdots(\mathfrak{D}-n+1) + P_1 \mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)\cdots(\mathfrak{D}-n+2) + \cdots\cdots\cdots + P_{n-1}\mathfrak{D} + P_n\}y = Q(e^z),$$

a linear equation with constant coefficients.

See Problems 1–3

THE LEGENDRE LINEAR EQUATION. Let $ax + b = e^z$; then

$$Dy = \frac{dy}{dz}\frac{dz}{dx} = \frac{a}{ax+b}\frac{dy}{dz} \qquad \text{and} \qquad (ax+b)Dy = a\frac{dy}{dz} = a\mathfrak{D}y,$$

$$D^2 y = \frac{a^2}{(ax+b)^2}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) \qquad \text{and} \qquad (ax+b)^2 D^2 y = a^2 \mathfrak{D}(\mathfrak{D}-1)y,$$

. . . . . .

. . . . . .

$$(ax+b)^r D^r y = a^r D(D-1)\cdots(D-r+1)y.$$

After making these replacements, 2) becomes

$$\{p_0 a^n D(D-1)(D-2)\cdots(D-n+1) + p_1 a^{n-1} D(D-1)(D-2)\cdots(D-n+2) + \cdots\cdots + p_{n-1} aD + p_n\}y = Q\left(\frac{e^z-b}{a}\right),$$

a linear equation with constant coefficients. See problems 4–5.

SOLVED PROBLEMS

1. Solve $(x^3D^3 + 3x^2D^2 - 2xD + 2)y = 0$.

The transformation $x = e^z$ reduces the equation to

$$\{D(D-1)(D-2) + 3D(D-1) - 2D + 2\}y = (D^3 - 3D + 2)y = 0$$

whose solution is $y = C_1e^z + C_2ze^z + C_3e^{-2z}$

Since $z = \ln x$, the complete solution of the given equation is $y = C_1x + C_2x\ln x + C_3/x^2$.

2. Solve $(x^3D^3 + 2xD - 2)y = x^2\ln x + 3x$.

The transformation $x = e^z$ reduces the equation to

$$\{D(D-1)(D-2) + 2D - 2\}y = (D-1)(D^2-2D+2)y = ze^{2z} + 3e^z.$$

The complementary function is $y = C_1e^z + e^z(C_2\cos z + C_3\sin z)$, and a particular integral is

$$y = \frac{1}{D^3-3D^2+4D-2}(ze^{2z}+3e^z) = e^{2z}\frac{1}{(D+2)^3-3(D+2)^2+4(D+2)-2}z + 3\frac{1}{(D-1)(D^2-2D+2)}e^z$$

$$= e^{2z}\frac{1}{D^3+3D^2+4D+2}z + 3\frac{1}{(D-1)(1)}e^z$$

$$= e^{2z}(\tfrac{1}{2}-D)z + 3e^z\int e^z\cdot e^{-z}\,dz = e^{2z}(\tfrac{1}{2}z-1) + 3ze^z.$$

Thus, the solution is $y = C_1e^z + e^z(C_2\cos z + C_3\sin z) + \frac{1}{2}e^{2z}(z-2) + 3ze^z$

$$= C_1x + x(C_2\cos\ln x + C_3\sin\ln x) + \tfrac{1}{2}x^2(\ln x - 2) + 3x\ln x.$$

3. Solve $(x^2D^2 - xD + 4)y = \cos\ln x + x\sin\ln x$.

The transformation $x = e^z$ reduces the equation to

$$\{D(D-1) - D + 4\}y = (D^2 - 2D + 4)y = \cos z + e^z\sin z.$$

The complementary function is $y = e^z(C_1\cos\sqrt{3}\,z + C_2\sin\sqrt{3}\,z)$, and a particular solution is

$$y = \frac{1}{D^2-2D+4}\cos z + \frac{1}{D^2-2D+4}e^z\sin z$$

$$= \frac{1}{3-2D}\cos z + e^z\frac{1}{D^2+3}\sin z = \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2}e^z\sin z.$$

Thus, the solution is

$$y = e^z(C_1\cos\sqrt{3}\,z + C_2\sin\sqrt{3}\,z) + \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2}e^z\sin z$$

$$= x(C_1\cos\sqrt{3}\cdot\ln x + C_2\sin\sqrt{3}\cdot\ln x) + \frac{1}{13}(3\cos\ln x - 2\sin\ln x) + \frac{1}{2}x\sin\ln x.$$

4. Solve $(x+2)^2\dfrac{d^2y}{dx^2} - (x+2)\dfrac{dy}{dx} + y = 3x + 4.$

Put $x + 2 = e^z$; then the given equation becomes

$$\{D(D-1) - D + 1\}y = (D-1)^2y = 3e^z - 2.$$

The complementary function is $y = C_1e^z + C_2ze^z$, and a particular integral is

$$y = \frac{1}{(D-1)^2}(3e^z - 2) = 3e^z\iint(dz)^2 - 2\frac{1}{(D-1)^2}e^{0z} = \frac{3}{2}z^2e^z - 2.$$

The solution is $y = C_1e^z + C_2ze^z + \dfrac{3}{2}z^2e^z - 2$ or, since $z = \ln(x+2)$,

$$y = (x+2)[C_1 + C_2\ln(x+2) + \frac{3}{2}\ln^2(x+2)] - 2.$$

5. Solve $\{(3x+2)^2D^2 + 3(3x+2)D - 36\}y = 3x^2 + 4x + 1.$

The transformation $3x + 2 = e^z$ reduces the equation to

$$\{9D(D-1) + 9D - 36\}y = 9(D^2-4)y = \frac{1}{3}(9x^2 + 12x + 3) = \frac{1}{3}(e^{2z} - 1) \quad \text{or} \quad (D^2-4)y = \frac{1}{27}(e^{2z} - 1).$$

The complete solution is $y = C_1e^{2z} + C_2e^{-2z} + \dfrac{1}{27}\left(\dfrac{1}{D^2-4}e^{2z} - \dfrac{1}{D^2-4}e^{0z}\right)$

$$= C_1e^{2z} + C_2e^{-2z} + \frac{1}{108}(ze^{2z} + 1)$$

or $$y = C_1(3x+2)^2 + C_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\ln(3x+2) + 1].$$

SUPPLEMENTARY PROBLEMS

Solve.

6. $(x^2D^2 - 3xD + 4)y = x + x^2\ln x$ *Ans.* $y = C_1x^2 + C_2x^2\ln x + x + \frac{1}{6}x^2\ln^3x$

7. $(x^2D^2 - 2xD + 2)y = \ln^2x - \ln x^2$ *Ans.* $y = C_1x + C_2x^2 + \frac{1}{2}(\ln^2x + \ln x) + \frac{1}{4}$

8. $(x^3D^3 + 2x^2D^2)y = x + \sin(\ln x)$ *Ans.* $y = C_1 + C_2x + C_3\ln x + x\ln x + \frac{1}{2}(\cos\ln x + \sin\ln x)$

9. $x^3y''' + xy' - y = 3x^4$ *Ans.* $y = C_1x + C_2x\ln x + C_3x\ln^2x + x^4/9$

10. $[(x+1)^2D^2 + (x+1)D - 1]y = \ln(x+1)^2 + x - 1$

Ans. $y = C_1(x+1) + C_2(x+1)^{-1} - \ln(x+1)^2 + \frac{1}{2}(x+1)\cdot\ln(x+1) + 2$

11. $(2x+1)^2y'' - 2(2x+1)y' - 12y = 6x$ *Ans.* $y = C_1(2x+1)^{-1} + C_2(2x+1)^3 - 3x/8 + 1/16$

CHAPTER 18

Linear Equations with Variable Coefficients

EQUATIONS OF THE SECOND ORDER

A LINEAR DIFFERENTIAL EQUATION of the second order has the form

$$1) \qquad \frac{d^2y}{dx^2} + R(x)\frac{dy}{dx} + S(x)\,y = Q(x).$$

If the coefficients R and S are constants, the equation can be solved by the methods of the preceding chapter; otherwise, no general method is known. In this chapter certain procedures are given which at times will yield a solution.

CHANGE OF DEPENDENT VARIABLE. Under the transformation

$$y = uv, \quad u = u(x) \quad \text{and} \quad v = v(x),$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + v\frac{d^2u}{dx^2},$$

1) becomes

$$2) \qquad \frac{d^2v}{dx^2} + R_1(x)\frac{dv}{dx} + S_1(x)\,v = Q_1(x)$$

with $$R_1(x) = \frac{2}{u}\frac{du}{dx} + R(x), \quad S_1(x) = \frac{1}{u}\left\{\frac{d^2u}{dx^2} + R(x)\frac{du}{dx} + S(x)\cdot u\right\}, \quad Q_1(x) = \frac{Q(x)}{u}.$$

a) If u is a particular integral of $\frac{d^2y}{dx^2} + R(x)\frac{dy}{dx} + S(x)\,y = 0$, then $S_1 = 0$ and 2) becomes

$$3) \qquad \frac{d^2v}{dx^2} + R_1(x)\frac{dv}{dx} = Q_1(x).$$

The further substitution $\frac{dv}{dx} = p$, $\frac{d^2v}{dx^2} = \frac{dp}{dx}$ reduces 3) to

$$4) \qquad \frac{dp}{dx} + R_1(x)\,p = Q_1(x),$$

a linear equation of the first order. See Problems 1-6.

b) If u is chosen so that $R_1(x) = \frac{2}{u}\frac{du}{dx} + R(x) = 0$ or $\frac{du}{u} = -\frac{1}{2}R(x)\,dx$, then

$$u = e^{-\frac{1}{2}\int R(x)\,dx}$$

Now $\frac{du}{dx} = -\frac{1}{2}uR(x)$ and $\frac{d^2u}{dx^2} = -\frac{1}{2}R(x)\frac{du}{dx} - \frac{1}{2}u\frac{dR}{dx}$ so that

$$S_1(x) = S(x) + \frac{R(x)}{u}\frac{du}{dx} + \frac{1}{u}\frac{d^2u}{dx^2} = S(x) + \frac{1}{2}\frac{R(x)}{u}\frac{du}{dx} - \frac{1}{2}\frac{dR}{dx} = S - \frac{1}{4}R^2 - \frac{1}{2}\frac{dR}{dx}$$

and $Q_1 = Q/u$.

If $S_1(x) = S - \frac{1}{4}R^2 - \frac{1}{2}\frac{dR}{dx} = A$, a constant, 2) becomes $\frac{d^2v}{dx^2} + Av = Q/u$, a linear equation with constant coefficients.

If $S_1(x) = A/x^2$, 2) becomes $x^2\frac{d^2v}{dx^2} + Av = Qx^2/u$, a Cauchy equation, and the substitution $x = e^z$ will reduce it to one with constant coefficients.

See Problems 7-10.

CHANGE OF INDEPENDENT VARIABLE. Let the transformation be $z = \theta(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}, \qquad \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2},$$

and 1) becomes

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \left(\frac{d^2z}{dx^2} + R\frac{dz}{dx}\right)\frac{dy}{dz} + Sy = Q$$

or

5) $$\frac{d^2y}{dz^2} + \frac{\frac{d^2z}{dx^2} + R\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Sy}{\left(\frac{dz}{dx}\right)^2} = \frac{Q}{\left(\frac{dz}{dx}\right)^2}.$$

Let $z = \theta(x)$ be chosen so that $\frac{dz}{dx} = \sqrt{\frac{\pm S}{a^2}}$, the sign being that which makes $\frac{dz}{dx}$ real and a^2 being any positive constant. (One may consistently take $a^2 = 1$.)

If now $\dfrac{\frac{d^2z}{dx^2} + R\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = A$, a constant, then 5) becomes $\frac{d^2y}{dz^2} + A\frac{dy}{dz} \pm a^2y = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$ a linear equation with constant coefficients.

See Problems 11-14.

OPERATIONAL FACTORING. It may be possible to separate the left member of

$$\{P(x)D^2 + R(x)D + S(x)\}y = Q(x)$$

into two linear operators $F_1(D)$ and $F_2(D)$ so that

6) $$\{F_1(D)\cdot F_2(D)\}y = F_1(D)\{F_2(D)y\} = \{P(x)D^2 + R(x)D + S(x)\}y = Q(x).$$

Then, setting $F_2(D)y = v$, 6) becomes $F_1(D)v = Q(x)$, a linear equation of order one.

The factorization in this section differs from that of Chapter 13. With possible exceptions, the factors here contain the independent variable x they are not commutative, and the factorization differs from that when D is treated as a variable. For example,

$$\{xD^2 - (x^2+2)D + x\}y = \{(xD-2)(D-x)\}y,$$

since
$$\{(xD-2)(D-x)\}y = (xD-2)(\frac{d}{dx}-x)y = (xD-2)(y'-xy)$$

$$= (x\frac{d}{dx}-2)(y'-xy) = x(y''-y-xy') - 2(y'-xy)$$

$$= xy'' - (x^2+2)y' + xy = \{(xD^2 - (x^2+2)D + x\}y.$$

The factors are not commutative, since

$$\{(D-x)(xD-2)\}y = (D-x)(xy'-2y) = xy'' + y' - 2y' - x^2y' + 2xy$$
$$= xy'' - (x^2+1)y' + 2xy = \{xD^2 - (x^2+1)D + 2x\}y.$$

Finally, when D is treated as a *variable* rather than an operator,

$$\{(xD-2)(D-x)\}y = \{xD^2 - (x^2+2)D + 2x\}y.$$ See Problems 15-17.

IN SUMMARY, the following procedure is suggested for solving

$$\frac{d^2y}{dx^2} + R(x)\frac{dy}{dx} + S(x)y = Q(x).$$

1) Find by inspection, or otherwise, a particular integral $u = u(x)$ of the equation when $Q(x) = 0$. The substitution $y = uv$ will yield a linear equation in which the dependent variable v does not appear. This equation is of the first order in $dv/dx = p$.

2) If a particular integral cannot be found, compute $S - \frac{1}{4}R^2 - \frac{1}{2}\frac{dR}{dx}$. If this is a constant K or K/x^2 the transformation $y = ve^{-\frac{1}{2}\int R\,dx}$ reduces the given equation to a linear equation with constant coefficients or to a Cauchy equation.

3) If the above procedure does not apply, put $\frac{dz}{dx} = \sqrt{\frac{\pm S}{a^2}}$ (choosing the sign so that the square root is real) and substitute in $\dfrac{\frac{d^2z}{dx^2} + R\frac{dz}{dx}}{(\frac{dz}{dx})^2}$. If this is a constant the transformation $z = \int\sqrt{\frac{\pm S}{a^2}}\,dx$ yields a linear equation with constant coefficients.

4) If the left member of the equation is operationally factorable, the problem is then reduced to that of solving two linear equations of order one.

Note. As a partial check on the work, it is desirable to know the type of equation which results when the transformations in 1)-3) are made.

SOLVED PROBLEMS

1. For the equation $(D^2 + RD + S)y = 0$, show that

a) $y = x$ is a particular integral if $R + xS = 0$,

b) $y = e^x$ is a particular integral if $1 + R + S = 0$,

c) $y = e^{-x}$ is a particular integral if $1 - R + S = 0$,

d) $y = e^{mx}$ is a particular integral if $m^2 + mR + S = 0$.

a) If $y = x$ is a particular integral of $(D^2 + RD + S)y = 0$ then, since $Dy = 1$ and $D^2y = 0$, $R + Sx = 0$.

d) If $y = e^{mx}$ is a particular integral of $(D^2 + RD + S)y = 0$ then, since $Dy = my$ and $D^2y = m^2y$, $(m^2 + mR + S)y = 0$ and $m^2 + mR + S = 0$. *b)* and *c)* are special cases ($m = 1$, $m = -1$) of *d)*.

2. Solve $(D^2 - \frac{3}{x}D + \frac{3}{x^2})y = 2x - 1$.

Here $R + Sx = 0$ and $y = x$ is a particular integral of $(D^2 - \frac{3}{x}D + \frac{3}{x^2})y = 0$.

The transformation $y = xv$, $Dy = x\frac{dv}{dx} + v$, $D^2y = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$ reduces the given equation to

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - 3\frac{dv}{dx} - \frac{3}{x}v + \frac{3}{x}v = x\frac{d^2v}{dx^2} - \frac{dv}{dx} = 2x - 1 \qquad \text{or} \qquad \frac{d^2v}{dx^2} - \frac{1}{x}\frac{dv}{dx} = 2 - \frac{1}{x}.$$

Putting $\frac{dv}{dx} = p$, $\frac{d^2v}{dx^2} = \frac{dp}{dx}$, this becomes $\frac{dp}{dx} - \frac{1}{x}p = 2 - \frac{1}{x}$ for which $e^{\int -dx/x} = 1/x$ is an integrating factor. Then

$$\frac{p}{x} = \int (\frac{2}{x} - \frac{1}{x^2})dx = 2\ln x + \frac{1}{x} + K, \qquad p = \frac{dv}{dx} = 2x\ln x + 1 + Kx,$$

$$v = \frac{y}{x} = \int (2x\ln x + 1 + Kx)dx = x^2\ln x + x + C_1x^2 + C_2, \quad \text{and} \quad y = C_1x^3 + C_2x + x^3\ln x + x^2.$$

3. Solve $x^2(x+1)\frac{d^2y}{dx^2} - x(2 + 4x + x^2)\frac{dy}{dx} + (2 + 4x + x^2)y = -x^4 - 2x^3$.

Here, $R + Sx = -\frac{x(2 + 4x + x^2)}{x^2(x+1)} + x\frac{2 + 4x + x^2}{x^2(x+1)} = 0$ and $y = x$ is a particular integral of the equation with its right member replaced by zero.

The transformation $y = xv$, $\frac{dy}{dx} = x\frac{dv}{dx} + v$, $\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$ reduces the given equation to $x^2(x+1)(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}) - x(2 + 4x + x^2)(x\frac{dv}{dx} + v) + (2 + 4x + x^2)xv = -x^4 - 2x^3$

or

$$\frac{d^2v}{dx^2} - \frac{x+2}{x+1}\frac{dv}{dx} = -\frac{x+2}{x+1}.$$

Putting $\frac{dv}{dx} = p$, this becomes $\frac{dp}{dx} - \frac{x+2}{x+1}p = -\frac{x+2}{x+1}$ for which $e^{-\int(1 + \frac{1}{x+1})dx} = \frac{e^{-x}}{x+1}$ is an integrating factor. Then

$$\frac{e^{-x}}{x+1}p = -\int \frac{(x+2)e^{-x}}{(x+1)^2}\,dx = \frac{e^{-x}}{x+1} + C_1, \qquad p = \frac{dv}{dx} = 1 + C_1(x+1)e^{x},$$

$$v = \frac{y}{x} = x + C_1xe^{x} + C_2, \quad \text{and} \quad y = C_1x^2e^{x} + C_2x + x^2.$$

4. Solve $x\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + (x+1)y = (x^2+x-1)e^{2x}$.

Here $1 + R + S = 1 - \frac{2x+1}{x} + \frac{x+1}{x} = 0$ and $y = e^{x}$ is a particular integral of the equation with its right member replaced by zero.

The transformation $y = e^{x}v$, $\frac{dy}{dx} = e^{x}(\frac{dv}{dx} + v)$, $\frac{d^2y}{dx^2} = e^{x}(\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v)$ reduces the given equation to $\frac{d^2v}{dx^2} - \frac{1}{x}\frac{dv}{dx} = (x + 1 - \frac{1}{x})e^{x}$.

Putting $\frac{dv}{dx} = p$, this becomes $\frac{dp}{dx} - \frac{1}{x}p = (x + 1 - \frac{1}{x})e^{x}$ for which $\frac{1}{x}$ is an integrating factor.

Then $\frac{p}{x} = \int (e^{x} + \frac{xe^{x} - e^{x}}{x^2})dx = e^{x} + \frac{e^{x}}{x} + K$, $p = \frac{dv}{dx} = xe^{x} + e^{x} + Kx$,

$$v = \frac{y}{e^{x}} = xe^{x} + C_1x^2 + C_2; \quad \text{and} \quad y = C_1x^2e^{x} + C_2e^{x} + xe^{2x}.$$

5. Solve $(x-2)\frac{d^2y}{dx^2} - (4x-7)\frac{dy}{dx} + (4x-6)y = 0$.

Here $m^2 + mR + S = m^2 - m\frac{4x-7}{x-2} + \frac{4x-6}{x-2} = 0$ when $m = 2$, and $y = e^{2x}$ is a particular integral.

The transformation $y = e^{2x}v$, $\frac{dy}{dx} = e^{2x}\frac{dv}{dx} + 2e^{2x}v$, $\frac{d^2y}{dx^2} = e^{2x}\frac{d^2v}{dx^2} + 4e^{2x}\frac{dv}{dx} + 4e^{2x}v$ reduces the given equation to

$$(x-2)(\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v) - (4x-7)(\frac{dv}{dx} + 2v) + (4x-6)v = 0 \quad \text{or} \quad \frac{d^2v}{dx^2} - \frac{1}{x-2}\frac{dv}{dx} = 0.$$

Putting $\frac{dv}{dx} = p$, this becomes $\frac{dp}{dx} - \frac{1}{x-2}p = 0$. Then

$$p = \frac{dv}{dx} = K(x-2), \qquad v = \frac{y}{e^{2x}} = C_1(x-2)^2 + C_2, \quad \text{and} \quad y = C_1e^{2x}(x-2)^2 + C_2e^{2x}.$$

6. Solve $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 3y = 2\sec x$.

By inspection, it is seen that $y = \sin x$ is a particular integral of $(D^2 - 2\tan x\,D + 3)y = 0$.

The transformation $y = v\sin x$ reduces the given equation to

$$\sin x\,\frac{d^2v}{dx^2} + 2(\cos x - \frac{\sin^2 x}{\cos x})\frac{dv}{dx} = 2\sec x, \quad \text{or} \quad \frac{d^2v}{dx^2} + 2(\cot x - \tan x)\frac{dv}{dx} = 4\csc 2x.$$

The substitution $\frac{dv}{dx} = p$ reduces this to $\frac{dp}{dx} + 2(\cot x - \tan x)p = 4\csc 2x$ for which an integrating factor is $\frac{1}{4}\sin^2 2x$. Then

$$\tfrac{1}{4}p\sin^2 2x = \int \sin 2x\,dx = -\tfrac{1}{2}\cos 2x + \tfrac{1}{4}K_1, \qquad p = \frac{dv}{dx} = -2\csc 2x\cot 2x + K_1\csc^2 2x,$$

$$v = \frac{y}{\sin x} = \csc 2x + K\cot 2x + C_2, \quad \text{and} \quad y = \tfrac{1}{2}\sec x + C_1(\cos x - \tfrac{1}{2}\sec x) + C_2\sin x.$$

7. Solve $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + (1 + \frac{2}{x^2})y = xe^x$.

Here $R = -\frac{2}{x}$, $S = 1 + \frac{2}{x^2}$, $S - \frac{1}{4}R^2 - \frac{1}{2}\frac{dR}{dx} = 1$ and $u = e^{-\frac{1}{2}\int R\,dx} = e^{\int dx/x} = x$.

The transformation $y = uv = xv$, $\frac{dy}{dx} = x\frac{dv}{dx} + v$, $\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$ reduces the given equation to $\frac{d^2v}{dx^2} + v = e^x$, a linear equation with constant coefficients, whose complete solution is

$$v = \frac{y}{x} = C_1\cos x + C_2\sin x + \frac{1}{D^2+1}e^x = C_1\cos x + C_2\sin x + \tfrac{1}{2}e^x.$$

Thus, $y = C_1x\cos x + C_2x\sin x + \tfrac{1}{2}xe^x$.

8. Solve $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = e^{\frac{1}{2}(x^2+2x)}$.

Here $R = -2x$, $S = x^2 + 2$, $S - \frac{1}{4}R^2 - \frac{1}{2}\frac{dR}{dx} = 3$, and $u = e^{-\frac{1}{2}\int R\,dx} = e^{\frac{1}{2}x^2}$.

The transformation $y = e^{\frac{1}{2}x^2}v$ reduces the equation to $\frac{d^2v}{dx^2} + 3v = e^x$ whose complete solution is

$$v = y/e^{\frac{1}{2}x^2} = C_1\cos\sqrt{3}\,x + C_2\sin\sqrt{3}\,x + \frac{1}{D^2+3}e^x = C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x + \tfrac{1}{4}e^x.$$

Thus, $y = e^{\frac{1}{2}x^2}(C_1\cos\sqrt{3}\,x + C_2\sin\sqrt{3}\,x) + \tfrac{1}{4}e^{\frac{1}{2}(x^2+2x)}$.

9. Solve $(D^2 - \frac{3}{x}D + \frac{3}{x^2})y = 2x - 1$. (Problem 2.)

Here $S - \frac{1}{4}R^2 - \frac{1}{2}\frac{dR}{dx} = \frac{3}{x^2} - \frac{9}{4x^2} - \frac{3}{2x^2} = -\frac{3}{4x^2}$ and $u = e^{-\frac{1}{2}\int R\,dx} = e^{-\frac{1}{2}\int -3\,dx/x} = x^{3/2}$.

The transformation $y = uv = x^{3/2}v$ reduces the equation to

$$\frac{d^2v}{dx^2} - \frac{3}{4x^2}v = \frac{2x-1}{x^{3/2}} \quad \text{or} \quad x^2\frac{d^2v}{dx^2} - \frac{3}{4}v = 2x^{3/2} - x^{1/2},$$ a Cauchy equation.

Putting $x = e^z$, we have $(D^2 - D - \frac{3}{4})v = 2e^{3z/2} - e^{z/2}$.

The complementary function is $v = C_1e^{-z/2} + C_2e^{3z/2}$, and a particular integral is

$$v = \frac{1}{D^2 - D - 3/4}(2e^{3z/2} - e^{z/2}) = \frac{1}{D - 3/2}e^{3z/2} + e^{z/2} = ze^{3z/2} + e^{z/2}.$$

The complete solution is $v = y/x^{3/2} = C_1x^{-1/2} + C_2x^{3/2} + x^{3/2}\ln x + x^{1/2}$

and $y = C_1x + C_2x^3 + x^3\ln x + x^2$.

10. Solve $\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 4x^2y = xe^{x^2}$.

Here $S - \frac{1}{2}R^2 - \frac{1}{2}\dfrac{dR}{dx} = 2$ and $u = e^{2\int x\,dx} = e^{x^2}$.

The transformation $y = ve^{x^2}$ reduces the equation to $\dfrac{d^2v}{dx^2} + 2v = x$ whose complete solution is

$$v = C_1\cos\sqrt{2}x + C_2\sin\sqrt{2}x + \tfrac{1}{2}x.$$

Then $y = ve^{x^2} = e^{x^2}(C_1\cos\sqrt{2}x + C_2\sin\sqrt{2}x) + \frac{1}{2}xe^{x^2}$.

11. Solve $\dfrac{d^2y}{dx^2} - (1 + 4e^x)\dfrac{dy}{dx} + 3e^{2x}y = e^{2(x + e^x)}$.

When $\dfrac{dz}{dx} = \sqrt{\dfrac{S}{a^2}} = \sqrt{\dfrac{3e^{2x}}{3}} = e^x$, $\dfrac{d^2z/dx^2 + R(dz/dx)}{(dz/dx)^2} = \dfrac{e^x - (1 + 4e^x)e^x}{(e^x)^2} = -4 = A$.

The introduction of $z = e^x$ as new independent variable leads to

$$\frac{d^2y}{dz^2} + A\frac{dy}{dz} + a^2y = \frac{Q}{(dz/dx)^2} \quad \text{or} \quad \frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 3y = \frac{e^{2(x + e^x)}}{e^{2x}} = e^{2e^x} = e^{2z}$$

whose complete solution is $y = C_1e^z + C_2e^{3z} + \dfrac{1}{D^2 - 4D + 3}e^{2z} = C_1e^z + C_2e^{3z} - e^{2z}$.

Replacing z by e^x, we have $y = C_1e^{e^x} + C_2e^{3e^x} - e^{2e^x}$.

Note. The choice of $a^2 = 3$ is one of convenience only. Taking $a^2 = 1$, $\dfrac{dz}{dx} = \sqrt{3}\,e^x$ and $A = \dfrac{-4}{\sqrt{3}}$.

The transformation $z = \sqrt{3}\,e^x$ yields $\dfrac{d^2y}{dz^2} - \dfrac{4}{\sqrt{3}}\dfrac{dy}{dz} + y = \dfrac{1}{3}e^{2z/\sqrt{3}}$ whose solution is

$y = C_1e^{z/\sqrt{3}} + C_2e^{\sqrt{3}z} - e^{2z/\sqrt{3}}$. Then $y = C_1e^{e^x} + C_2e^{3e^x} - e^{2e^x}$, as before.

12. Solve $\dfrac{d^2y}{dx^2} - \cot x\dfrac{dy}{dx} - \sin^2x\,y = \cos x - \cos^3x$.

Here $S = -\sin^2x$ and when $\dfrac{dz}{dx} = \sqrt{\dfrac{-S}{1}} = \sin x$, $\dfrac{d^2z/dx^2 + R(dz/dx)}{(dz/dx)^2} = \dfrac{\cos x + (-\cot x)(\sin x)}{\sin^2x} = 0$.

Thus the introduction of $z = -\cos x$ as new independent variable leads to

$\frac{d^2y}{dz^2} - y = \cos x = -z$ whose complete solution is $y = C_1e^z + C_2e^{-z} + z$.

Upon replacing z by $-\cos x$, we have $y = C_1e^{-\cos x} + C_2e^{\cos x} - \cos x$.

13. Solve $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{1}{x^4}y = \frac{2x^2+1}{x^6}$.

When $\frac{dz}{dx} = \sqrt{S} = \sqrt{\frac{1}{x^4}} = \frac{1}{x^2}$, $\frac{d^2z/dx^2 + R(dz/dx)}{(dz/dx)^2} = 0$. Thus the introduction of $z = -\frac{1}{x}$

as new independent variable leads to $\frac{d^2y}{dz^2} + y = 2 + z^2$ whose complete solution is

$y = C_1\cos z + K\sin z + z^2$.

Upon replacing z by $-1/x$, we have $y = C_1\cos(-1/x) + K\sin(-1/x) + 1/x^2$

$= C_1\cos(1/x) + C_2\sin(1/x) + 1/x^2$.

14. Solve $\frac{d^2y}{dx^2} + (4x - \frac{1}{x})\frac{dy}{dx} + 4x^2y = 3xe^{-x^2}$.

When $\frac{dz}{dx} = \sqrt{S} = \sqrt{4x^2} = 2x$, $\frac{d^2z/dx^2 + R(dz/dx)}{(dz/dx)^2} = \frac{2 + (4x - 1/x)2x}{(2x)^2} = 2$. Thus the introduction

of $z = x^2$ as new independent variable leads to $\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + y = \frac{3e^{-z}}{4\sqrt{z}}$ whose complete solution is

$y = C_1e^{-z} + C_2ze^{-z} + \frac{3/4}{(D+1)^2}e^{-z}z^{-1/2} = C_1e^{-z} + C_2ze^{-z} + z^{3/2}e^{-z}$.

Upon replacing z by x^2, we have $y = C_1e^{-x^2} + C_2x^2e^{-x^2} + x^3e^{-x^2}$.

15. Solve $(D^2 - \frac{3}{x}D + \frac{3}{x^2})y = 2x - 1$. (Problem 2.)

a. The equation is equivalent to $D^2y - D(\frac{3}{x}y) = D(D - \frac{3}{x})y = 2x - 1$.

Putting $(D - \frac{3}{x})y = v$, we have $Dv = 2x - 1$ and $v = x^2 - x + K$.

Now $(D - \frac{3}{x})y = x^2 - x + K$ for which $\frac{1}{x^3}$ is an integrating factor. Then

$\frac{y}{x^3} = \int(\frac{1}{x} - \frac{1}{x^2} + \frac{K}{x^3})dx = \ln x + \frac{1}{x} + \frac{C_1}{x^2} + C_2$ and $y = C_1x + C_2x^3 + x^2(1 + x\ln x)$.

b. The equation $(xD^2 - 3D + \frac{3}{x})y = 2x^2 - x$ is equivalent to $(D - \frac{3}{x})(xD - 1)y = 2x^2 - x$.

Putting $(xD - 1)y = v$, we have $(D - \frac{3}{x})v = 2x^2 - x$ for which $\frac{1}{x^3}$ is an integrating factor.

Then $\frac{v}{x^3} = \int(\frac{2}{x} - \frac{1}{x^2})dx = 2\ln x + \frac{1}{x} + K$ and

$$(xD-1)y = v = 2x^3 \ln x + x^2 + Kx^3 \quad \text{or} \quad (D-1/x)y = 2x^2 \ln x + x + Kx^2.$$

Here $1/x$ is an integrating factor so that

$$y/x = \int(2x \ln x + 1 + Kx)dx = x^2 \ln x - \tfrac{1}{2}x^2 + x + K_1x^2 + C_2 = x^2 \ln x + x + C_1x^2 + C_2$$

and $$y = C_1x^3 + C_2x + x^2(1 + x \ln x).$$

16. Solve $[xD^2 + (1-x)D - 2(1+x)]y = e^{-x}(1-6x)$.

The equation is equivalent to $[xD + (1+x)][D - 2]y = e^{-x}(1-6x)$.

Putting $(D-2)y = v$, we have $[xD + 1 + x]v = e^{-x}(1-6x)$ or $(D + \frac{1}{x} + 1)v = e^{-x}(\frac{1}{x} - 6)$.

Now xe^x is an integrating factor so that $vxe^x = \int(1-6x)dx = x - 3x^2 + K$

and $(D-2)y = v = (1-3x)e^{-x} + Ke^{-x}/x$.

Here e^{-2x} is an integrating factor so that

$$ye^{-2x} = \int[(1-3x)e^{-3x} + Ke^{-3x}/x]dx = xe^{-3x} + C_1\int\frac{e^{-3x}}{x}\,dx + C_2$$

and $$y = xe^{-x} + C_1e^{2x}\int\frac{e^{-3x}}{x}\,dx + C_2e^{2x}.$$

17. Solve $[(x+3)D^2 - (2x+7)D + 2]y = (x+3)^2e^x$.

The equation may be written as $[(x+3)D - 1][D - 2]y = (x+3)^2e^x$.

Putting $(D-2)y = v$, we have $[(x+3)D - 1]v = (x+3)^2e^x$ or $(D - \frac{1}{x+3})v = (x+3)e^x$.

Using the integrating factor $1/(x+3)$, we have $v/(x+3) = \int e^x\,dx = e^x + K$

so that $(D-2)y = v = (x+3)e^x + K(x+3)$.

Using the integrating factor e^{-2x}, we have

$$ye^{-2x} = \int[(x+3)e^{-x} + K(x+3)e^{-2x}]dx = -xe^{-x} - 4e^{-x} + K(-\tfrac{1}{2}xe^{-2x} - \tfrac{7}{4}e^{-2x}) + C_2$$

and $$y = -xe^x - 4e^x + C_1(2x+7) + C_2e^{2x}.$$

18. Show that the Riccati equation $\frac{dy}{dx} + yP(x) + y^2Q(x) = R(x)$, $Q(x) \neq 0$, is reduced to a linear equation of the second order by the substitution $y = \frac{1}{Qu}\frac{du}{dx}$.

Since $\frac{dy}{dx} = \frac{1}{Qu}\frac{d^2u}{dx^2} - \frac{1}{Qu^2}(\frac{du}{dx})^2 - \frac{1}{Q^2u}\frac{dQ}{dx}\frac{du}{dx}$, the substitution yields

$$\frac{1}{Qu}\frac{d^2u}{dx^2} - \frac{1}{Qu^2}(\frac{du}{dx})^2 - \frac{1}{Q^2u}\frac{dQ}{dx}\frac{du}{dx} + \frac{P}{Qu}\frac{du}{dx} + \frac{1}{Qu^2}(\frac{du}{dx})^2 - R = 0 \quad \text{or} \quad \frac{d^2u}{dx^2} + (P - \frac{1}{Q}\frac{dQ}{dx})\frac{du}{dx} - RQu = 0.$$

19. Use the procedure outlined in Problem 18 to solve $\frac{dy}{dx} + \frac{2}{x}y + \frac{1}{2}x^3y^2 = \frac{1}{2x}$.

The substitution $y = \frac{1}{Qu}\frac{du}{dx} = \frac{2}{x^3u}\frac{du}{dx}$ reduces the equation to

$$\frac{d^2u}{dx^2} + \left(\frac{2}{x} - \frac{3x^2/2}{x^3/2}\right)\frac{du}{dx} - \frac{1}{2x}\cdot\frac{x^3}{2}u = 0 \qquad \text{or} \qquad \frac{d^2u}{dx^2} - \frac{1}{x}\frac{du}{dx} - \frac{1}{4}x^2u = 0.$$

In turn, the substitution $\frac{dz}{dx} = \sqrt{\frac{-S}{\frac{1}{4}}} = \sqrt{\frac{\frac{1}{4}x^2}{\frac{1}{4}}} = x$ reduces this equation to $\frac{d^2u}{dz^2} - \frac{1}{4}u = 0$ whose solution is $u = C_1e^{\frac{1}{2}z} + C_2e^{-\frac{1}{2}z}$.

Then $y = \frac{1}{Qu}\frac{du}{dx} = \frac{2}{x^3}\cdot\frac{\frac{1}{2}(C_1e^{\frac{1}{2}z} - C_2e^{-\frac{1}{2}z})}{C_1e^{\frac{1}{2}z} + C_2e^{-\frac{1}{2}z}}x = \frac{1}{x^2}\cdot\frac{e^{\frac{1}{4}x^2} - ke^{-\frac{1}{4}x^2}}{e^{\frac{1}{4}x^2} + ke^{-\frac{1}{4}x^2}}$, where $k = \frac{C_2}{C_1}$.

20. Solve $\frac{dy}{dx} - (\tan x + 3\cos x)y + y^2\cos^2 x = -2$.

The substitution $y = \frac{1}{Qu}\frac{du}{dx} = \frac{\sec^2 x}{u}\frac{du}{dx}$ reduces the equation to

$$\frac{d^2u}{dx^2} + (\tan x - 3\cos x)\frac{du}{dx} + 2u\cos^2 x = 0.$$

In turn, the substitution $\frac{dz}{dx} = \sqrt{\frac{2\cos^2 x}{2}} = \cos x$, or $z = \sin x$, reduces this equation to $\frac{d^2u}{dz^2} - 3\frac{du}{dz} + 2u = 0$ whose solution is $u = C_1e^z + C_2e^{2z}$.

Then $y = \frac{1}{Qu}\frac{du}{dx} = \frac{\sec^2 x(C_1e^z + 2C_2e^{2z})}{C_1e^z + C_2e^{2z}}\cos x = \sec x\,\frac{e^{\sin x} + 2ke^{2\sin x}}{e^{\sin x} + ke^{2\sin x}}$.

SUPPLEMENTARY PROBLEMS

Solve.

		Ans.
21.	$xy'' - (x+2)y' + 2y = 0$	$y = C_1e^x + C_2(x^2 + 2x + 2)$
22.	$(1+x^2)y'' - 2xy' + 2y = 2$	$y = C_1x + C_2(x^2 - 1) + x^2$
23.	$(x^2+4)y'' - 2xy' + 2y = 8$	$y = C_1(x^2 - 4) + C_2x + x^2$
24.	$(x+1)y'' - (2x+3)y' + (x+2)y = (x^2 + 2x + 1)e^{2x}$	$y = C_1e^x + C_2e^x(x+1)^2 + xe^{2x}$
25.	$y'' - 2\tan x\, y' - 10y = 0$	$y = (C_1e^{3x} + C_2e^{-3x})\sec x$
26.	$x^2y'' - x(2x+3)y' + (x^2 + 3x + 3)y = (6 - x^2)e^x$	$y = C_1x^3e^x + C_2xe^x + e^x(x^2 + 2)$
27.	$4x^2y'' + 4x^3y' + (x^2+1)^2y = 0$	$y = \sqrt{x}\, e^{-x^2/4}(C_1 + C_2\ln x)$
28.	$x^2y'' + (x - 4x^2)y' + (1 - 2x + 4x^2)y = (x^2 - x + 1)e^x$	$y = e^{2x}(C_1\cos\ln x + C_2\sin\ln x) + e^x$
29.	$xy'' - y' + 4x^3y = 0$	$y = C_1\sin x^2 + C_2\cos x^2$
30.	$x^4y'' + 2x^3y' + y = (1+x)/x$	$y = C_1\cos(1/x) + C_2\sin(1/x) + (1+x)/x$

31. $x^8y'' + 4x^7y' + y = 1/x^5$ *Ans.* $y = C_1\cos(1/3x^3) + C_2\sin(1/3x^3) + 1/x^5$

32. $(x\sin x + \cos x)y'' - x\cos x\, y' + y\cos x = x$ $y = C_1x + C_2\cos x - \sin x$

33. $xy'' - 3y' + 3y/x = x + 2$ $y = C_1x + C_2x^3 - x^2 - x\ln x$

34. Solve Problem 21 by factoring.

35. $[(x+1)D^2 - (3x+4)D + 3]y = (3x+2)e^{3x}$ $y = C_1(3x+4) + C_2e^{3x} + xe^{3x}$

36. $x^2y'' - 4xy' + (6+9x^2)y = 0$ $y = x^2(C_1\cos 3x + C_2\sin 3x)$

37. $xy'' + 2y' + 4xy = 4$ $y = (C_1\cos 2x + C_2\sin 2x + 1)/x$

38. $(1+x^2)y'' - 2xy' + 2y = (1-x^2)/x$ $y = C_1(x^2-1) + C_2x + x\ln x$

CHAPTER 19

Linear Equations with Variable Coefficients

MISCELLANEOUS TYPES

IN THIS CHAPTER various types of differential equations of order higher than the first and with variable coefficients will be considered. There is no general procedure comparable to that for linear equations. However, for the types treated here, the procedure consists in obtaining from the given equation another of lower order. For example, if the given equation is of order three and if, by some means, an equation of order two, which is solvable by one of the methods of the previous chapters can be obtained from it, the given equation can be solved.

DEPENDENT VARIABLE ABSENT. If the equation is free of y, that is, is of the form

1) $$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \ldots\ldots, \frac{dy}{dx}, x\right) = 0,$$

the substitution $\frac{dy}{dx} = p$, $\frac{d^2 y}{dx^2} = \frac{dp}{dx}$, ... will reduce the order by one.

EXAMPLE. The equation $x^2 \frac{d^3 y}{dx^3} + 2 \frac{d^2 y}{dx^2} \frac{dy}{dx} - 3x\left(\frac{dy}{dx}\right)^2 + x^5 = 0$, of order three, is reduced to $x^2 \frac{d^2 p}{dx^2} + 2p \frac{dp}{dx} - 3xp^2 + x^5 = 0$, of order two, by the substitution $\frac{dy}{dx} = p$, $\frac{d^2 y}{dx^2} = \frac{dp}{dx}$, $\frac{d^3 y}{dx^3} = \frac{d^2 p}{dx^2}$.

If the given equation is of the form

2) $$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \ldots\ldots, \frac{d^k y}{dx^k}, x\right) = 0,$$

the substitution $\frac{d^k y}{dx^k} = q$, $\frac{d^{k+1} y}{dx^{k+1}} = \frac{dq}{dx}$, ... will reduce the order by k.

See Problems 1–5.

INDEPENDENT VARIABLE ABSENT. If the equation is free of x, that is, is of the form

3) $$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \ldots\ldots, \frac{dy}{dx}, y\right) = 0,$$

the substitution $\frac{dy}{dx} = p$, $\frac{d^2 y}{dx^2} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$,

$$\frac{d^3 y}{dx^3} = \frac{d}{dy}\left(p \frac{dp}{dy}\right)\frac{dy}{dx} = \left\{p \frac{d^2 p}{dy^2} + \left(\frac{dp}{dy}\right)^2\right\}\frac{dy}{dx} = p^2 \frac{d^2 p}{dy^2} + p\left(\frac{dp}{dy}\right)^2, \text{ etc.}$$

will reduce the order of the differential equation by one.

EXAMPLE. The substitution $\frac{dy}{dx} = p$, $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$, $\frac{d^3y}{dx^3} = p^2\frac{d^2p}{dy^2} + p(\frac{dp}{dy})^2$ reduces the equation $yy''' - y''(y')^2 = 1$, of order three, to $yp^2\frac{d^2p}{dy^2} + py(\frac{dp}{dy})^2 - p^3\frac{dp}{dy} = 1$, of order two.

See Problems 6–10.

LINEAR EQUATIONS WITH KNOWN PARTICULAR INTEGRAL. If a particular integral $y = u(x)$ of the equation

4) $$(P_0D^n + P_1D^{n-1} + \cdots\cdots + P_{n-1}D + P_n)y = 0$$

is known, then the substitution $y = uv$ will transform

5) $$(P_0D^n + P_1D^{n-1} + \cdots\cdots + P_{n-1}D + P_n)y = Q(x)$$

into an equation of the same order but with the dependent variable absent. In turn, the order of this equation may be reduced by the procedure of the first section of this chapter. Equation 4) is called the reduced equation of 5).

EXAMPLE. Since $y = x$ is a solution of $(D^2 - xD + 1)y = 0$, the substitution $y = vx$, $\frac{dy}{dx} = x\frac{dv}{dx} + v$, $\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$ reduces $(D^2 - xD + 1)y = e^{2x}$ to $\frac{d^2v}{dx^2} + \frac{2-x^2}{x}\frac{dv}{dx} = \frac{e^{2x}}{x}$.

Here, the dependent variable v is missing and the procedure of the first section above applies.

See Problems 11-14.

EXACT EQUATIONS. The differential equation

6) $$f(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \cdots\cdots, \frac{dy}{dx}, y, x) = Q(x)$$

is called an exact equation if it can be obtained by differentiating once an equation

7) $$g(\frac{d^{n-1}y}{dx^{n-1}}, \frac{d^{n-2}y}{dx^{n-2}}, \cdots\cdots, \frac{dy}{dx}, y, x) = Q_1(x) + C$$

of one lower order. For example, the equation

$$3y^2y''' + 14yy'y'' + 4(y')^3 + 12y'y'' = 2x$$

is an exact equation since it may be obtained by differentiating once the equation

$$3y^2y'' + 4y(y')^2 + 6(y')^2 = x^2 + C.$$

The linear equation 4) is exact provided

$P_n - P_{n-1}' + P_{n-2}'' + \cdots\cdots + (-1)^nP_0^{(n)} = 0$, identically.

EXAMPLE. Consider the equation $(x^3 - 2x)y''' + (8x^2 - 5)y'' + 15xy' + 5y = 0$ in which $P_3 = 5$, $P_2 = 15x$ and $P_2' = 15$, $P_1 = 8x^2 - 5$ and $P_1'' = 16$, and $P_0 = x^3 - 2x$ and $P_0''' = 6$. The equation is exact since $P_3 - P_2' + P_1'' - P_0''' = 5 - 15 + 16 - 6 = 0$. The given equation is the exact derivative of $(x^3 - 2x)y'' + (5x^2 - 3)y' + 5xy = C$.

If equation 6) is not linear no simple test for exactness can be stated. In this case, we show that 6) is exact by producing the equation of one lower order from which it may be obtained by a differentiation.

If 6) is not exact, it may be possible to find an integrating factor. Again, no general rule can be stated for determining an integrating factor.

See Problems 15–21.

SOLVED PROBLEMS

DEPENDENT VARIABLE ABSENT.

1. Solve $2\frac{d^2y}{dx^2} - (\frac{dy}{dx})^2 + 4 = 0$.

The substitution $\frac{dy}{dx} = p$ reduces the equation to $2\frac{dp}{dx} = p^2 - 4$ or $\frac{2\,dp}{p^2-4} = dx$.

Integrating, $\frac{1}{2}\ln\frac{p-2}{p+2} = x + \ln K$; $\frac{p-2}{p+2} = C_1e^{2x}$, $p = \frac{2(1+C_1e^{2x})}{1-C_1e^{2x}} = 2(1 + \frac{2C_1e^{2x}}{1-C_1e^{2x}})$,

and $y = 2x - 2\ln(1-C_1e^{2x}) + C_2$.

2. Solve $x\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 0$.

The substitution $\frac{d^2y}{dx^2} = q$ reduces the equation to $x\frac{dq}{dx} - 2q = 0$.

Then $\ln q = \ln x^2 + \ln K$, $q = \frac{d^2y}{dx^2} = Kx^2$, and $y = C_1x^4 + C_2x + C_3$.

3. Solve $\frac{d^4y}{dx^4}\cdot\frac{d^3y}{dx^3} = 1$.

The substitution $\frac{d^3y}{dx^3} = q$ reduces the equation to $q\frac{dq}{dx} = 1$ and $q^2 = 2x + C_1$.

Then $q = \frac{d^3y}{dx^3} = \pm(2x+C_1)^{1/2}$, $\frac{d^2y}{dx^2} = \pm\frac{1}{3}(2x+C_1)^{3/2} + K$, $\frac{dy}{dx} = \pm\frac{1}{15}(2x+C_1)^{5/2} + Kx + K_3$.

$y = \pm\frac{1}{105}(2x+C_1)^{7/2} + K_2x^2 + K_3x + K_4$ or $105\,y = \pm(2x+C_1)^{7/2} + C_2x^2 + C_3x + C_4$.

4. Solve $(\frac{d^3y}{dx^3})^2 + x\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 0$.

The substitution $\frac{d^2y}{dx^2} = q$ reduces the equation to $(\frac{dq}{dx})^2 + x\frac{dq}{dx} - q = 0$ or $q = x\frac{dq}{dx} + (\frac{dq}{dx})^2$,

a Clairaut equation.

Then $q = \frac{d^2y}{dx^2} = Kx + K^2$, $\frac{dy}{dx} = \frac{1}{2}Kx^2 + K^2x + C_2 = Cx^2 + 4C^2x + C_2$, and

$y = \frac{1}{3}Cx^3 + 2C^2x^2 + C_2x + C_3 = C_1x^3 + 18C_1^2x^2 + C_2x + C_3$.

5. Solve $(1+2x)\frac{d^3y}{dx^3} + 4x\frac{d^2y}{dx^2} - (1-2x)\frac{dy}{dx} = e^{-x}$.

The transformation $p = \frac{dy}{dx}$ reduces the equation to

$$(1+2x)p'' + 4xp' - (1-2x)p = e^{-x} \quad \text{or} \quad p'' + \frac{4x}{1+2x}p' - \frac{1-2x}{1+2x}p = \frac{e^{-x}}{1+2x}.$$

Since $1-R+S=0$, we use the substitution

$$p = e^{-x}v, \quad p' = e^{-x}(v'-v), \quad p'' = e^{-x}(v''-2v'+v)$$

to obtain $(1+2x)v''-2v' = 1$ or $(1+2x)^2v''-2(1+2x)v' = (1+2x)$, a Legendre linear equation.

The substitution $1+2x = e^t$ reduces the equation to $[4\mathcal{D}(\mathcal{D}-1)-4\mathcal{D}]v = e^t$ or $\mathcal{D}(\mathcal{D}-2)v = \frac{1}{4}e^t$.

Then $v = K_1 + K_2e^{2t} - \frac{1}{4}e^t = K_1 + K_2(1+2x)^2 - \frac{1}{4}(1+2x)$,

$$p = \frac{dy}{dx} = e^{-x}v = K_1e^{-x} + K_2(1+2x)^2e^{-x} - \frac{1}{4}(1+2x)e^{-x},$$

and $y = C_1e^{-x} + C_2(4x^2+12x+13)e^{-x} + C_3 + \frac{1}{4}(2x+3)e^{-x}$

or $y = Ae^{-x} + B(x^2+3x)e^{-x} + C + \frac{1}{2}xe^{-x}$.

INDEPENDENT VARIABLE ABSENT.

6. Solve $y'' = (y')^3 + y'$.

The substitution $y'=p$, $y''=p\frac{dp}{dy}$ reduces the equation to $p\frac{dp}{dy} = p^3+p$ or $\frac{dp}{dy} = p^2+1$.

Then $\frac{dp}{p^2+1} = dy$, $\arctan p = y + K_1$, and $p = \frac{dy}{dx} = \tan(y+K_1)$.

Now $\cot(y+K_1)dy = dx$, $\ln\sin(y+K_1) = x+K_2$, $\sin(y+K_1) = C_2e^x$, and $y = \arcsin C_2e^x +$

7. Solve $yy'' = 2(y')^2 - 2y'$.

The substitution $y'=p$, $y''=p\frac{dp}{dy}$ reduces the equation to $p(y\frac{dp}{dy} - 2p + 2) = 0$.

Here $p=0$ and $y=C$ is a solution, or

$$\frac{dp}{p-1} = 2\frac{dy}{y}, \quad \ln(p-1) = \ln A^2y^2, \quad p = A^2y^2+1, \quad \text{or} \quad \frac{dy}{1+A^2y^2} = dx.$$

Then $\frac{1}{A}\arctan Ay = x + K$, $\arctan Ay = Ax + B$, and $Ay = \tan(Ax+B)$.

8. Solve $yy'' - (y')^2 = y^2\ln y$.

The substitution $y'=p$, $y''=p\frac{dp}{dy}$ reduces the equation to

$$yp\frac{dp}{dy} - p^2 = y^2\ln y \quad \text{or} \quad \frac{2y^2p\,dp - 2p^2y\,dy}{y^4} = 2\ln y\frac{dy}{y}.$$

Then $\frac{p^2}{y^2} = \ln^2 y + C$, $\frac{dy}{y\sqrt{\ln^2 y + C}} = \pm dx$, and $\ln(\ln y + \sqrt{\ln^2 y + C}) = \pm x + \ln K$.

Now $\ln y + \sqrt{\ln^2 y + C} = Ke^{\pm x}$, $\sqrt{\ln^2 y + C} = Ke^{\pm x} - \ln y$, and $C = K^2 e^{\pm 2x} - 2Ke^{\pm x} \ln y$.

This may be written as $\ln y = C_1 e^{\pm x} + C_2 e^{\mp x}$ or, finally, $\ln y = C_1 e^{x} + C_2 e^{-x}$ since C_1 and C_2 are arbitrary constants.

9. Solve $yy'' + (y')^2 = y^2$.

The substitution $y' = p$, $y'' = p\dfrac{dp}{dy}$ reduces the equation to $py\dfrac{dp}{dy} + p^2 = y^2$ for which y is an integrating factor. The solution of $py^2\,dp + p^2 y\,dy = y^3\,dy$ is $2p^2y^2 = y^4 + C^2$.

Now $\sqrt{2}\,p = \sqrt{2}\dfrac{dy}{dx} = \pm\dfrac{\sqrt{y^4 + C^2}}{y}$ whose solution is $\sqrt{2}\sinh^{-1}\dfrac{y^2}{C} = \pm 2x + K\sqrt{2}$. Then $\sinh^{-1}\dfrac{y^2}{C} = \pm\sqrt{2}\,x + K$, $\dfrac{y^2}{C} = \sinh(\pm\sqrt{2}\,x + K) = \pm\sinh(\sqrt{2}\,x + K_1)$, and $y^2 = C_1\sinh(\sqrt{2}\,x + C_2)$.

10. Solve $\dfrac{d^2y}{dx^2} = e^{2y}$ given that $y = y' = 0$ when $x = 0$.

Putting $y'' = p\dfrac{dp}{dy}$, we have $2p\,dp = 2e^{2y}dy$ whose solution is $p^2 = e^{2y} + K$.

Using the initial conditions, $0 = 1 + K$ and $K = -1$. Now $p = \dfrac{dy}{dx} = \pm\sqrt{e^{2y} - 1}$ which, by the substitution $e^{2y} = z$, becomes $\dfrac{dz}{2z\sqrt{z-1}} = \pm\,dx$. The solution of this equation is $\arctan\sqrt{z-1} = \pm x + C$ or, in the original variables, $\arctan\sqrt{e^{2y} - 1} = \pm x + C$. Here, the initial conditions require $C = 0$ so that $\sqrt{e^{2y} - 1} = \tan(\pm x) = \pm\tan x$ and, finally, $e^{2y} = \sec^2 x$.

It should be noted that the form of the solution of the given equation depends on the sign of the first constant of integration. If in $p^2 = e^{2y} + K$, K is positive and $= A^2$, we solve

$\dfrac{dz}{2z\sqrt{z + A^2}} = \pm\,dx$, and obtain $\dfrac{1}{2A}\ln\dfrac{\sqrt{z + A^2} - A}{\sqrt{z + A^2} + A} = \pm x + C$. Then $\dfrac{\sqrt{z + A^2} - A}{\sqrt{z + A^2} + A} = Be^{\pm 2Ax}$ and

$\dfrac{A(1 + Be^{\pm 2Ax})}{1 - Be^{\pm 2Ax}} = \sqrt{z + A^2}$. Since A is arbitrary, we may write $z + A^2 = \dfrac{A^2(1 + Be^{2Ax})^2}{(1 - Be^{2Ax})^2}$ and

obtain $z = e^{2y} = \dfrac{4A^2Be^{2Ax}}{(1 - Be^{2Ax})^2}$ or $e^{y} = \dfrac{2ACe^{Ax}}{1 - C^2e^{2Ax}}$.

LINEAR EQUATIONS WITH KNOWN PARTICULAR INTEGRAL.

11. Solve $x^3(\sin x)y''' - (3x^2\sin x + x^3\cos x)y'' + (6x\sin x + 2x^2\cos x)y' - (6\sin x + 2x\cos x)y = 0$.

By inspection it is seen that $y = x$ is a particular integral.

By means of the substitution $y = xv$, $y' = xv' + v$, $y'' = xv'' + 2v'$, $y''' = xv''' + 3v''$, the equation is reduced to $\sin x\dfrac{d^3v}{dx^3} - \cos x\dfrac{d^2v}{dx^2} = 0$. In turn, the substitution $\dfrac{d^2v}{dx^2} = q$

reduces this equation to $\sin x \frac{dq}{dx} - q \cos x = 0$ or $\frac{dq}{q} = \cot x\, dx$.

Then $\ln q = \ln \sin x + \ln C$, $q = \frac{d^2v}{dx^2} = C \sin x$, and $v = \frac{y}{x} = C_1 \sin x + C_2 x + C_3$.

Thus, the solution is $y = C_1 x \sin x + C_2 x^2 + C_3 x$.

12. Solve $(x^3 - 3x^2 + 6x - 6)y^{iv} - x^3 y''' + 3x^2 y'' - 6xy' + 6y = 0$.

By inspection it is seen that $y = x$ is a particular integral.

The substitution $y = xv$, $y' = xv' + v$, $y'' = xv'' + 2v'$, $y''' = xv''' + 3v''$, $y^{iv} = xv^{iv} + 4v'''$ reduces the equation to $(x^4 - 3x^3 + 6x^2 - 6x)v^{iv} + (-x^4 + 4x^3 - 12x^2 + 24x - 24)v''' = 0$.

Putting $\frac{d^3v}{dx^3} = q$, this equation becomes

$$x(x^3 - 3x^3 + 6x - 6)\frac{dq}{dx} + (-x^4 + 4x^3 - 12x^2 + 24x - 24)q = 0 \quad \text{or} \quad \frac{dq}{q} + (-1 + \frac{4}{x} - \frac{3x^2 - 6x + 6}{x^3 - 3x^2 + 6x - 6})dx = 0.$$

Integrating, $\ln q = x - 4 \ln x + \ln(x^3 - 3x^2 + 6x - 6) + \ln A$ or $q = \frac{d^3v}{dx^3} = A \frac{x^3 - 3x^2 + 6x - 6}{x^4} e^x$.

Then $\frac{d^2v}{dx^2} = A \int \frac{x^3 - 3x^2 + 6x - 6}{x^4} e^x\, dx = A \cdot \frac{1}{D}(\frac{x^3 - 3x^2 + 6x - 6}{x^4} e^x)$

$$= Ae^x \frac{1}{D+1}(\frac{x^3 - 3x^2 + 6x - 6}{x^4}) = Ae^x \frac{1}{D+1}(\frac{1}{x} - \frac{3}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}).$$

Now $D(\frac{1}{x}) = -\frac{1}{x^2}$, $D^2(\frac{1}{x}) = \frac{2}{x^3}$, and $D^3(\frac{1}{x}) = -\frac{6}{x^4}$,

so that $\frac{1}{D+1}(\frac{1}{x} - \frac{3}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}) = \frac{1}{D+1}[\frac{1}{x} + 3D(\frac{1}{x}) + 3D^2(\frac{1}{x}) + D^3(\frac{1}{x})] = \frac{1}{D+1}(D+1)^3(\frac{1}{x})$

$$= (D^2 + 2D + 1)(\frac{1}{x}) = \frac{x^2 - 2x + 2}{x^3}.$$

Thus, $\frac{d^2v}{dx^2} = A \frac{x^2 - 2x + 2}{x^3} e^x + B$, $\frac{dv}{dx} = A \frac{(x-1)e^x}{x^2} + Bx + C$,

$$v = \frac{y}{x} = C_1 \frac{e^x}{x} + C_2 x^2 + C_3 x + C_4, \quad \text{and} \quad y = C_1 e^x + C_2 x^3 + C_3 x^2 + C_4 x.$$

In this example, it is fairly easy to see that $y = x$, $y = x^2$, $y = x^3$, and $y = e^x$ are particular integrals. Thus the complete solution could have been written down immediately.

13. Solve $(2 \sin x - x \sin x - x \cos x)y''' + (2x \cos x - \sin x - \cos x)y'' + x(\sin x - \cos x)y'$

$$+ (\cos x - \sin x)y = 2 \sin x - x \cos x - x \sin x.$$

By inspection it is seen that $y = x$, $y = e^x$ and $y = \sin x$ are particular integrals of the reduced equation. We shall obtain a particular integral of the given equation using the *method of variation of parameters.*

We take $y = L_1 x + L_2 e^x + L_3 \sin x$.

Then $y' = L_1 + L_2 e^x + L_3 \cos x + (L_1' x + L_2' e^x + L_3' \sin x)$

and we set

$A)$ $\qquad L_1' x + L_2' e^x + L_3' \sin x = 0.$

Now $y'' = L_2 e^x - L_3 \sin x + (L_1' + L_2' e^x + L_3' \cos x)$

and we set

$B)$ $\qquad L_1' + L_2' e^x + L_3' \cos x = 0.$

Then $y''' = L_2 e^x - L_3 \cos x + (L_2' e^x - L_3' \sin x)$

and we set

$C)$ $\qquad L_2' e^x - L_3' \sin x = 2 \sin x - x \cos x - x \sin x.$

Solving $A)$, $B)$, $C)$ simultaneously, we obtain

$L_1' = -\sin x + \cos x$ and $L_1 = \cos x + \sin x$,

$L_2' = -e^{-x}(x \cos x - \sin x)$ and $L_2 = \frac{1}{2} x e^{-x}(-\sin x + \cos x) - e^{-x} \sin x - \frac{1}{2} e^{-x} \cos x$,

$L_3' = -1 + x$ and $L_3 = -x + \frac{1}{2} x^2$.

Thus, the complete solution is

$$y = C_1 x + C_2 e^x + C_3 \sin x + \frac{1}{2} x^2 \sin x + \frac{3}{2} x \cos x - \frac{1}{2} x \sin x - \frac{1}{2} \cos x.$$

14. Solve $(x^2 + x)y''' - (x^2 + 3x + 1)y'' + (x + 4 + \frac{2}{x})y' - (1 + \frac{4}{x} + \frac{2}{x^2})y = 3x^2(x + 1)^2$.

By inspection it is seen that $y = x$ is a particular integral of the reduced equation. The substitution $y = xv$ reduces the given equation to

$$(x^2 + x)v''' - (x^2 - 2)v'' - (x + 2)v' = 3x(x + 1)^2$$

and, in turn, the substitution $v' = u$ reduces this to

$A)$ $\qquad (x^2 + x)u'' - (x^2 - 2)u' - (x + 2)u = 3x(x + 1)^2.$

Since the sum of the coefficients of the reduced equation of $A)$ is identically zero, $u = e^x$ is a particular integral and we use the substitution

$$u = e^x w, \quad u' = e^x w' + e^x w, \quad u'' = e^x w'' + 2e^x w' + e^x w$$

to reduce $A)$ to

$$(x^2 + x)w'' + (x^2 + 2x + 2)w' = 3xe^{-x}(x + 1)^2.$$

Using the substitution $w' = z$, this becomes

$$(x^2 + x)z' + (x^2 + 2x + 2)z = 3xe^{-x}(x + 1)^2$$

or $\frac{dz}{dx} + (1 + \frac{2}{x} - \frac{1}{x + 1})z = 3e^{-x}(x + 1)$ for which $\frac{x^2 e^x}{x + 1}$ is an integrating factor.

Then $z \frac{x^2 e^x}{x + 1} = \int 3x^2 dx = x^3 + K_1$, $\qquad \frac{dw}{dx} = z = x(x + 1)e^{-x} + K_1 \frac{x + 1}{x^2} e^{-x}$,

$\frac{u}{e^x} = w = -x^2 e^{-x} - 3xe^{-x} - 3e^{-x} + C_1 \frac{e^{-x}}{x} + C_2$, $\qquad \frac{dv}{dx} = u = -x^2 - 3x - 3 + \frac{C_1}{x} + C_2 e^x$,

and $\qquad y = xv = -\frac{x^4}{3} - \frac{3}{2} x^3 - 3x^2 + C_1 x \ln x + C_2 x e^x + C_3 x.$

EXACT EQUATIONS.

15. Show that $P_0(x)\,y^{IV} + P_1(x)\,y''' + P_2(x)\,y'' + P_3(x)\,y' + P_4(x)\,y = 0$ is exact if and only if $P_4 - P_3' + P_2'' - P_1''' + P_0^{IV} = 0$.

Let the given differential equation be obtained by differentiating

$$R_0(x)\,y''' + R_1(x)\,y'' + R_2(x)\,y' + R_3(x)\,y = C_1.$$

Since this differentiation yields $R_0 y^{IV} + (R_0' + R_1)y''' + (R_1' + R_2)y'' + (R_2' + R_3)y' + R_3' y = 0$,

we have $P_0 = R_0$, $P_1 = R_0' + R_1$, $P_2 = R_1' + R_2$, $P_3 = R_2' + R_3$, and $P_4 = R_3'$.

Now $P_4 - P_3' + P_2'' - P_1''' + P_0^{IV} = R_3' - (R_2'' + R_3') + (R_1''' + R_2'') - (R_0^{IV} + R_1''') + R_0^{IV} = 0$.

Conversely, suppose $P_4 - P_3' + P_2'' - P_1''' + P_0^{IV} = 0$. Since

$$\frac{d}{dx}[P_0 y''' + (P_1 - P_0')y'' + (P_2 - P_1' + P_0'')y' + (P_3 - P_2' + P_1'' - P_0''')y]$$

$$= P_0 y^{IV} + P_1 y''' + P_2 y'' + P_3 y' - (-P_3' + P_2'' - P_1''' + P_0^{IV})y$$

$$= P_0 y^{IV} + P_1 y''' + P_2 y'' + P_3 y' + P_4 y,$$ the given differential equation is exact.

16. Solve $xy''' + (x^2 + x + 3)y'' + (4x + 2)y' + 2y = 0$.

The equation is exact since $P_3 - P_2' + P_1'' - P_0''' = 2 - 4 + 2 - 0 = 0$.

Consider the left member $xy''' + (x^2 + x + 3)y'' + (4x + 2)y' + 2y$.

To obtain the first term we must differentiate xy''. Now $\frac{d}{dx}(xy'') = xy''' + y''$ and when this is removed, we have $(x^2 + x + 2)y'' + (4x + 2)y' + 2y$. To obtain the first term of the resulting relation, we must differentiate $(x^2 + x + 2)y'$. When $\frac{d}{dx}(x^2 + x + 2)y' = (x^2 + x + 2)y'' + (2x + 1)y'$ is removed, we have $(2x + 1)y' + 2y = \frac{d}{dx}(2x + 1)y$. Thus the given equation is the exact derivative of

$$A)\quad xy'' + (x^2 + x + 2)y' + (2x + 1)y = C_1.$$

Since $P_2 - P_1' + P_0'' = (2x + 1) - (2x + 1) + 0 = 0$, we now treat the left member of *A)* precisely as we did the corresponding member of the original equation.

We remove $\frac{d}{dx}(xy') = xy'' + y'$ and have $(x^2 + x + 1)y' + (2x + 1)y = \frac{d}{dx}(x^2 + x + 1)y$.

Hence *A)* is the exact derivative of

$$B)\quad xy' + (x^2 + x + 1)y = C_1 x + C_2,$$

a linear equation for which $xe^{\frac{1}{3}x(x+2)}$ is an integrating factor.

Thus, the complete solution of the given equation is

$$xye^{\frac{1}{3}x(x+2)} = C_1 \int xe^{\frac{1}{3}x(x+2)}\,dx + C_2 \int e^{\frac{1}{3}x(x+2)}\,dx + C_3.$$

The following scheme will be found convenient.

$$xy''' + (x^2 + x + 3)y'' + (4x + 2)y' + 2y = 0$$

xy'' $\quad xy''' + y''$

$$(x^2 + x + 2)y'' + (4x + 2)y' + 2y$$

$(x^2 + x + 2)y'$ $\quad (x^2 + x + 2)y'' + (2x + 1)y'$

$$(2x + 1)y' + 2y$$

$(2x + 1)y$ $\quad (2x + 1)y' + 2y$

A) $\quad xy'' + (x^2 + x + 2)y' + (2x + 1)y = C_1$

xy' $\quad xy'' + y'$

$$(x^2 + x + 1)y' + (2x + 1)y$$

$(x^2 + x + 1)y$ $\quad (x^2 + x + 1)y' + (2x + 1)y$

B) $\quad xy' + (x^2 + x + 1)y = C_1x + C_2.$

17. Solve $\quad 2y\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2}\dfrac{dy}{dx} = -\dfrac{1}{x^2}.$

We write $\quad 2y\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2}\dfrac{dy}{dx}$

$2y\dfrac{d^2y}{dx^2}$ $\quad 2y\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2}\dfrac{dy}{dx}$

$2\left(\dfrac{dy}{dx}\right)^2$ $\quad 4\dfrac{d^2y}{dx^2}\dfrac{dy}{dx}$

Thus, the given equation is exact, being obtained by differentiating

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = -\int \frac{1}{x^2}\,dx = \frac{1}{x} + K_1.$$

A second integration yields $\quad 2y\dfrac{dy}{dx} = \ln x + K_1x + K_2$ $\quad$ whose solution is

$$y^2 = x\ln x + C_1x^2 + C_2x + C_3.$$

18. Solve $\quad (1 + 3xy^2)y''' + 9(y^2 + 2xyy')y'' + 18y(y')^2 + 6x(y')^3 = 6.$

We write $\quad (1 + 3xy^2)y''' + 9y^2y'' + 18xyy'y'' + 18y(y')^2 + 6x(y')^3$

$(1 + 3xy^2)y''$ $\quad (1 + 3xy^2)y''' + 3y^2y'' + 6xyy'y''$

$$6y^2y'' + 12xyy'y'' + 18y(y')^2 + 6x(y')^3$$

$6y^2y'$ $\quad 6y^2y'' + 12y(y')^2$

$$12xyy'y'' + 6y(y')^2 + 6x(y')^3$$

$6xy(y')^2$ $\quad 12xyy'y'' + 6y(y')^2 + 6x(y')^3$

The given equation is exact, being obtained by differentiating

$$(1 + 3xy^2)y'' + 6y^2y' + 6xy(y')^2 = 6x + K$$

$(1 + 3xy^2)y'$ $\quad (1 + 3xy^2)y'' + 3y^2y' + 6xy(y')^2$

y^3 $\quad 3y^2y'$

$3y^2y'$

and this equation is obtained by differentiating $(1+3xy^2)y' + y^3 = 3x^2 + Kx + C_2$.

In turn, this equation is exact and we have $xy^3 + y = x^3 + C_1x^2 + C_2x + C_3$.

19. Solve $x^3y''' + 5x^2y'' + (2x - x^3)y' - (2+x^2)y = 40x^3 - 4x^5$.

It is readily verified that this linear equation is not exact. To test whether or not it has an integrating factor of the form x^n, we multiply by x^n to get

$$x^{n+3}y''' + 5x^{n+2}y'' + (2x^{n+1} - x^{n+3})y' - (2x^n + x^{n+2})y = (40x^3 - 4x^5)x^n$$

and write the condition

$$-(2x^n + x^{n+2}) - 2(n+1)x^n + (n+3)x^{n+2} + 5(n+2)(n+1)x^n - (n+3)(n+2)(n+1)x^n$$

$$= (n+2)x^{n+2} + (n+2)(n-n^2)x^n = 0, \text{ for all values of } x.$$

Then $n = -2$, and x^{-2} is an integrating factor. Using it, we have

$$xy''' + 5y'' + (\frac{2}{x} - x)y' - (\frac{2}{x^2} + 1)y = 40x - 4x^3$$

xy'' $\qquad xy''' + y''$

$$4y'' + (\frac{2}{x} - x)y' - (\frac{2}{x^2} + 1)y$$

$4y' + (\frac{2}{x} - x)y$ $\qquad 4y'' + (\frac{2}{x} - x)y' - (\frac{2}{x^2} + 1)y$

and $$xy'' + 4y' + (\frac{2}{x} - x)y = 20x^2 - x^4 + K.$$

The transformation $y = \frac{v}{x^2}$ reduces this equation to $v'' - v = (D^2 - 1)v = 20x^3 - x^5 + Kx$,

and the complete solution is $v = x^2y = C_1e^x + C_2e^{-x} - (1 + D^2 + D^4 + D^6 + \cdots)(20x^3 - x^5 + Kx)$

$$= C_1e^x + C_2e^{-x} + C_3x + x^5.$$

20. Solve $2yy''' + 2(y + 3y')y'' + 2(y')^2 = 2$.

We write $\qquad 2yy''' + 2yy'' + 6y'y'' + 2(y')^2 = 2$

$2yy''$ $\qquad 2yy''' + 2y'y''$

$$2yy'' + 4y'y'' + 2(y')^2$$

$2yy' + 2(y')^2$ $\qquad 2yy'' + 4y'y'' + 2(y')^2$

and thus obtain by integration

$$2yy'' + 2(y')^2 + 2yy' = 2x + K_1$$

$2yy'$ $\qquad 2yy'' + 2(y')^2$

$$2yy'$$

y^2 $\qquad 2yy'$

and obtain $2yy' + y^2 = x^2 + K_1x + K_2$. By inspection, e^x is an integrating factor; then

$$y^2e^x = x^2e^x - 2xe^x + 2e^x + K_1(xe^x - e^x) + K_2e^x + C_3 \quad \text{or} \quad y^2 = x^2 + C_1 + C_2x + C_3e^{-x}.$$

21. Solve $x\cos y\,y''' - 3x\sin y\,y'y'' - \cos y\,y'' - x\cos y\,(y')^3 + \sin y\,(y')^2 + x\cos y\,y' - \sin y = 0$.

Since $\frac{d}{dx}\left(\frac{\sin y}{x}\right) = \frac{x\cos y\,y' - \sin y}{x^2}$, the last two terms of the given equation suggest $\frac{1}{x^2}$ as a possible integrating factor. Using it and integrating,

$$\frac{\cos y\,y'' - \sin y\,(y')^2}{x} + \frac{\sin y}{x} = C_1 \quad \text{or} \quad \cos y\,y'' - \sin y\,(y')^2 + \sin y = C_1 x.$$

The substitution $\sin y = z$ reduces this equation to $z'' + z = C_1 x$ whose complete solution is

$$z = \sin y = C_1 x + C_2\cos x + C_3\sin x.$$

SUPPLEMENTARY PROBLEMS

Solve.

22. $y'' + (y')^2 + 1 = 0$ *Ans.* $y = \ln\cos(x - C_1) + C_2$

23. $(1+x^2)y'' + 2xy' = 2x^{-3}$ $y = C_1 + C_2\arctan x + 1/x$

24. $xy'' - y' = -2/x - \ln x$ $y = C_1x^2 + C_2 + (x+1)\ln x$

25. $y''' + y'' = x^2$ $y = C_1e^{-x} + C_2x + C_3 + x^2(x^2 - 4x + 12)/12$

26. $yy'' + (y')^3 = 0$ $x = C_1 + C_2y + y\ln y$

27. $yy'' + (y')^2 = 2$ $y^2 = 2x^2 + C_1x + C_2$

28. $yy'' = (y')^2(1 - y'\cos y + yy'\sin y)$ $x = C_1 + C_2\ln y + \sin y$

29. $(2x-3)y''' - (6x-7)y'' + 4xy' - 4y = 8$ $y = C_1x + C_2e^x + C_3e^{2x} - 2$

Hint: $y = x$ is a particular integral of the reduced equation.

30. $(2x^3 - 1)y''' - 6x^2y'' + 6xy' = 0$ $y = C_1(x^4 + 4x) + C_2x^2 + C_3$

31. $yy'' - (y')^2 = y^2\ln y$ $\ln y = C_1e^x + C_2e^{-x}$

Hint: Use $\ln y = z$.

32. $(x + 2y)y'' + 2(y')^2 + 2y' = 2$ $y(x+y) = x^2 + C_1x + C_2$

33. $(1 + 2y + 3y^2)y''' + 6y'[y'' + (y')^2 + 3yy''] = x$ $y + y^2 + y^3 = C_1x^2 + C_2x + C_3 + x^4/24$

34. $3x[y^2y''' + 6yy'y'' + 2(y')^3] - 3y[yy'' + 2(y')^2] = -2/x$

Hint: $1/x^2$ is an integrating factor. *Ans.* $y^3 = C_1x^3 + C_2x + C_3 + x\ln x$

35. $yy''' + 3y'y'' - 2yy'' - 2(y')^2 + yy' = e^{2x}$ *Ans.* $y^2 = C_1 + C_2e^x + C_3xe^x + e^{2x}$

Hint: e^{-x} is an integrating factor. Solve also using $y^2 = v$.

36. $2(y+1)y'' + 2(y')^2 + y^2 + 2y = 0$ *Ans.* $y^2 + 2y = C_1\cos x + C_2\sin x$

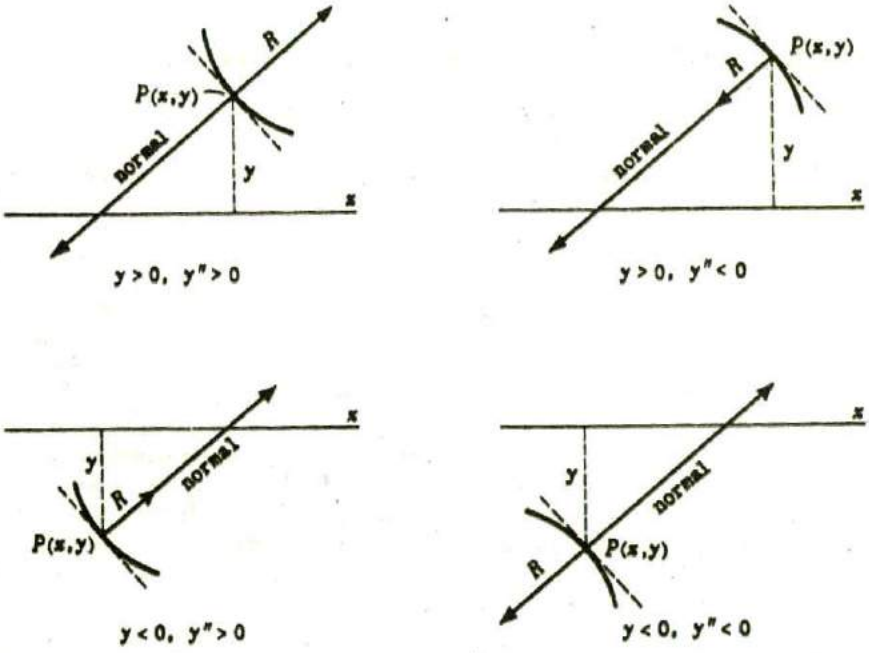
Hint: Use $y^2 + 2y = v$.

CHAPTER 20

Applications of Linear Equations

GEOMETRICAL APPLICATIONS. In rectangular coordinates the radius of curvature R of a curve $y = f(x)$ at a general point on it is given by

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$



Let the normal at the point be drawn toward the x-axis. It is clear from the figures that the normal and radius of curvature at any point have the same direction when y and d^2y/dx^2 have opposite signs and have opposite directions when y and d^2y/dx^2 have the same signs.

PHYSICAL APPLICATIONS. OSCILLATORY MOTION. Consider a ball bobbing up and down at the end of a rubber string.

If the other end of the string is held fixed and no external force is applied to the ball to keep it moving once it has been started, and if the mass of the string and the resistance offered by the air are such that they may be neglected, the ball will move with *simple harmonic motion*

$$x = A \cos \omega t + B \sin \omega t$$

where x is the displacement of the ball at time t from its position of rest or equilibrium.

For simple harmonic motion:

a) The *amplitude* or maximum displacement from equilibrium position is $\sqrt{A^2 + B^2}$, since when $dx/dt = 0$, $\tan \omega t = B/A$, and $x = \sqrt{A^2 + B^2}$.

b) The *period* or number of units (sec) of time for a complete oscillation is $2\pi/\omega$ sec, since when t is changed by $2\pi/\omega$ sec the values of x and dx/dt are unchanged, while for any change of t less than this amount one (or both) of x and dx/dt is changed.

c) The *frequency* or number of oscillations (cycles) per sec is $\omega/2\pi$ cycles/sec.

d) The *differential equation* of simple harmonic motion is $m\dfrac{d^2x}{dt^2} = -kx$, where k is a positive quantity. In the above illustration

$$m\frac{d^2x}{dt^2} = -m\omega^2(A\cos\omega t + B\sin\omega t) = -kx$$

where m is the mass of the ball and $k = m\omega^2$.

If the above assumptions are modified so that the resistance of the air cannot be neglected, the ball will move with *free damped motion*

$$x = e^{-st}(A\cos\omega t + B\sin\omega t).$$

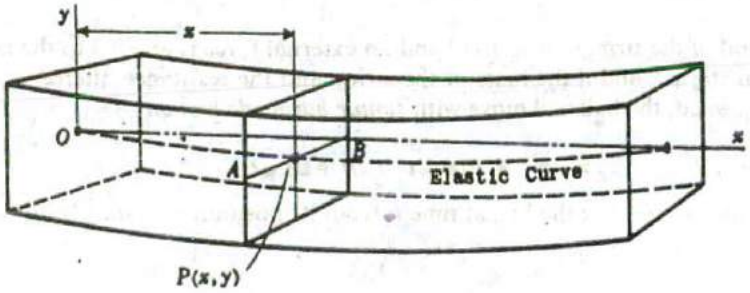
The motion is oscillatory as before but never repeats itself. Since the *damping factor* e^{-st} decreases as t increases, the amplitude of each oscillation is less than that of the preceding one. The frequency is $\omega/2\pi$ cycles/sec. See Problem 8*a*.

If the resistance offered to the motion is sufficiently great, other cases will arise.

See Problem 8*b*.

If in addition to a resistance, there is an external force acting on the ball or the complete system is given a motion, the motion of the ball is said to be *forced*. If the forcing function is harmonic with period $2\pi/\lambda$, the motion of the ball is the result of two motions – a free damping motion which dies out as time increases (called the *transient phenomenon*) and a simple harmonic motion with period that of the forcing function (called the *steady-state phenomenon.*) See Problem 9.

HORIZONTAL BEAMS. The problem is that of determining the deflection (bending) of a beam under given loadings. Only beams which are uniform in material and shape will be considered. It is convenient to think of the beam as consisting of fibres running lengthwise. In the bent beam shown, the fibres of the upper half are compressed and those of the lower half stretched, the two halves being separated by a neutral surface whose fibres are neither compressed nor stretched. The fibre which originally coincided with the horizontal axis of the beam now lies in the neutral surface along a curve (the elastic curve or curve of deflection). We seek the equation of this curve.



Consider a cross section of the beam at a distance x from one end. Let AB be its intersection with the neutral surface and P its intersection with the elastic curve. It is shown in Mechanics that the moment M with respect to AB of all external forces acting on either of the two segments into which the beam is separated by the cross section *(a)* is independent of the segment considered and *(b)* is given by

A) $$EI/R = M.$$

Here, E = the modulus of elasticity of the beam and I = the moment inertia of the cross section with respect to AB are constants associated with the beam, and R is the radius of curvature of the elastic curve at P.

For convenience, think of the beam as replaced by its elastic curve and the cross section by the point P. Take the origin at the left end of the beam with the x axis horizontal and let P have coordinates (x, y). Since the slope dy/dx of the elastic curve at all of its points is numerically small,

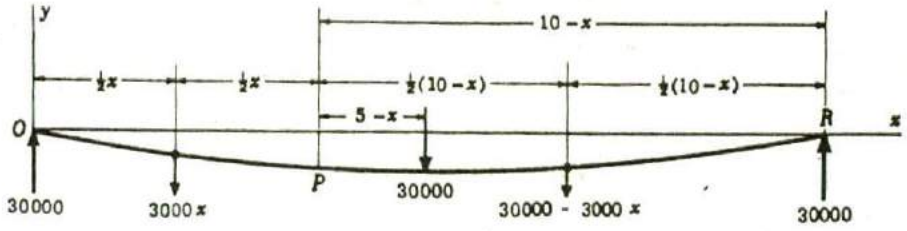
$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{1}{\frac{d^2y}{dx^2}}, \quad \text{approximately,}$$

and *A)* reduces to

B) $$EI\,\frac{d^2y}{dx^2} = M.$$

The bending moment M at the cross section (point P of the elastic curve) is the algebraic sum of the moments of the external forces acting on the segment of the beam (segment of the elastic curve) about the line AB in the cross section (about the point P of the elastic curve). We shall assume here that upward forces give positive moments and downward forces give negative moments.

EXAMPLE. Consider a 10 metre beam resting on two vertical supports, as in the figure below. Suppose the beam carries a uniform load of 3000 $\mathrm{N\,m^{-1}}$ of length and a load of 30000 N at its middle.



The external forces acting on OP are *(a)* an upward thrust at O, x m from P, equal to one-half the total load, i.e., $\frac{1}{2}(30000 + 10 \times 3000) = 30000$ N and *(b)* a downward force of $3000x$ N thought of as concentrated at the middle of OP and thus $\frac{1}{2}x$ m from P. The bending moment at P is

$$M = 30000x - 3000x(\tfrac{1}{2}x) = 30000x - 1500x^2.$$

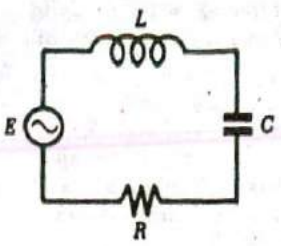
To show that the bending moment at P is independent of the segment used, consider the forces acting on PR: *(a)* an upward thrust of 30000N at R, $10 - x$ m from P, *(b)* the load of 30000 N acting downward at the middle of the beam, $5 - x$ m from P, and *(c)* $3000(10 - x)$ N downward thought of as concentrated at the

middle of PR, $\frac{1}{2}(10-x)$m from P. Then

$$M = 30000(10-x) - 30000(5-x) - 3000(10-x)\cdot\tfrac{1}{2}(10-x)$$
$$= 30000x - 1500x^2, \qquad \text{as before.}$$

A beam is said to be *fixed* at one end if it is held horizontal there by the masonry. In the example above the beam is not horizontal at O and is said to be freely *supported* there.

SIMPLE ELECTRIC CIRCUITS. The sum of the voltage drops across the elements of a closed circuit is equal to the total electromotive force E in the circuit. The voltage drop across a resistance R ohms is Ri, across a coil of inductance L henries is $L\,di/dt$, and across a condenser of capacitance (capacity) C farads is q/C. Here, the current i amperes and the charge q coulombs are related by $i = dq/dt$. We will consider R, L, and C as constants.



The differential equation of an electric circuit containing an inductance L, a resistance R, a condenser of capacitance C, and an electromotive force $E(t)$ is therefore

$C')$ $$L\frac{di}{dt} + Ri + \frac{q}{C} = E(t)$$

or, since $i = dq/dt$, $di/dt = d^2q/dt^2$,

$C)$ $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

from which $q = q(t)$ may be found.

By differentiating $C')$ and using $\frac{dq}{dt} = i$, we have

$D)$ $$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = E'(t)$$

from which $i = i(t)$ may be found.

SOLVED PROBLEMS

GEOMETRIC APPLICATIONS.

1. Determine the curve whose radius of curvature at any point $P(x,y)$ is equal to the normal at P and *(a)* in the same direction, *(b)* in the opposite direction.

a) Here $$\frac{[1+(y')^2]^{3/2}}{y''} = -y[1+(y')^2]^{1/2} \quad \text{or} \quad yy'' + (y')^2 + 1 = 0.$$

The equation is exact and an integration yields $yy' + x - C_1 = 0$ or $y\,dy + (x - C_1)dx = 0$.

Integrating again $\frac{1}{2}y^2 + \frac{1}{2}(x - C_1)^2 = K$ or $y^2 + (x - C_1)^2 = C_2$, a family of circles with centres on the x-axis.

b) Here $\dfrac{[1 + (y')^2]^{3/2}}{y''} = y[1 + (y')^2]^{1/2}$ or $yy'' - (y')^2 - 1 = 0$.

The substitution $y' = p$, $y'' = p\dfrac{dp}{dy}$ of Chapter 19 reduces the equation to

$$yp\frac{dp}{dy} - p^2 - 1 = 0 \qquad \text{or} \qquad \frac{p\,dp}{1 + p^2} = \frac{dy}{y}.$$

Then $\ln(1 + p^2) = \ln y^2 + \ln C_1^2$, $1 + p^2 = C_1^2 y^2$, or $\dfrac{dy}{\sqrt{C_1^2 y^2 - 1}} = \pm dx$.

Integrating, $\cosh^{-1} C_1 y = \pm C_1 x + C_2$, $C_1 y = \cosh(\pm C_1 x + C_2)$, or

$$y = \frac{1}{2C_1}[e^{(\pm C_1 x + C_2)} + e^{-(\pm C_1 x + C_2)}].$$

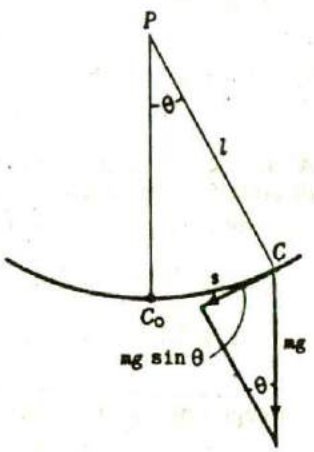
The curves are catenaries and the equation may be written in the form

$$y = \tfrac{1}{2}A[e^{(B \pm x)/A} + e^{-(B \pm x)/A}], \text{ where } A = \frac{1}{C_1} \text{ and } B = \frac{C_2}{C_1}.$$

PHYSICAL APPLICATIONS

MOTION OF A PENDULUM.

2. A pendulum, of length l and mass m, suspended at P (see figure) moves in a vertical plane through P. Disregarding all forces except that of gravity, find its motion.



Under the assumptions, the centre of gravity C of the bob moves on a circle with centre P and radius l, Let θ, positive when measured counterclockwise, be the angle which the string makes with the vertical at time t. The only force is gravity, positive when measured downward, and its component along the tangent to the path of the bob is $mg \sin\theta$. If s denotes the length of arc C_0C, then $s = l\theta$ and the acceleration along the arc is

$$\frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}.$$

Thus $m \cdot l\dfrac{d^2\theta}{dt^2} = -mg \sin\theta$ or $l\dfrac{d^2\theta}{dt^2} = -g \sin\theta$.

Multiplying by $2\dfrac{d\theta}{dt}$ and integrating, $l\left(\dfrac{d\theta}{dt}\right)^2 = 2g\cos\theta + C_1$ or $\dfrac{d\theta}{\sqrt{2g\cos\theta + C_1}} = \pm\dfrac{dt}{\sqrt{l}}$.

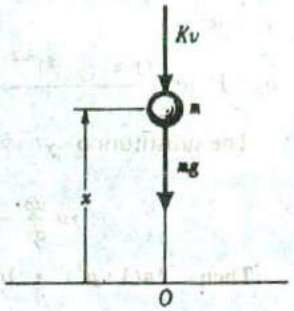
This integral cannot be expressed in terms of elementary functions.

When θ is small, $\sin\theta = \theta$, approximately. When this replacement is made in the original differential equation, we have $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{l}\theta = 0$ whose solution is $\theta = C_1\cos\sqrt{\dfrac{g}{l}}\,t + C_2\sin\sqrt{\dfrac{g}{l}}\,t$.

This is an example of simple harmonic motion. The amplitude is $\sqrt{C_1^2 + C_2^2}$ and the period is $2\pi\sqrt{\dfrac{l}{g}}$.

MOTION ALONG A STRAIGHT LINE.

3. A mass m is projected vertically upward from O with initial velocity v_0. Find the maximum height reached, assuming that the resistance of the air is proportional to the velocity.



Take upward direction from O as positive, and let x denote the distance of the mass from O at time t. The mass is acted upon by two forces, the gravitational force of magnitude mg and the resistance of magnitude $Kv = K\dfrac{dx}{dt}$ each directed down.

Hence, using mass $\times$ acceleration = net force,

$$m\frac{d^2x}{dt^2} = -mg - K\frac{dx}{dt} \quad \text{or} \quad \frac{d^2x}{dt^2} + k\frac{dx}{dt} = -g, \text{ where } K = mk.$$

Integrating, 1) $x = C_1 + C_2e^{-kt} - \dfrac{g}{k}t$, and then differentiating once with respect to t,

2) $v = \dfrac{dx}{dt} = -kC_2e^{-kt} - \dfrac{g}{k}$.

When $t = 0$, $x = 0$ and $v = v_0$. Then $C_1 + C_2 = 0$, $v_0 = -kC_2 - \dfrac{g}{k}$, and $C_1 = -C_2 = \dfrac{v_0}{k} + \dfrac{g}{k^2}$.

Making these replacements in 1), we have $x = \dfrac{1}{k^2}(g + kv_0)(1 - e^{-kt}) - \dfrac{g}{k}t$.

The maximum height is reached when $v = 0$. From 2), $e^{-kt} = \dfrac{-g}{k^2C_2} = \dfrac{g}{g + kv_0}$ and $t = \dfrac{1}{k}\ln\dfrac{g + kv_0}{g}$.

Then the maximum height is $x = \dfrac{1}{k^2}(g + kv_0)\left(1 - \dfrac{g}{g + kv_0}\right) - \dfrac{g}{k}\left(\dfrac{1}{k}\ln\dfrac{g + kv_0}{g}\right) = \dfrac{1}{k}\left(v_0 - \dfrac{g}{k}\ln\dfrac{g + kv_0}{g}\right)$.

4. A mass m, free to move along the x-axis, is attracted toward the origin with a force proportional to its distance from the origin. Find the motion *(a)* if it starts from rest at $x = x_0$ and *(b)* if it starts at $x = x_0$ with initial velocity v_0 moving away from the origin.

Let x denote the distance from the origin to the mass at time t.

Then $m\dfrac{d^2x}{dt^2} = -Kx$ or $\dfrac{d^2x}{dt^2} + k^2x = 0$, where $K = mk^2$.

Integrating, 1) $x = C_1 \sin kt + C_2 \cos kt$, and differentiating once with respect to t,

2) $v = -kC_2 \sin kt + kC_1 \cos kt$.

a) When $t = 0$, $x = x_0$ and $v = 0$. Then $C_1 = 0$ from 2), $C_2 = x_0$ from 1), and

$$x = x_0 \cos kt.$$

b) When $t = 0$, $x = x_0$ and $v = v_0$. Then $C_2 = x_0$, $C_1 = v_0/k$, and

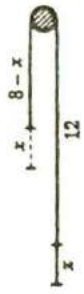
$$x = \frac{v_0}{k}\sin kt + x_0 \cos kt.$$

In *a)* the motion is simple harmonic motion of amplitude x_0 and period $2\pi/k$.

In *b)* the motion is simple harmonic of amplitude $\dfrac{\sqrt{v_0^2 + k^2x_0^2}}{k}$ and period $2\pi/k$.

MOTION OF A COMPLEX SYSTEM.

5. A chain hangs over a smooth peg, 8 metres being on one side and 12 metres on the other. Find the time required for it to slide off *(a)* neglecting friction and *(b)* if the friction is equal to the weight of 1 metre of the chain.



a) Denote the total mass of the chain by m and the length (metres) of the chain which has moved over the peg at time t by x. At time t there are $(8 - x)$ metres of chain on one side and $(12 + x)$ metres on the other. The excess $(4 + 2x)$ metres on one side produces an unbalanced force of $(4 + 2x)\frac{mg}{20}$ newtons. Thus,

$$m\frac{d^2x}{dt^2} = (4 + 2x)\frac{mg}{20} \qquad \text{or} \qquad 10\frac{d^2x}{dt^2} = gx + 2g.$$

Solution 1.

Integrating $\frac{d^2x}{dt^2} - \frac{g}{10}x = \frac{g}{5}$, we have $x = C_1e^{\sqrt{g/10}\,t} + C_2e^{-\sqrt{g/10}\,t} - 2.$

Differentiating once with respect to t, $v = \sqrt{\frac{g}{10}}\,(C_1e^{\sqrt{g/10}\,t} - C_2e^{-\sqrt{g/10}\,t})$.

When $t = 0$, $x = 0$ and $v = 0$. Then $C_1 = C_2 = 1$ and $x = e^{\sqrt{g/10}\,t} + e^{-\sqrt{g/10}\,t} - 2 = 2\cosh\sqrt{\frac{g}{10}}\,t - 2.$

Hence $t = \sqrt{\frac{10}{g}}\cosh^{-1}\tfrac{1}{2}(x + 2) = \sqrt{\frac{10}{g}}\ln\frac{x + 2 + \sqrt{x^2 + 4x}}{2}$.

When $x = 8$ m has moved over the peg, $t = \sqrt{\frac{10}{g}}\ln(5 + 2\sqrt{6})$ sec.

Solution 2. Multiplying the equation by $\frac{dx}{dt}$ and integrating, we have

$$10\frac{dx}{dt}\frac{d^2x}{dt^2} = gx\frac{dx}{dt} + 2g\frac{dx}{dt} \quad \text{and} \quad 5(\frac{dx}{dt})^2 = \tfrac{1}{2}gx^2 + 2gx + C_1.$$

When $t = 0$, $x = 0$ and $dx/dt = 0$. Then $C_1 = 0$ and

$$5(\frac{dx}{dt})^2 = \tfrac{1}{2}gx^2 + 2gx \qquad \text{or} \qquad dt = \sqrt{\frac{10}{g}}\,\frac{dx}{\sqrt{x^2 + 4x}}.$$

(The positive square root is used here since x increases with t.)

Integrating, $t = \sqrt{\frac{10}{g}}\int\frac{dx}{\sqrt{(x + 2)^2 - 4}} = \sqrt{\frac{10}{g}}\ln(x + 2 + \sqrt{x^2 + 4x}) + C_2.$

When $t = 0$, $x = 0$. Then $C_2 = -\sqrt{\frac{10}{g}}\ln 2$, and $t = \sqrt{\frac{10}{g}}\ln\frac{x + 2 + \sqrt{x^2 + 4x}}{2}$ as before.

b) Here $m\frac{d^2x}{dt^2} = (4 + 2x)\frac{mg}{20} - \frac{mg}{20}$ or $20\frac{d^2x}{dt^2} = (2x + 3)g.$

Multiplying by $\frac{dx}{dt}$ and integrating, we have $10(\frac{dx}{dt})^2 = gx^2 + 3gx + C_1.$

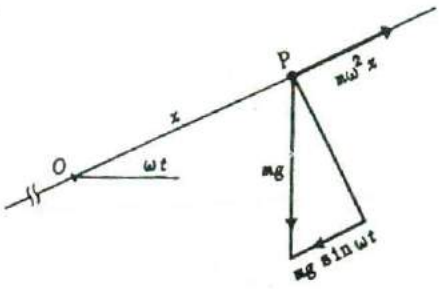
When $t = 0$, $x = 0$ and $v = 0$. Then $C_1 = 0$ and $\frac{dx}{dt} = \sqrt{\frac{g}{10}(x^2 + 3x)}$ or $dt = \sqrt{\frac{10}{g}}\,\frac{dx}{\sqrt{x^2 + 3x}}$.

Then $t = \sqrt{\frac{10}{g}}\ln(x + \frac{3}{2} + \sqrt{x^2 + 3x}) + C_2.$

When $t = 0$, $x = 0$. Then $C_2 = -\sqrt{\frac{10}{g}} \ln \frac{3}{2}$ and $t = \sqrt{\frac{10}{g}} \ln \frac{2}{3}(x + \frac{3}{2} + \sqrt{x^2 + 3x})$.

When $x = 8$, $t = \sqrt{\frac{10}{g}} \ln \frac{19 + 4\sqrt{22}}{3} = 1.4$ sec.

6. A bead slides without friction along a straight rod of negligible mass as the rod rotates with constant angular velocity ω about its midpoint O. Determine the motion *(a)* if the bead is initially at rest at O and *(b)* if the bead is initially at O moving with velocity $g/2\omega$.



Let the bead be x units from O at time t. It is being acted upon by two forces, (i) gravity and (ii) the centrifugal force $m\omega^2 x$ acting along the rod and directed away from O. Since the rod has rotated through an angle ωt, the component of the gravitational force along the rod has magnitude $mg \sin \omega t$; its direction is toward O. Hence

$$m\frac{d^2x}{dt^2} = m\omega^2 x - mg \sin \omega t \quad \text{or} \quad \frac{d^2x}{dt^2} - \omega^2 x = -g \sin \omega t.$$

Integrating, 1) $x = C_1 e^{\omega t} + C_2 e^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t$. Differentiating once with respect to t,

2) $v = \omega C_1 e^{\omega t} - \omega C_2 e^{-\omega t} + \frac{g}{2\omega} \cos \omega t$.

a) When $t = 0$, $x = 0$ and $v = 0$.

Then $C_1 + C_2 = 0$ from 1), $C_1 - C_2 + \frac{g}{2\omega^2} = 0$ from 2), $C_1 = -C_2 = -\frac{g}{4\omega^2}$, and

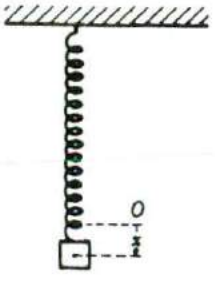
$$x = \frac{g}{4\omega^2}(e^{-\omega t} - e^{\omega t}) + \frac{g}{2\omega^2} \sin \omega t = -\frac{g}{2\omega^2} \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t.$$

b) When $t = 0$, $x = 0$ and $v = g/2\omega$.

Then $C_1 + C_2 = 0$, $C_1 - C_2 = 0$, $C_1 = C_2 = 0$, and $x = \frac{g}{2\omega^2} \sin \omega t$.

SPRINGS.

7. A spring, for which $k = 700 N\,m^{-1}$ hangs in a vertical position with its upper end fixed. A mass of 7 kg is attached to the lower end. After coming to rest, the mass is pulled down 0·05 m and released. Discuss the resulting motion of the mass, neglecting air resistance.



Take the origin at the centre of gravity of the mass after coming to rest, and let x, positive when measured downward, be the change in position of the mass at time t. When the mass is at rest the spring force is equal to but opposite in direction to the gravitational force. The net force at time t is the spring force $-kx$ corresponding to the change x in the position of the mass. Then

$$7\frac{d^2x}{dt^2} = -700x \quad \text{or} \quad \frac{d^2x}{dt^2} + 100x = 0.$$

Integrating, $x = C_1 \sin 10t + C_2 \cos 10t$.

Differentiating once with respect to t, $v = \frac{dx}{dt} = 10(C_1 \cos 10t - C_2 \sin 10t)$.

When $t = 0$, $x = 0,05$ and $v = 0$. Then $C_2 = 0,05$, $C_1 = 0$ and $x = 0,05 \cos 10t$.

This represents a simple harmonic motion. The period is $\frac{2\pi}{10} = 0,628$ sec the frequency is $\frac{10}{2\pi} = 1,59$ cycles/sec, and the amplitude is 0·05 m.

8. Solve Problem 7 if the medium offers a resistance (N) equal to *(a)* $v/4$ and *(b)* $980v$, where v is expressed in ms^{-1}

a) Here $7\frac{d^2x}{dt^2} = -700x - \frac{1}{4}\frac{dx}{dt}$ or $\frac{d^2x}{dt^2} + \frac{1}{28}\frac{dx}{dt} + 100x = 0$ Using the D notation,

$$(D^2 + \frac{1}{28}D + 100)x = [D - (-0,0179 + 10i)][D - (-0,0179 - 10i)]x = 0,$$

and $x = e^{-0,0179t}(C_1 \cos 10t + C_2 \sin 10t)$.

Differentiating once with respect to t,

$$\frac{dx}{dt} = v = e^{-0,0179t}[(10C_2 - 0,0179C_1)\cos 10t - (10C_1 + 0,0179C_2)\sin 10t].$$

When $t = 0$, $v = 0$ and $x = 0,05$. $C_1 = 0,05$, $0 = 10C_2 - 0,0179C_1$ and $C_2 = 0,000895$.

Thus, $x = e^{-0,0179t}(0,05 \cos 10t + 0,000895 \sin 10t)$.

This represents a damped oscillatory motion. Note that the frequency $10/2\pi = 1,59$ cycles/sec remains constant throughout the motion, while the amplitude of each oscillation is smaller than the preceding one due to the damping factor $e^{-0,0179t}$ At $t = 0$ the magnitude of the damping factor is 1. It will be 2/3 when $e^{-0,0179t} = 2/3$ or after $t = 22,7$ sec. It will be 1/3 when $e^{-0,0179t} = 1/3$ or after $t = 61,4$ sec

b) Here $7\frac{d^2x}{dt^2} = -700x - 980\frac{dx}{dt}$ or $(D^2 + 140D + 100)x = 0$

Integrating, $x = C_1e^{-0,7t} + C_2e^{-139,3t}$

Differentiating once with respect to t,

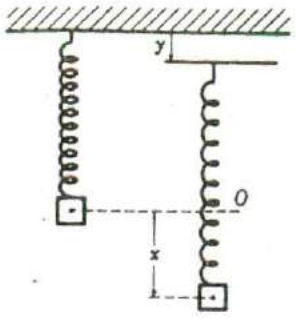
$$v = -0,7C_1e^{-0,7t} - 139,3C_2e^{-139,3t}.$$

When $t = 0$, $x = 0,05$ and $v = 0$. Then $C_1 + C_2 = 0,05$, $-0,7C_1 - 139,3C_2 = 0$, $C_1 = 0,0503$, $C_2 = -0,0003$ and

$$x = 0,0503e^{-0,7t} - 0,0003e^{-139,3t}.$$

The motion is not vibratory. After the initial displacement, the mass moves slowly toward the position of equilibrium as t increases.

9. Solve Problem 8*a* if, in addition, the support of the spring is given a motion $y = 0,3 \cos 4t$ metres.



Take the origin as in Problem 8 and let x represent the change in position of the mass after t sec. From the figure, it is seen that the stretch in the spring is $(x - y)$ and the spring force is $-700(x - y) = -700(x - 0,3 \cos 4t)$ N. Hence,

$$7\frac{d^2x}{dt^2} = -700(x - 0.3\cos 4t) - \frac{1}{4}\frac{dx}{dt}$$

or $$(D^2 + \frac{1}{28}D + 100)x = 30\cos 4t.$$

Integrating, $$x = e^{-0.0179t}(C_1 \cos 10t + C_2 \sin 10t) + \frac{30}{D^2 + D/28 + 100}\cos 4t$$

$$= e^{-0.0179t}(C_1 \cos 10t + C_2 \sin 10t) + 0.0006 \sin 4t + 0.36\cos 4t.$$

Differentiating once with respect to t,

$$v = e^{-0.0179t}[(10C_2 - 0,0179\,C_1)\cos 10t - (10C_1 + 0,0179C_2)\sin 10t] + 0,0024\cos 4t - 1,44\sin 4t.$$

When $t = 0$, $v = 0$ and $x = 0,3 + 0,05 = 0,35$. Then $C_1 = -0.01$, $C_2 = -0.00026$ and

$$x = e^{-0.0179t}(-0.01\cos 10t - 0.00026\sin 10t) + 0\,0006\sin 4t + 0.36\cos 4t.$$

The motion consists of a damped harmonic motion which gradually dies away (transient phenomenon) and a harmonic motion which remains (steadystate – phenomenon). After a time the only effective motion is that of the steady-state. These steady-state oscillations will have a period and a frequency equal to those of the forcing function $y = 0.3 \cos 4t$, namely a period of $2\pi/4 = 1.57$ sec and a frequency of $4/2\pi = 0.637$ cycles/sec.

The amplitude is $\sqrt{(0.0006)^2 + (0.36)^2} = 0.36$ m.

10. A mass of 10 kg is suspended from a spring which is thereby stretched 81 mm. The upper end of the spring is then given a motion $y = \sin 2t + \cos 2t$ m. Find the equation of the motion, neglecting air resistance.

Take the origin at the centre of gravity of the mass when at rest. Let x represent the change in position of the mass at time t. The change in the length of the spring is $(x - y)$, the spring constant is $10(9,8)/0,081 = 1210$ N m^{-1} and the net spring force is $-1210(x - y)$. Then

$$10\frac{d^2x}{dt^2} = -1210(x - \sin 2t - \cos 2t) \quad \text{or} \quad \frac{d^2x}{dt^2} + 121x = 121(\sin 2t + \cos 2t).$$

Integrating, $$x = C_1 \cos 11t + C_2 \sin 11t + \frac{121}{117}(\sin 2t + \cos 2t).$$

Differentiating once with respect to t,

$$v = -11C_1 \sin 11t + 11C_2 \cos 11t + \frac{242}{117}(-\sin 2t + \cos 2t).$$

When $t = 0$, $x = 1$ and $v = 0$.

Then $$1 = C_1 + \frac{121}{117}, \quad C_1 = -0,034 \quad \text{and} \quad 11C_2 + \frac{242}{117} = 0, \quad C_2 = -0.188.$$

Hence, $$x = -0.034\cos 11t - 0.188\sin 11t + 1.034(\sin 2t + \cos 2t)$$

11. A mass of 30 kg is attached to a spring for which $k = 750\,\text{N m}^{-1}$ and brought to rest. Find the position of the mass at time t if a force equal to $20 \sin 2t$ is applied to it.

Take the origin at the centre of gravity of the mass when at rest. The equation of motion is then

$$30\,\frac{d^2x}{dt^2} + 750x = 20 \sin 2t \quad \text{or} \quad \frac{d^2x}{dt^2} + 25x = \frac{2}{3}\sin 2t.$$

Integrating, $\quad x = C_1 \cos 5t + C_2 \sin 5t + \frac{2}{63}\sin 2t$

Differentiating once with respect to t, $\quad v = -5C_1 \sin 5t + 5C_2 \cos 5t + \frac{4}{63}\cos 2t.$

Using the initial conditions $x = 0$, $v = 0$ when $t = 0$, $\quad C_1 = 0$, $\quad C_2 = -\frac{4}{315}$ and

$$x = -0{,}013 \sin 5t + 0{,}032 \sin 2t.$$

The displacement here is the algebraic sum of two harmonic displacements of *different* periods.

12. A mass of 9 kg is attached to a spring for which $k = 729\,\text{N m}^{-1}$ and brought to rest. Find the motion of the mass if the support of the spring is given a motion $y = 0{,}3 \sin 9t$ m

Take the origin at the centre of gravity of the mass when at rest and let x represent the change in position of the mass at time t.

The stretch in the spring is $(x - y)$ and the spring force is $-729(x - y)$. Thus,

$$9\,\frac{d^2x}{dt^2} = -729(x - 0.3 \sin 9t) \quad \text{or} \quad \frac{d^2x}{dt^2} + 81x = 24{\cdot}3 \sin 9t.$$

Integrating, $\quad x = C_1 \cos 9t + C_2 \sin 9t - 1{,}35t \cos 9t,$

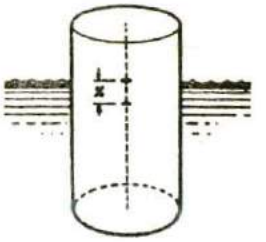
and $\quad v = -9C_1 \sin 9t + 9C_2 \cos 9t - 1{,}35 \cos 9t + 12{,}15t \sin 9t.$

Using the initial conditions $x = 0$, $v = 0$ when $t = 0$, $\quad C_1 = 0$, $C_2 = 0{,}15$ and

$$x = 0{,}15 \sin 9t - 1{,}35t \cos 9t.$$

The first term represents a simple harmonic motion while the second represents a vibratory motion with increasing amplitude (because of the factor t). As t increases, the amplitude of the oscillation increases until there is a mechanical breakdown.

13. A cylindrical buoy 0·5 m in diameter stands in water (density $1000\,\text{kg m}^{-3}$) with its axis vertical. When depressed slightly and released, it is found that the period of vibration is 2 seconds. Find the mass of the cylinder.



Take the origin at the intersection of the axis of the cylinder and the surface of the water when the buoy is in equilibrium, and take the downward direction as positive.

Let x (m) denote the change in the position of the buoy at time t. By Archimedes' Principle, a body partly or totally submerged in a fluid is buoyed up by a force equal to the weight of the fluid it displaces. Thus the corresponding change in the buoying force is $1000(0{,}25)^2 \pi g x$ and

$$W\,\frac{d^2x}{dt^2} = -62{,}5\pi g x \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{62{,}5\pi}{W}x = 0$$

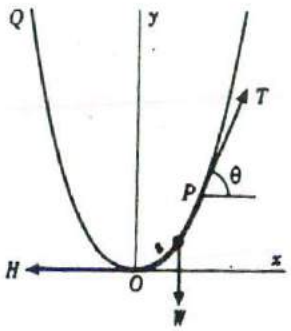
where W (kg) is the mass of the buoy and $g = 9{,}8\,\text{m s}^{-2}$

Integrating, $x = C_1 \sin \sqrt{\frac{62,5\pi}{W}}\, t + C_2 \cos \sqrt{\frac{62,5\pi}{W}}\, t$.

Since the period is $\frac{2\pi\sqrt{W}}{\sqrt{62,5\pi}} = 2$, $W = \frac{62,5}{\pi} = 195\ \text{kg}$

HANGING CABLE.

14. Determine the shape of a uniform cable which hangs under its own weight, w N m^{-1} of length.



Choose the coordinate axes as in the figure, the origin being at the lowest point of the cable. Consider the part between O and a variable point $P(x,y)$. This part is in equilibrium under the action of (1) a horizontal force of magnitude H at O, (2) the tension T along the tangent at P, and (3) the weight W of OP.

Since OP is in equilibrium, all force acting horizontally toward the right and all force acting horizontally toward the left must be equal in magnitude, and, also, all force acting vertically upward and all force acting vertically downward.

Hence, $T\cos\theta = H$, $T\sin\theta = W$, and $\tan\theta = \frac{dy}{dx} = \frac{W}{H}$.

Now H is constant, being due to the part OQ of the cable, while $W = ws$, where s is the length of OP. Thus,

$$\frac{d^2y}{dx^2} = \frac{1}{H}\frac{dW}{dx} = \frac{w}{H}\frac{ds}{dx} = \frac{w}{H}\sqrt{1 + (dy/dx)^2}.$$

To solve the above equation, write $\frac{dy}{dx} = p$ and obtain

$$\frac{dp}{dx} = \frac{w}{H}\sqrt{1+p^2} \quad \text{or} \quad \frac{dp}{\sqrt{1+p^2}} = \frac{w}{H}\,dx.$$

Integrating between the limits $x = 0,\ p = 0$ and $x = x,\ p = p$,

$$\sinh^{-1} p = \frac{w}{H}x \quad \text{and} \quad p = \frac{dy}{dx} = \sinh\frac{w}{H}x.$$

Integrating $dy = \sinh\frac{w}{H}x\,dx$ between the limits $x = 0,\ y = 0$ and $x = x,\ y = y$,

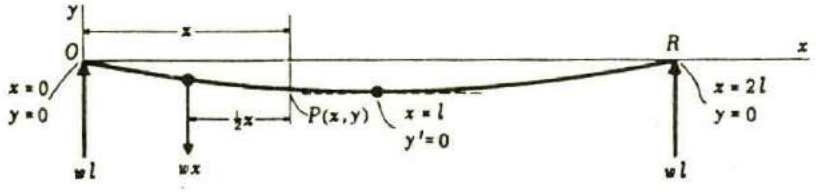
$$y = \frac{H}{w}\left(\cosh\frac{w}{H}x - 1\right), \quad \text{a catenary.}$$

If the origin had been taken at a distance H/w under the lowest point of the cable (thus making H/w the y- intercept of the curve) the equation of the curve would have been

$$y = \frac{H}{w}\cosh\frac{w}{H}x.$$

HORIZONTAL BEAMS.

15. A horizontal beam of length $2l$ metres is freely supported at both ends. Find the equation of its elastic curve and its maximum deflection when the load is w N m^{-1} of length.



Take the origin at the left end of the beam with the x- axis horizontal as in the figure. Let P, any point on the elastic curve, have coordinates (x, y).

Consider the segment OP of the beam. There is an upward thrust wl N at 0, x m from P, and the load wx N at the midpoint of OP, $\frac{1}{2}x$ m from P. Then, since $EI\, d^2y/dx^2 = M$,

1) $$EI\frac{d^2y}{dx^2} = wlx - wx(\tfrac{1}{2}x) = wlx - \tfrac{1}{2}wx^2.$$

Solution 1. Integrating 1) once, $$EI\frac{dy}{dx} = \frac{1}{2}wlx^2 - \frac{1}{6}wx^3 + C_1.$$

At the middle of the beam $x = l$ and $dy/dx = 0$. Then $C_1 = -\frac{1}{3}wl^3$ and

2) $$EI\frac{dy}{dx} = \frac{1}{2}wlx^2 - \frac{1}{6}wx^3 - \frac{1}{3}wl^3.$$

Integrating 2), $EIy = \frac{1}{6}wlx^3 - \frac{1}{24}wx^4 - \frac{1}{3}wl^3x + C_2$. At O, $x = y = 0$. Then $C_2 = 0$ and

3) $$y = \frac{w}{24EI}(4lx^3 - x^4 - 8l^3x).$$

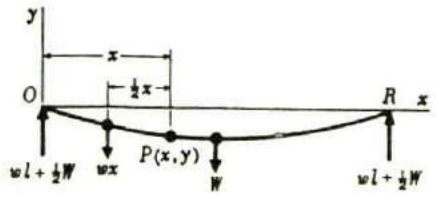
Solution 2. Integrating 1) twice, $$EIy = \frac{1}{6}wlx^3 - \frac{1}{24}wx^4 + C_1x + C_2.$$

At O, $x = y = 0$, while at R, $x = 2l$, $y = 0$. Using these boundary conditions in turn, we find $C_2 = 0$ and $C_1 = -\frac{1}{3}wl^3$, as before.

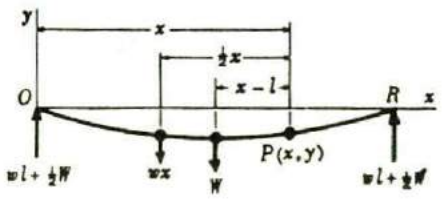
The deflection of the beam at any distance x from O is given by $-y$. The maximum deflection occurs at the middle of the beam ($x = l$) and is, from 3),

$$-y_{max} = -\frac{w}{24EI}(4l^4 - l^4 - 8l^4) = \frac{5wl^4}{24EI}.$$

16. Solve Problem 15 if there is in addition a load of W N at the middle of the beam.



$0 < x < l$



$l < x < 2l$

Choose the coordinate system as in Problem 15. Since the forces acting on a segment OP of the beam differ according as P lies to the left or right of the midpoint, two cases must be considered.

When $0 < x < l$, the forces acting on OP are an upward thrust of $(wl + \frac{1}{2}W)$ N at O, x m from P, and the load wx acting downward at the midpoint of OP, $\frac{1}{2}x$ m from P. The bending moment is then

1) $M = (wl + \frac{1}{2}W)x - wx(\frac{1}{2}x) = wlx + \frac{1}{2}Wx - \frac{1}{2}wx^2.$

When $l < x < 2l$ there is an additional force – the load W N at the midpoint of the beam, $(x - l)$ m from P. The bending moment is then

2) $M = (wl + \frac{1}{2}W)x - wx(\frac{1}{2}x) - W(x - l) = wlx + \frac{1}{2}Wx - \frac{1}{2}wx^2 - W(x - l).$

Both 1) and 2) yield the bending moment $M = \frac{1}{2}wl^2 + \frac{1}{2}Wl$ when $x = l$. The two cases may be treated at the same time by noting that for 1)

$$wlx + \tfrac{1}{2}Wx - \tfrac{1}{2}wx^2 = wlx - \tfrac{1}{2}wx^2 - \tfrac{1}{2}W(l - x) + \tfrac{1}{2}Wl$$

and for 2) $wlx + \frac{1}{2}Wx - \frac{1}{2}wx^2 - W(x - l) = wlx - \frac{1}{2}wx^2 + \frac{1}{2}W(l - x) + \frac{1}{2}Wl.$ Then

3) $EI\dfrac{d^2y}{dx^2} = wlx - \frac{1}{2}wx^2 \mp \frac{1}{2}W(l - x) + \frac{1}{2}Wl$

with the understanding that the upper sign holds for $0 < x < l$ and the lower for $l < x < 2l$.

Integrating 3) twice, $EIy = \dfrac{1}{6}wlx^3 - \dfrac{1}{24}wx^4 \mp \dfrac{1}{12}W(l - x)^3 + \dfrac{1}{4}Wlx^2 + C_1x + C_2.$

Using the boundary conditions $x = y = 0$ at O and $x = 2l$, $y = 0$ at R,

$C_2 = \dfrac{1}{12}Wl^3$, $2lC_1 + C_2 = -\dfrac{4}{3}wl^4 + \dfrac{2}{3}wl^4 + \dfrac{1}{12}Wl^3 - Wl^3$, and $C_1 = -\dfrac{1}{3}wl^3 - \dfrac{1}{2}Wl^2$. Then

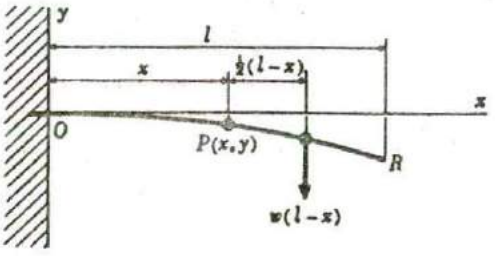
$$EIy = \frac{1}{6}wlx^3 - \frac{1}{24}wx^4 - \frac{1}{3}wl^3x \mp \frac{1}{12}W(l - x)^3 + \frac{1}{4}Wlx^2 - \frac{1}{2}Wl^2x + \frac{1}{12}Wl^3$$

$$= \frac{1}{6}wlx^3 - \frac{1}{24}wx^4 - \frac{1}{3}wl^3x - \frac{1}{12}W|l - x|^3 + \frac{1}{4}Wlx^2 - \frac{1}{2}Wl^2x + \frac{1}{12}Wl^3,$$

and $$y = \frac{w}{24EI}(4lx^3 - x^4 - 8l^3x) + \frac{W}{12EI}(3lx^2 - |l - x|^3 - 6l^2x + l^3).$$

The maximum deflection, occurring at the middle of the beam, is $-y_{max} = \dfrac{5wl^4}{24EI} + \dfrac{Wl^3}{6EI}$.

17. A horizontal beam of length l metres is fixed at one end but otherwise unsupported. Find the equation of its elastic curve and the maximum deflection when the uniform load is w N m^{-1} of length.



Take the origin at the fixed end and let P have coordinates (x, y). Consider the segment PR. The only force is the weight $w(l - x)$ N at the midpoint of PR $\frac{1}{2}(l - x)$ m from P. Then

$EI\dfrac{d^2y}{dx^2} = -w(l - x)\cdot\frac{1}{2}(l - x) = -\frac{1}{2}w(l - x)^2.$ Integrating once, $EI\dfrac{dy}{dx} = \dfrac{1}{6}w(l - x)^3 + C_1.$

At O: $x = 0$, $\dfrac{dy}{dx} = 0$; $C_1 = -\dfrac{1}{6}wl^3$ and $EI\dfrac{dy}{dx} = \dfrac{1}{6}w(l - x)^3 - \dfrac{1}{6}wl^3.$

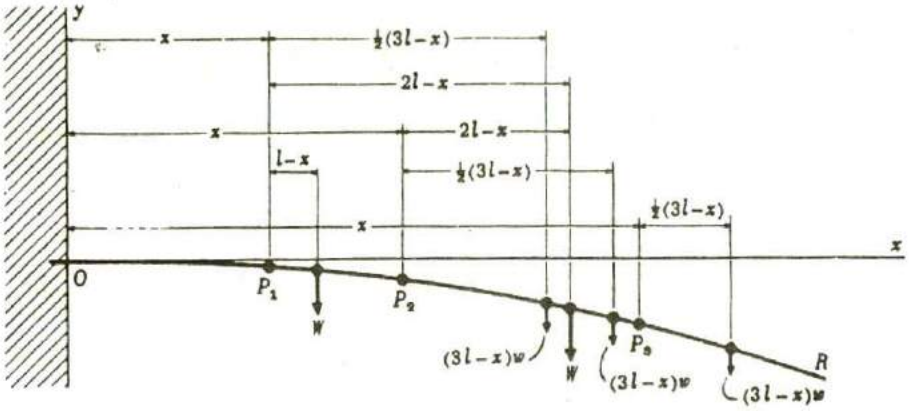
Integrating again, $EIy = -\dfrac{1}{24}w(l - x)^4 - \dfrac{1}{6}wl^3x + C_2.$

At O: $x = y = 0$; then $C_2 = \frac{1}{24} wl^4$, $EIy = -\frac{1}{24} w(l-x)^4 - \frac{1}{6} wl^3 x + \frac{1}{24} wl^4$, and

$$y = \frac{w}{24EI}(4lx^3 - 6l^2x^2 - x^4).$$

The maximum deflection, occurring at $R\,(x = l)$, is $-y_{max} = \frac{1}{8}\frac{wl^4}{EI}$. Note that this is not a relative minimum as in Problem 16 but an absolute minimum occuring at an end of the interval $0 \leqq x \leqq l$.

18. A horizontal beam of lenght $3l$ metres is fixed at one end but otherwise unsupported. There is a uniform load of w N m of length and two loads of W N each at points l and $2l$ m from the fixed end. Find the equation of the elastic curve and the maximum deflection.



Take the origin at the fixed end and let P have coordinates (x, y). There are three cases to be considered according as P is on the interval $(0 < x < l)$, $(l < x < 2l)$, or $(2l < x < 3l)$. In each case, use will be made of the right hand segment of the beam in computing the three bending moments.

When $0 < x < l$, ($P = P_1$ in the figure), there are three forces acting on P_1R: the weight $(3l-x)w$ N taken at the midpoint of P_1R, $\frac{1}{2}(3l-x)$ m from P_1; the load W N, $(l-x)$ m from P_1; and the load W N, $(2l-x)$ m from P_1. The bending moment about P_1 is

$$M_1 = -(3l-x)w \cdot \tfrac{1}{2}(3l-x) - W(l-x) - W(2l-x) = -\tfrac{1}{2}w(3l-x)^2 - W(l-x) - W(2l-x),$$

and

$$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}w(3l-x)^2 - W(l-x) - W(2l-x).$$

Integrating, $$EI\frac{dy}{dx} = \frac{1}{6}w(3l-x)^3 + \frac{1}{2}W(l-x)^2 + \frac{1}{2}W(2l-x)^2 + C_1.$$

At O: $x = 0$ and $dy/dx = 0$; then $C_1 = -\frac{9}{2}wl^3 - \frac{5}{2}Wl^2$,

$$EI\frac{dy}{dx} = \frac{1}{6}w(3l-x)^3 + \frac{1}{2}W(l-x)^2 + \frac{1}{2}W(2l-x)^2 - \frac{9}{2}wl^3 - \frac{5}{2}Wl^2,$$

and

$$EIy = -\frac{1}{24}w(3l-x)^4 - \frac{1}{6}W(l-x)^3 - \frac{1}{6}W(2l-x)^3 - \frac{9}{2}wl^3x - \frac{5}{2}Wl^2x + C_2.$$

At O: $x = y = 0$; then $C_2 = \frac{27}{8}wl^4 + \frac{3}{2}Wl^3$ and

A) $$EIy = -\frac{1}{24}w(3l-x)^4 - \frac{1}{6}W(l-x)^3 - \frac{1}{6}W(2l-x)^3 - \frac{9}{2}wl^3x - \frac{5}{2}Wl^2x + \frac{27}{8}wl^4 + \frac{3}{2}Wl^3.$$

When $l < x < 2l$, ($P = P_2$ in the figure), the bending moment about P_2 is

and $$M_2 = -\tfrac{1}{2}w(3l-x)^2 - W(2l-x),$$

$$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}w(3l-x)^2 - W(2l-x).$$ Integrating twice, we obtain

B′) $$EIy = -\frac{1}{24}w(3l-x)^4 - \frac{1}{6}W(2l-x)^3 + C_3x + C_4.$$

When $x = l$, *B′)* and *A)* must agree in deflection and slope $\frac{dy}{dx}$ so that $C_3 = C_1$ and $C_4 = C_2$. Thus,

B) $$EIy = -\frac{1}{24}w(3l-x)^4 - \frac{1}{6}W(2l-x)^3 - \frac{9}{2}wl^3x - \frac{5}{2}Wl^2x + \frac{27}{8}wl^4 + \frac{3}{2}Wl^3.$$

When $2l < x < 3l$, ($P = P_3$ in the figure), the bending moment about P_3 is $M_3 = -\tfrac{1}{2}w(3l-x)^2$,

and $$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}w(3l-x)^2.$$ Then

C) $$EIy = -\frac{1}{24}w(3l-x)^4 + C_5x + C_6 = -\frac{1}{24}w(3l-x)^4 - \frac{9}{2}wl^3x - \frac{5}{2}Wl^2x + \frac{27}{8}wl^4 + \frac{3}{2}Wl^3,$$

since, when $x = 2l$, there must be agreement with *B)* in deflection and slope.

A), B), C) may be written in the form

$$y = \frac{w}{24EI}(12lx^3 - 54l^2x^2 - x^4) + \frac{W}{6EI}(2x^3 - 9lx^2), \qquad 0 \leqq x \leqq l,$$

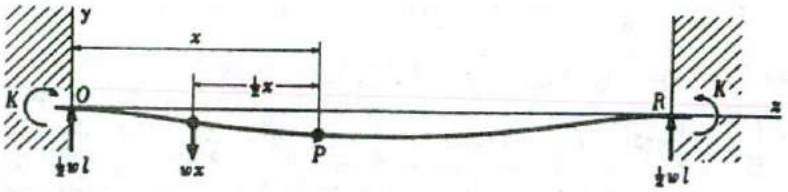
$$y = \frac{w}{24EI}(12lx^3 - 54l^2x^2 - x^4) + \frac{W}{6EI}(x^3 - 6lx^2 - 3l^2x + l^3), \qquad l \leqq x \leqq 2l,$$

$$y = \frac{w}{24EI}(12lx^3 - 54l^2x^2 - x^4) + \frac{W}{2EI}(3l^3 - 5l^2x), \qquad 2l \leqq x \leqq 3l.$$

The maximum deflection, occurring at R $(x = 3l)$, is $-y_{max} = \frac{1}{8EI}(81wl^4 + 48Wl^3)$.

Note that the elastic curve consists of arcs of three distinct curves, the slopes of each pair of arcs at a junction point being equal.

19. A horizontal beam of length l m is fixed at both ends. Find the equation of the elastic curve and the maximum deflection if it carries a uniform load of w N m^{-1} of length.



Take the origin at the left end of the beam and let P have coordinates (x, y).

The external forces acting on the segment OP are: a couple of unknown moment K exerted by the wall to keep the beam horizontal at O; an upward thrust of $\frac{1}{2}wl$ N at O, x m from P; and the load wx N acting downward at the midpoint of OP, $\frac{1}{2}x$ m from P. Thus,

$$EI\frac{d^2y}{dx^2} = K + \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2.$$

Integrating once and using $x = 0$, $dy/dx = 0$ at O, $EI\frac{dy}{dx} = Kx + \frac{1}{4}wlx^2 - \frac{1}{6}wx^3$.

At R: $x = l$, $dy/dx = 0$ since the beam is fixed there. Then

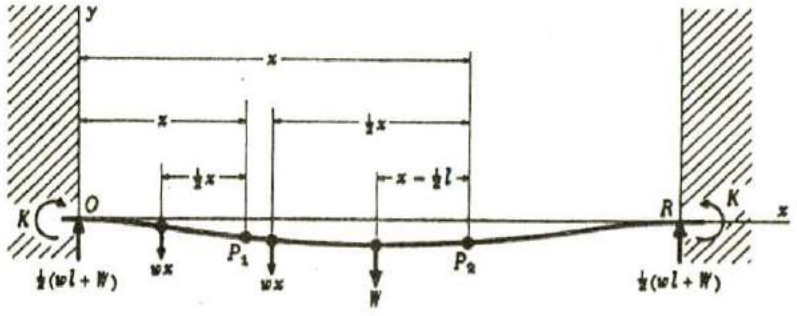
$$Kl + \tfrac{1}{4}wl^3 - \tfrac{1}{6}wl^3 = 0, \quad K = -\frac{1}{12}wl^2, \quad \text{and} \quad EI\frac{dy}{dx} = -\frac{1}{12}wl^2x + \frac{1}{4}wlx^2 - \frac{1}{6}wx^3.$$

Integrating and using $x = y = 0$ at O,

$$EIy = -\frac{1}{24}wl^2x^2 + \frac{1}{12}wlx^3 - \frac{1}{24}wx^4 \quad \text{and} \quad y = \frac{wx^2}{24EI}(2lx - l^2 - x^2).$$

The maximum deflection, occurring at the middle of the beam ($x = \frac{1}{2}l$), is $-y_{max} = \dfrac{wl^4}{384EI}$.

20. Solve Problem 19 when in addition there is a weight W N at the middle of the beam.



Using the coordinate system of Problem 19, there are two cases to be considered: from $x = 0$ to $x = \frac{1}{2}l$ and from $x = \frac{1}{2}l$ to $x = l$.

When $0 < x < \frac{1}{2}l$, the external forces on the segment to the left of $P_1(x, y)$ are: a couple of unknown moment K at O; an upward thrust of $\frac{1}{2}(wl + W)$ N at O, x m from P_1; and the load wx N, $\frac{1}{2}x$ m from P_1. Thus,

$$EI\frac{d^2y}{dx^2} = K + \frac{1}{2}(wl + W)x - \frac{1}{2}wx^2 = K + \frac{1}{2}wlx - \frac{1}{2}wx^2 + \frac{1}{2}Wx.$$

Integrating once and using $x = 0$, $dy/dx = 0$, at O,

A′) $$EI\frac{dy}{dx} = Kx + \frac{1}{4}wlx^2 - \frac{1}{6}wx^3 + \frac{1}{4}Wx^2.$$

Integrating and using $x = y = 0$ at O,

A) $$EIy = \frac{1}{2}Kx^2 + \frac{1}{12}wlx^3 - \frac{1}{24}wx^4 + \frac{1}{12}Wx^3.$$

When $\frac{1}{2}l < x < l$, there is in addition the weight W N at the middle of the beam, $(x - \frac{1}{2}l)$ m from P_2. Thus,

$$EI\frac{d^2y}{dx^2} = K + \frac{1}{2}wlx - \frac{1}{2}wx^2 + \frac{1}{2}Wx - W(x - \tfrac{1}{2}l).$$ Integrating twice,

$$B') \qquad EIy = \frac{1}{2}Kx^2 + \frac{1}{12}wlx^3 - \frac{1}{24}wx^4 + \frac{1}{12}Wx^3 - \frac{1}{6}W(x - \tfrac{1}{2}l)^3 + C_1x + C_2.$$

When $x = \frac{1}{2}l$, the values of y and dy/dx for $B')$ must agree respectively with those for $A)$. Thus, $C_1 = C_2 = 0$ and

$$B) \qquad EIy = \frac{1}{2}Kx^2 + \frac{1}{12}wlx^3 - \frac{1}{24}wx^4 + \frac{1}{12}Wx^3 - \frac{1}{6}W(x - \tfrac{1}{2}l)^3.$$

To determine K, use $x = \frac{1}{2}l$, $dy/dx = 0$ in $A')$. Then

$\frac{1}{2}lK + \frac{1}{16}wl^3 - \frac{1}{48}wl^3 + \frac{1}{16}Wl^2 = 0$ and $K = -\frac{1}{12}wl^2 - \frac{1}{8}Wl$. Substituting in $A)$ and $B)$,

$$EIy = -\frac{1}{24}wl^2x^2 + \frac{1}{12}wlx^3 - \frac{1}{24}wx^4 + \frac{1}{12}Wx^3 - \frac{1}{16}Wlx^2 \quad \text{and}$$

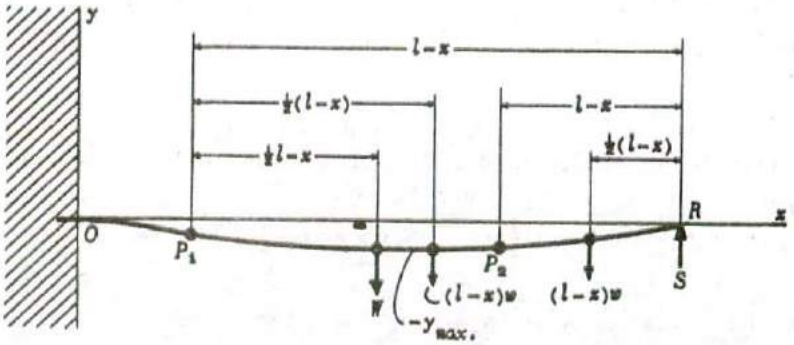
$$y = \frac{w}{24EI}(2lx^3 - l^2x^2 - x^4) + \frac{W}{48EI}(4x^3 - 3lx^2), \qquad 0 \leq x \leq \tfrac{1}{2}l,$$

$$EIy = -\frac{1}{24}wl^2x^2 + \frac{1}{12}wlx^3 - \frac{1}{24}wx^4 + \frac{1}{12}Wx^3 - \frac{1}{6}W(x - \tfrac{1}{2}l)^3 - \frac{1}{16}Wlx^2 \quad \text{and}$$

$$y = \frac{w}{24EI}(2lx^3 - l^2x^2 - x^4) + \frac{W}{48EI}(l^3 - 6l^2x + 9lx^2 - 4x^3), \qquad \tfrac{1}{2}l \leq x \leq l.$$

The maximum deflection, occurring at the middle of the beam, is $-y_{max} = \frac{1}{384EI}(wl^4 + 2Wl^3)$.

21. A horizontal beam of length l m is fixed at one end and freely supported at the other end. *(a)* Find the equation of the elastic curve if the beam carries a uniform load w N m^{-1} of length and a weight W N at the middle. *(b)* Locate the point of maximum deflection when $l = 3$ and $W = 10w$.



Take the origin at the fixed end and let P have coordinates (x, y). There are two cases to be considered.

When $0 < x < \frac{1}{2}l$, the external forces acting on the segment P_1R are: an unknown upward thrust S N at R, $(l - x)$ m from P_1; the load $w(l - x)$ N at the midpoint of P_1R, $\frac{1}{2}(l - x)$ m from P_1; and W N, $(\frac{1}{2}l - x)$ m from P_1. Thus,

$$EI\frac{d^2y}{dx^2} = S(l - x) - w(l - x)\cdot\tfrac{1}{2}(l - x) - W(\tfrac{1}{2}l - x) = S(l - x) - \tfrac{1}{2}w(l - x)^2 - W(\tfrac{1}{2}l - x).$$

Integrating once and using $x = 0$, $dy/dx = 0$ at O,

$$EI\frac{dy}{dx} = -\frac{1}{2}S(l - x)^2 + \frac{1}{6}w(l - x)^3 + \frac{1}{2}W(\tfrac{1}{2}l - x)^2 + \frac{1}{2}Sl^2 - \frac{1}{6}wl^3 - \frac{1}{8}Wl^2.$$

Integrating again and using $x = y = 0$ at O,

$$A)\quad EIy = \frac{1}{6}S(l-x)^3 - \frac{1}{24}w(l-x)^4 - \frac{1}{6}W(\tfrac{1}{2}l-x)^3 + (\frac{1}{2}Sl^2 - \frac{1}{6}wl^3 - \frac{1}{8}Wl^2)x - \frac{1}{6}Sl^3 + \frac{1}{24}wl^4 + \frac{1}{48}Wl^3.$$

When $\frac{1}{2}l < x < l$, the forces acting on P_2R are the unknown upward thrust S at R, $(l-x)$ m from P_2 and the load $w(l-x)$ N, $\frac{1}{2}(l-x)$ m from P_2. Thus,

$$EI\frac{d^2y}{dx^2} = S(l-x) - \frac{1}{2}w(l-x)^2 \qquad \text{and}$$

$$B')\qquad EIy = \frac{1}{6}S(l-x)^3 - \frac{1}{24}w(l-x)^4 + C_1x + C_2.$$

When $x = \frac{1}{2}l$, the values of EIy and $EI\frac{dy}{dx}$ as given by $A)$ and $B')$ must agree. Hence, C_1 and C_2 in $B')$ have the values of the corresponding constants of integration found in determining $A)$, and $B')$ becomes

$$B)\quad EIy = \frac{1}{6}S(l-x)^3 - \frac{1}{24}w(l-x)^4 + (\frac{1}{2}Sl^2 - \frac{1}{6}wl^3 - \frac{1}{8}Wl^2)x - \frac{1}{6}Sl^3 + \frac{1}{24}wl^4 + \frac{1}{48}Wl^3.$$

To determine S, use $x = l$, $y = 0$ at R in $B)$; then $S = \frac{3}{8}wl + \frac{5}{16}W$. Making this replacement in $A)$ and $B)$,

$$y = \frac{w}{48EI}(5lx^3 - 3l^2x^2 - 2x^4) + \frac{W}{96EI}(11x^3 - 9lx^2), \qquad 0 \leq x \leq \tfrac{1}{2}l, \qquad \text{and}$$

$$y = \frac{w}{48EI}(5lx^3 - 3l^2x^2 - 2x^4) + \frac{W}{96EI}(2l^3 - 12l^2x + 15lx^2 - 5x^3), \qquad \tfrac{1}{2}l \leq x \leq l.$$

It is clear that the maximum deflection occurs to the right of the midpoint of the beam. When $l = 3$, $W = 10w$, the equation immediately above becomes

$$y = \frac{w}{48EI}(-2x^4 - 10x^3 + 198x^2 - 540x + 270).$$

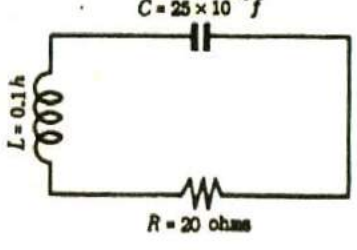
Since $\frac{dy}{dx} = 0$ at the point of maximum deflection, we solve

$$8x^3 + 30x^2 - 396x + 540 = 0$$

for the real root $x = 1.67$, approximately. Thus, the maximum deflection occurs at the point approximately 1.67 m from the fixed end.

ELECTRIC CIRCUITS.

22. An electric circuit consists of an inductance of 0.1 henry a resistance of 20 ohms and a condenser of capacitance 25 microfarads (1 microfarad = 10^{-6} farad). Find the charge q and the current i at time t, given the initial conditions *(a)* $q = 0.05$ coulomb, $i = dq/dt = 0$ when $t = 0$,
(b) $q = 0.05$ coulomb, $i = -0.2$ ampere when $t = 0$.



Since $L = 0.1$, $R = 20$, $C = 25\cdot10^{-6}$, $E(t) = 0$,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

reduces to

$$\frac{d^2q}{dt^2} + 200\frac{dq}{dt} + 400{,}000q = 0.$$

Integrating, $q = e^{-100t}(A \cos 100\sqrt{39}\,t + B \sin 100\sqrt{39}\,t)$.

Differentiating once with respect to t,

$$i = \frac{dq}{dt} = 100e^{-100t}[(\sqrt{39}\,B - A)\cos 100\sqrt{39}\,t - (\sqrt{39}\,A + B)\sin 100\sqrt{39}\,t].$$

a) Using the initial conditions $q = 0.05$, $i = 0$ when $t = 0$, $A = 0.05$ and $B = \frac{0.05}{\sqrt{39}} = 0.008$.

Hence, $q = e^{-100t}(0.05 \cos 624.5t + 0.008 \sin 624.5t)$

and $i = -0.32e^{-100t} \sin 624.5t$.

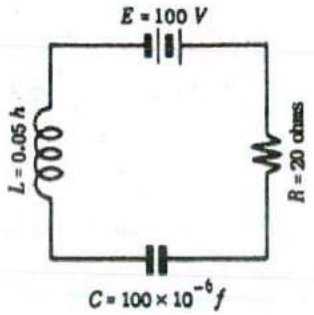
b) Using the initial conditions $q = 0.05$, $i = -0.2$ when $t = 0$, $A = 0.05$ and $B = 0.0077$.

Hence, $q = e^{-100t}(0.05 \cos 624.5t + 0.0077 \sin 624.5t)$

and $i = e^{-100t}(-0.2 \cos 624.5t - 32.0 \sin 624.5t)$

Note that q and i are transients, each becoming negligible very quickly.

23. A circuit consists of an inductance of 0.05 henry, a resistance of 20 ohms, a condenser of capacitance 100 microfarads, and an emf of $E = 100$ volts. Find i and q, given the initial conditions $q = 0$, $i = 0$ when $t = 0$.



Here $0.05\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + \frac{q}{100 \cdot 10^{-6}} = 100$

or $\frac{d^2q}{dt^2} + 400\frac{dq}{dt} + 200{,}000q = 2000.$

Integrating, $q = e^{-200t}(A \cos 400t + B \sin 400t) + 0.01.$

Differentiating once with respect to t,

$$i = \frac{dq}{dt} = 200e^{-200t}[(-A + 2B)\cos 400t + (-B - 2A)\sin 400t].$$

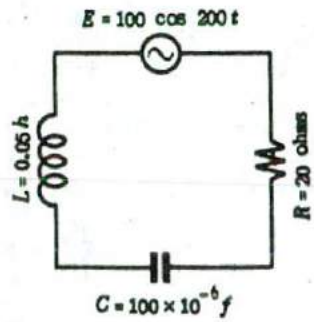
Using the initial conditions: $A = -0.01$, $-A + 2B = 0$, and $B = -0.005$.

Then $q = e^{-200t}(-0.01 \cos 400t - 0.005 \sin 400t) + 0.01$

and $i = 5e^{-200t} \sin 400t$.

Here i becomes negligible very soon while q, for all purposes, becomes $q = 0.01$.

24. Solve Problem 23 assuming that there is a variable emf of $E(t) = 100 \cos 200t$.



The differential equation is now

$$\frac{d^2q}{dt^2} + 400\frac{dq}{dt} + 200{,}000q = 2000 \cos 200t.$$ Then

$$q = e^{-200t}(A \cos 400t + B \sin 400t) + 0.01 \cos 200t + 0.005 \sin 200t$$ and

$$i = e^{-200t}[(-200A + 400B)\cos 400t + (-200B - 400A)\sin 400t] - 2 \sin 200t + \cos 200t.$$

Using the initial conditions: $A = -0.01$, $-200A + 400B + 1 = 0$ and $B = -0.0075$. Then

$$q = e^{-200t}(-0.01\cos 400t - 0.0075\sin 400t) + 0.01\cos 200t + 0.005\sin 200t$$

and

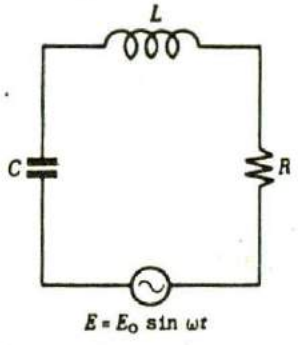
$$i = e^{-200t}(-\cos 400t + 5.5\sin 400t) - 2\sin 200t + \cos 200t.$$

Here the transient parts of q and i very quickly become negligible. For this reason, when the transients may be neglected, one needs find only the steady-state solutions

$$q = 0.01\cos 200t + 0.005\sin 200t \quad \text{and} \quad i = \cos 200t - 2\sin 200t.$$

The frequency $200/2\pi$ cycles/sec of the steady-state solutions is equal to the frequency of the applied emf. (See also Problem 25.)

25. For a circuit consisting of an inductance L, a resistance R, a capacitance C, and an emf $E(t) = E_0 \sin \omega t$, derive the formula for the steady state current



$$i = \frac{E_0}{Z}\left(\frac{R}{Z}\sin\omega t - \frac{X}{Z}\cos\omega t\right) = \frac{E_0}{Z}\sin(\omega t - \theta),$$

where $X = L\omega - \dfrac{1}{C\omega}$, $Z = \sqrt{X^2 + R^2}$, and θ is determined from $\sin\theta = \dfrac{X}{Z}$ and $\cos\theta = \dfrac{R}{Z}$.

By differentiating $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = E_0 \sin\omega t$

and using $i = \dfrac{dq}{dt}$, we obtain

1) $$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = (LD^2 + RD + 1/C)i = \omega E_0 \cos\omega t.$$

The required steady-state solution is the particular integral of 1):

$$i = \frac{\omega E_0}{LD^2 + RD + 1/C}\cos\omega t = \frac{\omega E_0}{RD - (L\omega - \frac{1}{C\omega})\omega}\cos\omega t$$

$$= \frac{\omega E_0 (RD + X\omega)}{R^2D^2 - X^2\omega^2}\cos\omega t = \frac{E_0}{R^2 + X^2}(R\sin\omega t - X\cos\omega t)$$

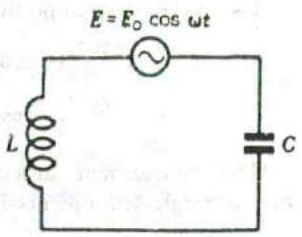
$$= \frac{E_0}{Z}\left(\frac{R}{Z}\sin\omega t - \frac{X}{Z}\cos\omega t\right) = \frac{E_0}{Z}\sin(\omega t - \theta).$$

X is called the *reactance* of the circuit; when $X = 0$ the amplitude of i is greatest (the circuit is in resonance). Z, called the *impedance* of the circuit, is also the ratio of the amplitudes of the emf and the current. θ is called the phase angle.

At times $t = \pi/2\omega, 3\pi/2\omega, \cdots\cdots$ the emf attains maximum amplitude, while at times given by $\omega t - \theta = \pi/2, 3\pi/2, \cdots\cdots$, that is, when $t = \dfrac{\pi/2 + \theta}{\omega}, \dfrac{3\pi/2 + \theta}{\omega}, \cdots\cdots$ the current attains maximum amplitude. Thus the voltage *leads* the current by a time θ/ω or the current and voltage are out of phase by the phase angle θ.

Note that $\theta = 0$ when $X = 0$ that is, $\theta = 0$ if there is resonance.

26. The circuit consisting of an inductance L, a condenser of capacitance C, and an emf E is known as an harmonic oscillator. Find q and i when $E = E_0 \cos \omega t$ and the initial conditions are $q = q_0$, $i = i_0$ when $t = 0$.



Since $R = 0$, the differential equation is

$$\frac{d^2q}{dt^2} + \frac{q}{CL} = \frac{E_0}{L} \cos \omega t.$$

There are two cases to be considered:

$$(a)\ \omega \neq \frac{1}{\sqrt{CL}} \qquad \text{and} \qquad (b)\ \omega = \frac{1}{\sqrt{CL}}.$$

a) $$q = A \cos \frac{1}{\sqrt{CL}} t + B \sin \frac{1}{\sqrt{CL}} t + \frac{E_0}{L} \frac{1}{D^2 + 1/CL} \cos \omega t$$

$$= A \cos \frac{1}{\sqrt{CL}} t + B \sin \frac{1}{\sqrt{CL}} t + \frac{E_0 C}{1 - \omega^2 CL} \cos \omega t$$

and $$i = \frac{1}{\sqrt{CL}} \left(-A \sin \frac{1}{\sqrt{CL}} t + B \cos \frac{1}{\sqrt{CL}} t\right) - \frac{E_0 C \omega}{1 - \omega^2 CL} \sin \omega t.$$

Using the initial conditions: $A = q_0 - \dfrac{E_0 C}{1 - \omega^2 CL}$ and $B = \sqrt{CL}\, i_0$. Then

$$q = \left(q_0 - \frac{E_0 C}{1 - \omega^2 CL}\right) \cos \frac{1}{\sqrt{CL}} t + \sqrt{CL}\, i_0 \sin \frac{1}{\sqrt{CL}} t + \frac{E_0 C}{1 - \omega^2 CL} \cos \omega t$$

and $$i = i_0 \cos \frac{1}{\sqrt{CL}} t - \frac{1}{\sqrt{CL}} \left(q_0 - \frac{E_0 C}{1 - \omega^2 CL}\right) \sin \frac{1}{\sqrt{CL}} t - \frac{E_0 C \omega}{1 - \omega^2 CL} \sin \omega t.$$

b) Here $$\frac{d^2q}{dt^2} + \omega^2 q = \frac{E_0}{L} \cos \omega t.$$

Then $$q = A \cos \omega t + B \sin \omega t + \frac{E_0}{2L\omega} t \sin \omega t$$

and $$i = \omega(-A \sin \omega t + B \cos \omega t) + \frac{E_0}{2L} \left(\frac{1}{\omega} \sin \omega t + t \cos \omega t\right).$$

Using the initial conditions: $A = q_0$ and $B = i_0/\omega$.

Then $$q = q_0 \cos \omega t + \frac{i_0}{\omega} \sin \omega t + \frac{E_0}{2L\omega} t \sin \omega t$$

and $$i = i_0 \cos \omega t - q_0 \omega \sin \omega t + \frac{E_0}{2L} \left(\frac{1}{\omega} \sin \omega t + t \cos \omega t\right).$$

Note that in *(b)* the frequency of the emf is the natural frequency of the oscillator, that is, the frequency when there is no emf. The circuit is in resonance since the reactance $X = L\omega - \dfrac{1}{C\omega} = 0$ when $\omega = \dfrac{1}{\sqrt{CL}}$.

The presence of the term $\dfrac{E_0 t}{2L} \cos \omega t$, whose amplitude increases with t, indicates that eventually such a circuit will be destroyed.

SUPPLEMENTARY PROBLEMS

27. Determine the curve for which the radius of curvature is proportional to the slope of the tangent.

Ans. $y = \pm\left(\sqrt{k^2-(x+C_1)^2} + k \ln\dfrac{\sqrt{k^2-(x+C_1)^2}-k}{x+C_1}\right) + C_2$

28. A 15 cm pendulum is released with a velocity of 1/2 rad/sec, toward the vertical, from a position 1/5 rad from the vertical. Find the equation of motion. (Assume that $g = 9{\cdot}6\ \mathrm{ms}^{-2}$)

Ans. $\theta = \dfrac{1}{5}\cos 8t - \dfrac{1}{16}\sin 8t$

29. A particular of mass m is repelled from O with a force equal to $k>0$ times the distance from O. If the particle starts from rest at a distance a from O, find its position t sec later.

Ans. $x = \frac{1}{2}a(e^{\sqrt{k/m}\,t} + e^{-\sqrt{k/m}\,t})$

30. If, in Problem 29, $k = m$ and $a = 4$ m determine *a)* the distance from O and the velocity when $t = 2$ sec, *b)* when it will be 6 m from O and its velocity then.

Ans. *a)* $x = 15$ m $14{\cdot}5\ \mathrm{ms}^{-1}$ *b)* $t = 0.96$ sec, $v = 4.47\ \mathrm{ms}^{-1}$

31. A chain hangs over a smooth peg, 2 m on one side and 3 m on the other. If the force of friction is equal to the weight of 0·25 m of chain, find the time required for it to slide off.

Ans. $\dfrac{1}{\sqrt{0{\cdot}4g}} \ln\dfrac{1}{3}(19+4\sqrt{22})$ sec

32. When the inner of two concentric spheres of radii r_1 and r_2, $r_1 < r_2$, carries an electric charge, the differential equation for the potential V at any point between the two spheres at a distance r from their common centre is

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$$

Solve for V given $V = V_1$ when $r = r_1$ and $V = V_2$ when $r = r_2$.

Ans. $V = \dfrac{V_2 r_2(r-r_1) - V_1 r_1(r-r_2)}{r(r_2-r_1)}$

33. A spring is such that it would be stretched 6 cm by a mass of 4 kg. A mass of 10 kg is attached and brought to rest. Find the equation of the motion if the mass is then

a) pulled down 8 cm and released.

b) pulled down 4 cm and given an upward velocity of $0{\cdot}5\ \mathrm{ms}^{-1}$

c) pulled down 6 cm and given a downward velocity of $1\ \mathrm{ms}^{-1}$

d) pushed up 6 cm and released.

e) pushed up 8 cm and given an upward velocity of $1{\cdot}25\ \mathrm{ms}^{-1}$

Ans. *a)* $x = 0{,}08\cos 7t$, *b)* $x = 0{,}04\cos 7t - 0{,}071\sin 7t$, *c)* $x = 0{,}06\cos 7t + 0{,}143\sin 7t$,

d) $x = -0{,}06\cos 7t$, *e)* $x = -0{,}08\cos 7t - 0{,}178\sin 7t$.

34. A spring is such that it would be stretched 72 mm by a mass of 15 kg. A mass of 30 kg is attached and brought to rest. The resistance of the medium is numerically equal to 120 dx/dt N. Find the equation of the motion of the weight if

a) it is started downward with velocity 3 ms^{-1}

b) it is pulled down 140 mm and given an upward velocity of 3 ms^{-1}

Ans. *a)* $x = \frac{3}{8}e^{-2t}\sin 8t$, *b)* $x = e^{-2t}(0,14\cos 8t - 0,34\sin 8t)$.

35. A spring is such that it would be stretched 77 mm by a mass of 1·5 kg. A mass of 1·5 kg is attached and brought to rest. The mass is then pulled down 4 cm and released. Determine the equation of motion if

a) an impressed force 7 sin 6 t acts on the spring. *Ans.* $x = 0,04\cos 8t - \frac{1}{8}\sin 8t + \frac{1}{6}\sin 6t$

b) an impressed force 8 sin 8t acts on the spring. *Ans.* $x = 0,04\,(1 - 12,5t)\cos 8t + \frac{1}{16}\sin 8t$

36. A beam of length l m is fixed at one end and otherwise unsupported. Find the equation of the elastic curve and the maximum deflection if there is a uniform load of w N m^{-1} of length and a load W N at the free end.

Ans. $y = \frac{w}{24EI}(4lx^3 - 6l^2x^2 - x^4) + \frac{W}{6EI}(x^3 - 3lx^2)$, $-y_{max} = \frac{1}{24EI}(3wl^4 + 8Wl^3)$

37. A beam of length $2l$ m is freely supported at both ends and carries a uniform load of w N m^{-1} of length. Taking the origin at the midpoint (low point) of the beam, find the equation of the elastic curve and the maximum deflection. Compare with Problem 15.

Hint: $EIy'' = wl(l-x) - \frac{1}{2}w(l-x)^2 = \frac{1}{2}w(l^2 - x^2)$ and $y = y' = 0$ when $x = 0$.

Ans. $y = \frac{w}{24EI}(6l^2x^2 - x^4)$, $y_{max} = \frac{5wl^4}{24EI}$

38. A beam of length 3 m is freely supported at both ends. There is a uniform load of w N m^{-1} of length and loads of W N at a distance $\frac{1}{2}$ m from each end. Taking the origin as in Problem 37, find the maximum deflection.

Hint $M = \frac{w}{2}(\frac{9l^2}{4} - x^2) + W(\frac{3l}{2} - x)$, $\frac{l}{2} < x < \frac{3l}{2}$; and $M = \frac{w}{2}(\frac{9l^2}{4} - x^2) + Wl$, $0 < x < \frac{l}{2}$.

Ans. $y_{max} = \frac{1}{384EI}(405wl^4 + 368Wl^3)$

39. A circuit consists of an inductance of 0.05 henry, a resistance of 5 ohms, and a condenser of capacitance $4(10)^{-4}$ farad. If $q = i = 0$ when $t = 0$, find q and i in terms of t when *a)* there a constant emf 110 volts, *b)* there is an alternating emf 200 cos 100 t. Find the steady-state solutions in *b)*.

Ans. *a)* $q = e^{-50t}(-\frac{11}{250}\cos 50\sqrt{19}t - \frac{11\sqrt{19}}{4750}\sin 50\sqrt{19}t) + \frac{11}{250}$, $i = \frac{44\sqrt{19}}{19}e^{-50t}\sin 50\sqrt{19}t$

b) $q = e^{-50t}(-\frac{16}{170}\cos 50\sqrt{19}t - \frac{12\sqrt{19}}{1615}\sin 50\sqrt{19}t) + \frac{4}{170}(4\cos 100t + \sin 100t)$,

$i = e^{-50t}(-\frac{40}{17}\cos 50\sqrt{19}t + \frac{1640\sqrt{19}}{323}\sin 50\sqrt{19}t) + \frac{40}{17}(\cos 100t - 4\sin 100t)$

40. Solve Problem 39 after replacing the 5 ohm resistance with a 50 ohm resistance.

Ans. *a)* $q = -0.047e^{-55t} + 0.0026e^{-947t} + 0.044$, $i = 2.46(e^{-55t} - e^{-947t})$

b) $q = -0.018e^{-55t} + 0.005e^{-947t} + 0.034\sin 100t + 0.014\cos 100t$,

$i = 0.98e^{-55t} - 4.43e^{-947t} + 3.45\cos 100t - 1.38\sin 100t$

CHAPTER 21

Systems of Simultaneous Linear Equations

IN PREVIOUS CHAPTERS, differential equations involving only two variables have been treated. In the next several chapters, equations involving more than two variables will be considered. If but one of the variables is independent, the equations are ordinary differential equations; if more of the variables are independent, the equations are called partial differential equations. In this chapter we shall be concerned with systems of ordinary linear differential equations with *constant* coefficients such as

$$A)\quad \begin{cases} 2\dfrac{dx}{dt} + \dfrac{dy}{dt} - 4x - y = e^t \\ \dfrac{dx}{dt} + 3x + y = 0 \end{cases} \qquad \text{or} \quad A')\quad \begin{cases} 2(D-2)x + (D-1)y = e^t \\ (D+3)x + y = 0, \end{cases} \text{ where } D = \frac{d}{dt}$$

and

$$B)\quad \begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} + y = 1 \\ \dfrac{dx}{dt} - \dfrac{dz}{dt} + 2x + z = 1 \\ \dfrac{dy}{dt} + \dfrac{dz}{dt} + y + 2z = 0 \end{cases} \qquad \text{or} \quad B')\quad \begin{cases} Dx + (D+1)y = 1 \\ (D+2)x - (D-1)z = 1 \\ (D+1)y + (D+2)z = 0 \end{cases}$$

in which the number of simultaneous equations is equal to the number of dependent variables.

THE BASIC PROCEDURE for solving a system of n ordinary differential equations in n dependent variables consists in obtaining, by differentiating the given equations, a set from which all but one of the dependent variables, say x, can be eliminated. The equation resulting from the elimination is then solved for this variable x. Each of the dependent variables is obtained in a similar manner.

EXAMPLE. Consider system $A)$: 1) $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - 4x - y = e^t$, 2) $\dfrac{dx}{dt} + 3x + y = 0$.

Solution.

First, we note that the general solution $x = x(t)$, $y = y(t)$ of this system will also satisfy

3) $$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + \frac{dy}{dt} = 0$$

obtained by differentiating 2). Moreover, multiplying 1) by -1, 2) by -1, 3) by 1, and adding, we obtain

4) $$\frac{d^2x}{dt^2} + x = -e^t$$

which is also satisfied by $x = x(t)$, $y = y(t)$. This latter differential equation, being free of y and its derivatives, may be solved readily; thus,

$$x = C_1 \cos t + C_2 \sin t - \frac{1}{D^2+1} e^t = C_1 \cos t + C_2 \sin t - \tfrac{1}{2}e^t.$$

To find y in a similar manner, we differentiate 1) to obtain

5) $$2\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} - 4\frac{dx}{dt} - \frac{dy}{dt} = e^t$$

and between this and equations 1), 2), 3) eliminate x and its derivatives. However, it is simpler here to proceed as follows. From 2), we have

$$y = -\frac{dx}{dt} - 3x = -(-C_1 \sin t + C_2 \cos t - \tfrac{1}{2}e^t) - 3(C_1 \cos t + C_2 \sin t - \tfrac{1}{2}e^t)$$

$$= (C_1 - 3C_2)\sin t - (3C_1 + C_2)\cos t + 2e^t.$$

Thus, $x = C_1 \cos t + C_2 \sin t - \frac{1}{2}e^t$, $y = (C_1 - 3C_2)\sin t - (3C_1 + C_2)\cos t + 2e^t$ is the general solution.

When the equations are written in the D notation, there is a striking similarity between the procedures used here and the method of solving a system of n equations in n unknowns. This is due to the fact, noted in previous chapters, that the operator D may at times be treated as a variable (letter).

Solution 2. Consider the system A'): 1) $2(D-2)x + (D-1)y = e^t$

2) $(D+3)x + y = 0.$

Proceeding as in the case of two equations in two unknowns x and y, we multiply 2) by $D-1$. Actually, we operate on 2) with $D - 1 = (\frac{d}{dt} - 1)$, to get

$$(D-1)(D+3)x + (D-1)y = 0$$

and subtract 1) from it to obtain

$$[(D-1)(D+3) - 2(D-2)]x = -e^t \quad \text{or} \quad (D^2+1)x = -e^t.$$

Now this is 4) above as might have been anticipated, since operating on 2) with $D-1$ is equivalent to differentiating 2) and adding -1 times 2) as in the previous solution. The general solution is obtained as in Solution 1.

Solution 3. We may also effect a solution using determinants. From system A') we obtain

$$\begin{vmatrix} 2(D-2) & D-1 \\ D+3 & 1 \end{vmatrix} x = \begin{vmatrix} e^t & D-1 \\ 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 2(D-2) & D-1 \\ D+3 & 1 \end{vmatrix} y = \begin{vmatrix} 2(D-2) & e^t \\ D+3 & 0 \end{vmatrix}$$

or $$(D^2+1)x = -e^t \quad \text{and} \quad (D^2+1)y = 4e^t.$$

The first of these equations is 4) above, and the second would have been obtained by the procedure rejected in Solution 1. We shall now show why it was rejected. When the two equations are solved, we have

6) $x = C_1 \cos t + C_2 \sin t - \frac{1}{2}e^t$ and 7) $y = C_3 \cos t + C_4 \sin t + 2e^t.$

We know from Solution 1 that 6) and 7) contain extraneous solutions. To eliminate them (that is, to reduce the number of arbitrary constants), we substitute in 2) and see that

$$(C_2 + 3C_1 + C_3)\cos t + (3C_2 - C_1 + C_4)\sin t = 0$$

for every value of t. Thus,

$$C_3 = -(3C_1 + C_2) \quad \text{and} \quad C_4 = C_1 - 3C_2.$$

When these values are substituted in 6) and 7), we obtain the general solution found above.

THE NUMBER OF INDEPENDENT ARBITRARY CONSTANTS appearing in the general solution of the system

$$f_1(D)x + g_1(D)y = h_1(t)$$
$$f_2(D)x + g_2(D)y = h_2(t)$$

is equal to the degree in D of the determinant $\Delta = \begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix}$,

provided Δ does not vanish identically. If $\Delta = 0$, the system is dependent; such systems will not be considered here.

For the system A'), $\Delta = \begin{vmatrix} 2(D-2) & D-1 \\ D+3 & 1 \end{vmatrix} = -(D^2+1)$.

The degree (2) in D agrees with the number of arbitrary constants appearing in the general solution.

The theorem may be extended readily to the case of n equations in n dependent variables.

SOLVED PROBLEMS

1. Solve the system: 1) $(D-1)x + Dy = 2t + 1$
   2) $(2D+1)x + 2Dy = t$.

   Subtracting twice 1) from 2), we have $3x = -3t - 2$. Substituting $x = -t - 2/3$ in 1), we obtain $Dy = 2t + 1 - (D-1)x = t + \frac{4}{3}$ and $y = \frac{1}{2}t^2 + \frac{4}{3}t + C_1$.

   The complete solution is $x = -t - \frac{2}{3}$, $y = \frac{1}{2}t^2 + \frac{4}{3}t + C_1$.

   Note that $\begin{vmatrix} D-1 & D \\ 2D+1 & 2D \end{vmatrix}$ is of degree 1 in D and there is but one arbitrary constant.

2. Solve the system: 1) $(D+2)x + 3y = 0$
   2) $3x + (D+2)y = 2e^{2t}$.

   Operating on 1) with $D+2$, multiplying 2) by -3, and adding: $(D^2+4D-5)x = -6e^{2t}$.

   Then $x = C_1e^{t} + C_2e^{-5t} - \frac{6}{7}e^{2t}$. From 1), $y = -\frac{1}{3}(D+2)x = -C_1e^{t} + C_2e^{-5t} + \frac{8}{7}e^{2t}$.

3. Solve the system: 1) $(D-3)x + 2(D+2)y = 2 \sin t$
   2) $2(D+1)x + (D-1)y = \cos t$.

   Operating on 1) with $D-1$ and on 2) with $2(D+2)$, we have

   3) $(D-1)(D-3)x + 2(D-1)(D+2)y = (D-1)[2 \sin t] = 2 \cos t - 2 \sin t$
   4) $4(D+2)(D+1)x + 2(D+2)(D-1)y = 2(D+2)\cos t = 4 \cos t - 2 \sin t$.

   Subtracting 3) from 4) and noting that $(D-1)(D+2) = (D+2)(D-1)$, since the operators have constant coefficients,

$$[4(D^2+3D+2) - (D^2-4D+3)]x = (3D^2+16D+5)x = 2\cos t$$

and $$x = C_1e^{-5t} + C_2e^{-t/3} + \frac{2}{3D^2+16D+5}\cos t = C_1e^{-5t} + C_2e^{-t/3} + \frac{1}{8D+1}\cos t$$

$$= C_1e^{-5t} + C_2e^{-t/3} + (8\sin t + \cos t)/65.$$

From 2), $$(D-1)y = \cos t + 2(D+1)x$$

$$= \cos t + 8C_1e^{-5t} - \frac{4}{3}C_2e^{-t/3} - (18\cos t + 14\sin t)/65$$

$$= 8C_1e^{-5t} - \frac{4}{3}C_2e^{-t/3} + (47\cos t - 14\sin t)/65.$$

Then $$ye^{-t} = \int(8C_1e^{-6t} - \frac{4}{3}C_2e^{-4t/3} + \frac{47\cos t - 14\sin t}{65}e^{-t})dt$$

$$= -\frac{4}{3}C_1e^{-6t} + C_2e^{-4t/3} + \frac{61\sin t - 33\cos t}{130}e^{-t} + C_3$$

and $$y = -\frac{4}{3}C_1e^{-5t} + C_2e^{-t/3} + \frac{61\sin t - 33\cos t}{130} + C_3e^{t}.$$

Since the degree of Δ is 2, the general solution has but two arbitrary constants. Hence, when these expressions are substituted for x and y in 1) it is found that $C_3 = 0$. Then

$$x = C_1e^{-5t} + C_2e^{-t/3} + \frac{8\sin t + \cos t}{65}, \qquad y = -\frac{4}{3}C_1e^{-5t} + C_2e^{-t/3} + \frac{61\sin t - 33\cos t}{130}$$

is the general solution.

4. Solve the system: 1) $(D^2-2)x - 3y = e^{2t}$

2) $(D^2+2)y + x = 0.$

Find the particular solution satisfying the conditions $x = y = 1$, $Dx = Dy = 0$ when $t = 0$.

Operating on 1) with D^2 to obtain $D^4x - 2D^2x - 3D^2y = 4e^{2t}$ and making replacements $D^2x = 2x + 3y + e^{2t}$ from 1) and $D^2y = -x - 2y$ from 2), we have $(D^4-1)x = 6e^{2t}$

Then $x = C_1e^{t} + C_2e^{-t} + C_3\cos t + C_4\sin t + \frac{2}{5}e^{2t}$ and, using 1),

$$y = \frac{1}{3}[(D^2-2)x - e^{2t}] = -\frac{1}{3}(C_1e^{t} + C_2e^{-t}) - (C_3\cos t + C_4\sin t) - \frac{1}{15}e^{2t}.$$

Note that x could also be obtained by the use of determinants. Thus,

$$\begin{vmatrix} D^2-2 & -3 \\ 1 & D^2+2 \end{vmatrix} x = \begin{vmatrix} e^{2t} & 3 \\ 0 & D^2+2 \end{vmatrix} \quad \text{or} \quad (D^4-1)x = 6e^{2t}, \text{ etc.}$$

When $t = 0$, $x = C_1 + C_2 + C_3 + \frac{2}{5} = 1$ and $Dx = C_1 - C_2 + C_4 + \frac{4}{5} = 0$,

$$y = -\frac{1}{3}(C_1 + C_2) - C_3 - \frac{1}{15} = 1 \quad \text{and} \quad Dy = -\frac{1}{3}(C_1 - C_2) - C_4 - \frac{2}{15} = 0.$$

Then $C_1 = 3/4$, $C_2 = 7/4$, $C_3 = -19/10$, $C_4 = 1/5$, and the required particular solution is

$$x = \frac{1}{4}(3e^{t} + 7e^{-t}) - \frac{1}{10}(19\cos t - 2\sin t) + \frac{2}{5}e^{2t},$$

$$y = -\frac{1}{12}(3e^{t} + 7e^{-t}) + \frac{1}{10}(19\cos t - 2\sin t) - \frac{1}{15}e^{2t}.$$

5. Solve the system: 1) $(D+1)x + (D-1)y = e^t$, 2) $(D^2+D+1)x + (D^2-D+1)y = t^2$.

Operating on 1) with D^2+D+1 and on 2) with $D+1$, and subtracting, we have

$$2y = t^2 + 2t - 3e^t \quad \text{and} \quad y = \frac{1}{2}t^2 + t - \frac{3}{2}e^t.$$

Operating on 1) with D^2-D+1 and on 2) with $D-1$, and subtracting, we have

$$2x = t^2 - 2t + e^t \quad \text{and} \quad x = \frac{1}{2}t^2 - t + \frac{1}{2}e^t.$$

Note that $\begin{vmatrix} D+1 & D-1 \\ D^2+D+1 & D^2-D+1 \end{vmatrix} = 2$ is of degree 0 in D; hence, there are no arbitrary constants in the solution.

6. Solve the system: 1) $D^2x - m^2y = 0$, 2) $D^2y + m^2x = 0$.

Operating on 1) with D^2 and substituting $D^2y = -m^2x$ from 2), we obtain

$$D^4x - m^2(-m^2x) = D^4x + m^4x = (D^4 + m^4)x = 0. \text{ Then } D = \pm\frac{m}{\sqrt{2}}(1 \pm i)$$

and $x = e^{mt/\sqrt{2}}(C_1 \cos mt/\sqrt{2} + C_2 \sin mt/\sqrt{2}) + e^{-mt/\sqrt{2}}(C_3 \cos mt/\sqrt{2} + C_4 \sin mt/\sqrt{2})$.

Substituting for x in 1) and solving,

$$y = \frac{1}{m^2}D^2x = e^{mt/\sqrt{2}}(C_2 \cos mt/\sqrt{2} - C_1 \sin mt/\sqrt{2}) + e^{-mt/\sqrt{2}}(C_3 \sin mt/\sqrt{2} - C_4 \cos mt/\sqrt{2}).$$

7. Solve the system: 1) $(D^2+4)x - 3Dy = 0$, 2) $3Dx + (D^2+4)y = 0$.

Operating on 1) with D^2+4 and on 2) with $3D$, and adding, we have

$[(D^2+4)^2 + 9D^2]x = (D^2+16)(D^2+1)x = 0$ and $x = C_1 \cos 4t + C_2 \sin 4t + C_3 \cos t + C_4 \sin t$.

Operating on 1) with $-3D$ and on 2) with D^2+4, and adding, we have

$(D^2+16)(D^2+1)y = 0$ and $y = K_1 \cos 4t + K_2 \sin 4t + K_3 \cos t + K_4 \sin t$.

To eliminate the extraneous solutions, substitute for x and y in 1). We have

$$-12C_1 \cos 4t - 12C_2 \sin 4t + 3C_3 \cos t + 3C_4 \sin t + 12K_1 \sin 4t - 12K_2 \cos 4t + 3K_3 \sin t - 3K_4 \cos t = 0$$

for all values of t; thus, $K_1 = C_2$, $K_2 = -C_1$, $K_3 = -C_4$, $K_4 = C_3$.

The complete solution is: $x = C_1 \cos 4t + C_2 \sin 4t + C_3 \cos t + C_4 \sin t$,

$y = C_2 \cos 4t - C_1 \sin 4t - C_4 \cos t + C_3 \sin t$.

8. Solve the system:

1) $Dx + (D+1)y = 1$
2) $(D+2)x - (D-1)z = 1$
3) $(D+1)y + (D+2)z = 0$.

Subtracting 3) from 1), we have 4) $Dx - (D+2)z = 1$ which is free of y.
Operating on 2) with D and on 4) with $D+2$, and subtracting, we have $(5D+4)z = -2$; then $z = -\frac{1}{2} + C_1e^{-4t/5}$. Substituting for z in 3), $(D+1)y = -(D+2)z = 1 - \frac{6}{5}C_1e^{-4t/5}$; then

$$y = e^{-t}\int (e^{t} - \frac{6}{5}C_1 e^{t/5})dt = e^{-t}(e^{t} - 6C_1 e^{t/5} + C_2) = 1 - 6C_1 e^{-4t/5} + C_2 e^{-t}.$$

Substituting for y in 1), $Dx = 1 - (D+1)y = \frac{6}{5}C_1 e^{-4t/5}$; then $x = -\frac{3}{2}C_1 e^{-4t/5} + C_3$.

Since $\begin{vmatrix} D & D+1 & 0 \\ D+2 & 0 & -(D-1) \\ 0 & D+1 & D+2 \end{vmatrix} = -(5D^2 + 9D + 4)$ is of degree 2 in D, there are but two arbitrary constants in the general solution. Substituting for x and z in 2), we obtain

$(\frac{6}{5}C_1 e^{-4t/5} - 3C_1 e^{-4t/5} + 2C_3) - (-\frac{4}{5}C_1 e^{-4t/5} + \frac{1}{2} - C_1 e^{-4t/5}) = 1$ and, hence, $C_3 = \frac{3}{4}$. Thus,

$$x = \frac{3}{4} - \frac{3}{2}C_1 e^{-4t/5}, \quad y = 1 - 6C_1 e^{-4t/5} + C_2 e^{-t}, \quad z = -\frac{1}{2} + C_1 e^{-4t/5}$$

is the general solution.

9. Solve the system:

$$\begin{aligned} &1) \quad (D+1)^2 x + 2Dy + 3Dz = 1 \\ &2) \quad Dx + z = 0 \\ &3) \quad x - Dy - Dz = 0. \end{aligned}$$

Find the particular solution for which $x = z = 1$, $y = 0$ when $t = 0$.

First, operate on 2) with D to obtain 4) $D^2 x + Dz = 0$.

Next, add twice 3) to 1) and subtract 4) to get $(2D+3)x = 1$; then

$$xe^{3t/2} = \frac{1}{2}\int e^{3t/2}\,dt = \frac{1}{3}e^{3t/2} + C_1 \quad \text{and} \quad x = \frac{1}{3} + C_1 e^{-3t/2}.$$

From 2), $z = -Dx = \frac{3}{2}C_1 e^{-3t/2}$.

From 3), $Dy = x - Dz = \frac{1}{3} + C_1 e^{-3t/2} + \frac{9}{4}C_1 e^{-3t/2} = \frac{1}{3} + \frac{13}{4}C_1 e^{-3t/2}$; then

$$y = \frac{1}{3}t - \frac{13}{6}C_1 e^{-3t/2} + C_2.$$

Since $\begin{vmatrix} (D+1)^2 & 2D & 3D \\ D & 0 & 1 \\ 1 & -D & -D \end{vmatrix} = 2D^2 + 3D$ is of degree 2 in D, there are 2 arbitrary constants and the general solution is $x = \frac{1}{3} + C_1 e^{-3t/2}$, $y = \frac{1}{3}t - \frac{13}{6}C_1 e^{-3t/2} + C_2$, $z = \frac{3}{2}C_1 e^{-3t/2}$.

When $t = 0$: $x = \frac{1}{3} + C_1 = 1$ and $C_1 = \frac{2}{3}$; $y = (-\frac{13}{6})(\frac{2}{3}) + C_2 = 0$ and $C_2 = \frac{13}{9}$.

Thus, the required particular solution is

$$x = \frac{1}{3} + \frac{2}{3}e^{-3t/2}, \quad y = \frac{1}{3}t - \frac{13}{9}e^{-3t/2} + \frac{13}{9}, \quad z = e^{-3t/2}.$$

Note that a particular solution satisfying a given set of initial conditions cannot always be found. For example, there is no solution satisfying the conditions $x = 1$, $y = z = 0$ when $t = 0$ since $x = 1$, $y = 0$ contradicts $x = 1/3 + 2z/3$. Similarly, $y = 0$, $z = 1$, $dx/dt = 1$ when $t = 0$ contradicts $dx/dt = -z$.

SUPPLEMENTARY PROBLEMS

Solve the following simultaneous equations.

10. $Dx - (D+1)y = -e^t$
$x + (D-1)y = e^{2t}$

Ans. $x = (C_1 - C_2)\cos t + (C_1 + C_2)\sin t + 3e^{2t}/5$
$y = C_1\cos t + C_2\sin t + 2e^{2t}/5 + e^t/2$

11. $(D+2)x + (D+1)y = t$
$5x + (D+3)y = t^2$

$x = \frac{C_1 - 3C_2}{5}\sin t - \frac{3C_1 + C_2}{5}\cos t - t^2 + t + 3$
$y = C_1\cos t + C_2\sin t + 2t^2 - 3t - 4$

12. $(D+1)x + (2D+7)y = e^t + 2$
$-2x + (D+3)y = e^t - 1$

$x = \frac{1}{2}C_1e^{-4t}[\cos(t + C_2) - \sin(t + C_2)] - \frac{5e^t}{26} + \frac{13}{17}$
$y = C_1e^{-4t}\sin(t + C_2) + \frac{2e^t}{13} + \frac{3}{17}$

13. $(D-1)x + (D+3)y = e^{-t} - 1$
$(D+2)x + (D+1)y = e^{2t} + t$

$x = 2C_1e^{-7t/5} + \frac{5}{17}e^{2t} + \frac{3}{7}t - \frac{1}{49}$
$y = 3C_1e^{-7t/5} - \frac{1}{17}e^{2t} + \frac{1}{2}e^{-t} + \frac{1}{7}t - \frac{26}{49}$

14. $(D^2 + 16)x - 6Dy = 0$
$6Dx + (D^2 + 16)y = 0$

$x = C_1\cos 2t - C_2\sin 2t + C_3\cos 8t + C_4\sin 8t$
$y = C_2\cos 2t + C_1\sin 2t + C_4\cos 8t - C_3\sin 8t$

15. $(D^2 + 4)x + y = \sin^2 z$
$(D^2 + 1)y - 2x = \cos^2 z$

$x = C_1\cos(\sqrt{2}z + C_2) + C_3\cos(\sqrt{3}z + C_4) + \frac{1}{2}\cos 2z$
$y = -2C_1\cos(\sqrt{2}z + C_2) - C_3\cos(\sqrt{3}z + C_4) + \frac{1}{2} - \frac{1}{2}\cos 2z$

16. $(D^2 + D + 1)x + (D^2 + 1)y = e^t$
$(D^2 + D)x + D^2y = e^{-t}$

$x = -e^t - 2e^{-t} - C_1$
$y = 2e^t + e^{-t} + C_1$

17. $(D-1)x + (D+2)y = 1 + e^t$
$(D+2)y + (D+1)z = 2 + e^t$
$(D-1)x + (D+1)z = 3 + e^t$

$x = -1 + te^t/2 + C_2e^t$
$y = e^t/6 + C_1e^{-2t}$
$z = 2 + e^t/4 + C_3e^{-t}$

CHAPTER 22

Total Differential Equations

THE DIFFERENTIAL EQUATIONS

A) $(3x^2y^2 - e^x z)dx + (2x^3y + \sin z)dy + (y\cos z - e^x)dz = 0,$

B) $(3xz + 2y)dx + x\,dy + x^2\,dz = 0,$

C) $y\,dx + dy + dz = 0,$

being of the general form

$$P(x, y, z, \cdots, t)dx + Q(x, y, z, \cdots, t)dy + \cdots\cdots + S(x, y, z, \cdots, t)dt = 0,$$

are called *total differential equations.*

It may be verified readily that *A)* is the exact differential of

$$f(x, y, z) = x^3y^2 - e^x z + y\sin z = C,$$

C being an arbitrary constant. Such an equation is called *exact.*

Equation *B)* is not exact, but the use of x as an integrating factor yields

$$(3x^2z + 2xy)dx + x^2dy + x^3\,dz = 0$$

which is the exact differential of $x^3z + x^2y = C$. Equations *A)* and *B)* are called *integrable.*

Equation *C)* is not integrable; that is, no primitive

1) $$f(x, y, z) = C$$

can be found for it. It will be shown later (Problem 32) that for such equations a solution 1) can be obtained consistent with any prescribed relation $g(x, y, z) = 0$ of the variables.

THE CONDITION OF INTEGRABILITY of the total differential equation

2) $$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0$$

is

3) $$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0, \text{ identically.}$$ See Problem 1.

EXAMPLE 1. For equation *B)*,

$P = 3xz + 2y$, $\frac{\partial P}{\partial y} = 2$, $\frac{\partial P}{\partial z} = 3x$; $Q = x$, $\frac{\partial Q}{\partial x} = 1$, $\frac{\partial Q}{\partial z} = 0$; $R = x^2$, $\frac{\partial R}{\partial x} = 2x$, $\frac{\partial R}{\partial y} = 0$, and

3) becomes $(3xz + 2y)(0 - 0) + x(2x - 3x) + x^2(2 - 1) = 0 - x^2 + x^2 = 0$. The equation is integrable.

EXAMPLE 2. For equation *C)*,

$P = y$, $\frac{\partial P}{\partial y} = 1$, $\frac{\partial P}{\partial z} = 0$; $Q = 1$, $\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial z} = 0$; $R = 1$, $\frac{\partial R}{\partial x} = \frac{\partial R}{\partial y} = 0$, and 3) becomes

$y(0 - 0) + 1(0 - 0) + 1(1 - 0) \neq 0$. The equation is not integrable.

THE CONDITIONS FOR EXACTNESS of 2) are

4) $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

EXAMPLE 3. For equation *A)*,

$$P = 3x^2y^2 - e^x z, \quad \frac{\partial P}{\partial y} = 6x^2 y, \quad \frac{\partial P}{\partial z} = -e^x;$$

$$Q = 2x^3 y + \sin z, \quad \frac{\partial Q}{\partial x} = 6x^2 y, \quad \frac{\partial Q}{\partial z} = \cos z;$$

$$R = y\cos z - e^x, \quad \frac{\partial R}{\partial x} = -e^x, \quad \frac{\partial R}{\partial y} = \cos z,$$

and the conditions 4) are satisfied. The equation is exact.

EXAMPLE 4. From Example 1 it is readily seen that 4) is not satisfied; hence, equation *B)* is not exact.

TO SOLVE AN INTEGRABLE TOTAL DIFFERENTIAL EQUATION in three variables:

a) If 2) is exact, the solution is evident after, at most, a regrouping of terms. See Problem 3.

b) If 2) is not exact, it may be possible to find an integrating factor. See Problems 4–6.

c) If 2) is homogeneous, one variable, say z, can be separated from the others by the transformation $x = uz$, $y = vz$. See Problems 7–10.

d) If no integrating factor can be found, consider one of the variables, say z, as a constant. Integrate the resulting equation, denoting the arbitrary constant of integration by $\phi(z)$. Take the total differential of the integral just obtained and compare the coefficients of its differentials with those of the given differential equation, thus determining $\phi(z)$. This procedure is illustrated in Problem 13. See also Problems 14–16.

PAIRS OF TOTAL DIFFERENTIAL EQUATIONS IN THREE VARIABLES. The solution of the simultaneous total differential equations

5) $$P_1\,dx + Q_1\,dy + R_1\,dz = 0$$

6) $$P_2\,dx + Q_2\,dy + R_2\,dz = 0$$

consists of a pair of relations

7) $$f(x, y, z) = C_1$$

8) $$g(x, y, z) = C_2.$$

To solve a given pair of equations:

e) If 5) and 6) are both integrable, each may be solved by one or more of the procedures *a) – d)*. Then, 7), say is the complete solution (primitive) of 5), and 8) is the complete solution of 6). See Problem 18.

f) If 5) is integrable but 6) is not, then 7), say, is the complete solution of 5). To obtain 8), we use 5), 6), 7) to eliminate one variable and its differential, and integrate the resulting equation. See Problem 19.

g) If neither equation is integrable, we may use the method of chapter 21, treating two of the variables, say x and y, as functions of the third variable z.

At times it may be simpler to proceed as follows: Eliminate in turn dy and dz (or any other pair) between 5) and 6) to obtain

$$\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix} dx - \begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} dz = 0, \qquad \begin{vmatrix} R_1 & P_1 \\ R_2 & P_2 \end{vmatrix} dx - \begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} dy = 0$$

and express them in the symmetric form

9) $$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z},$$

where $$X = \lambda \begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix}, \quad Y = \lambda \begin{vmatrix} R_1 & P_1 \\ R_2 & P_2 \end{vmatrix}, \quad Z = \lambda \begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}, \quad \lambda \neq 0.$$

(Note that this is the procedure for obtaining the symmetric form of the equations of a straight line when the two-plane form is given.)

Of the three equations

9′) $$Y\,dx = X\,dy, \qquad X\,dz = Z\,dx, \qquad Z\,dy = Y\,dz$$

given by 9), any one may be obtained from the other two. Hence, in obtaining 9), we merely replace the original pair of differential equations by an equivalent pair, that is, any two of 9′).

If two of 9′) are integrable, we proceed as in *e)*. See Problem 20.

If but one of 9′) is integrable, we proceed as in *f)*. See Problem 21.

If no one of 9′) is integrable, we increase the number of possible equations. By a well known principle,

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = \frac{l_1\,dx + m_1\,dy + n_1\,dz}{l_1X + m_1Y + n_1Z} = \frac{l_2\,dx + m_2\,dy + n_2\,dz}{l_2X + m_2Y + n_2Z}$$

where the l, m, n arbitrary functions of the variables such that

$$lX + mY + nZ \neq 0.$$

By a proper choice of the multipliers, it may be possible to obtain an integrable equation, say

$$\frac{dy}{Y} = \frac{l\,dx + m\,dy + n\,dz}{lX + mY + nZ} \qquad \text{or} \qquad \frac{a\,dx + b\,dy + c\,dz}{aX + bY + cZ} = \frac{p\,dx + q\,dy + r\,dz}{pX + qY + rZ}.$$

If so, we proceed as in *f)*. See Problem 22.

In actual practice, it may be simpler at times to find by means of multipliers a second integrable equation rather than to proceed as in *f)*. See Problems 23–24.

If $lX + mY + nZ = 0$, then also $l\,dx + m\,dy + n\,dz = 0$.
If now $l\,dx + m\,dy + n\,dz = 0$ is integrable, we integrate and have one of the required relations.

See Problems 25–29.

SOLVED PROBLEMS

1. Obtain the condition of integrability of $P\,dx + Q\,dy + R\,dz = 0$.

Suppose that the given equation is obtained by differentiating

1) $$f(x,y,z) = C$$

and, perhaps, removing a common factor $\mu(x,y,z)$. Since from 1) $\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 0$, it follows that $\frac{\partial f}{\partial x} = \mu P$, $\frac{\partial f}{\partial y} = \mu Q$, and $\frac{\partial f}{\partial z} = \mu R$.

Now assuming the existence and continuity conditions,

$$A) \quad \frac{\partial^2 f}{\partial y\,\partial x} = \mu\frac{\partial P}{\partial y} + P\frac{\partial \mu}{\partial y} = \mu\frac{\partial Q}{\partial x} + Q\frac{\partial \mu}{\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$B) \quad \frac{\partial^2 f}{\partial z\,\partial y} = \mu\frac{\partial Q}{\partial z} + Q\frac{\partial \mu}{\partial z} = \mu\frac{\partial R}{\partial y} + R\frac{\partial \mu}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial z}$$

$$C) \quad \frac{\partial^2 f}{\partial x\,\partial z} = \mu\frac{\partial R}{\partial x} + R\frac{\partial \mu}{\partial x} = \mu\frac{\partial P}{\partial z} + P\frac{\partial \mu}{\partial z} = \frac{\partial^2 f}{\partial z\,\partial x}.$$

Upon multiplying these relations by R, P, Q, respectively and adding,

$$\mu\left(R\frac{\partial P}{\partial y} + P\frac{\partial Q}{\partial z} + Q\frac{\partial R}{\partial x}\right) = \mu\left(R\frac{\partial Q}{\partial x} + P\frac{\partial R}{\partial y} + Q\frac{\partial P}{\partial z}\right)$$

and the condition $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$ follows.

2. If $\mu(x,y,z) = 1$ in Problem 1, the differential equation is exact. Show that this implies

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

These relations follow from $A)$, $B)$, $C)$ in Probl. 1. For example, if $\mu = 1$, $A)$ yields $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

3. Solve $(x-y)\,dx - x\,dy + z\,dz = 0$.

Since $\frac{\partial P}{\partial y} = -1 = \frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x} = 0 = \frac{\partial P}{\partial z}$, the equation is exact.

Upon regrouping thus $x\,dx - (x\,dy + y\,dx) + z\,dz = 0$ and integrating, we have

$$\tfrac{1}{2}x^2 - xy + \tfrac{1}{2}z^2 = K \quad \text{or} \quad x^2 - 2xy + z^2 = C.$$

4. Solve $y^2\,dx - z\,dy + y\,dz = 0$.

Here $P = y^2$, $\frac{\partial P}{\partial y} = 2y$, $\frac{\partial P}{\partial z} = 0$; $Q = -z$, $\frac{\partial Q}{\partial x} = 0$, $\frac{\partial Q}{\partial z} = -1$; $R = y$, $\frac{\partial R}{\partial x} = 0$, $\frac{\partial R}{\partial y} = 1$;

then $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = y^2(-1-1) - z(0-0) + y(2y-0) = 0$ and the equation is integrable. The integrating factor $1/y^2$ reduces the equation to $dx + \frac{y\,dz - z\,dy}{y^2} = 0$ whose solution is $x + z/y = C$.

5. Solve $(2x^3y + 1)dx + x^4\,dy + x^2 \tan z\,dz = 0$.

The condition of integrability is satisfied since

$$(2x^3y + 1)(0 - 0) + x^4(2x \tan z - 0) + x^2 \tan z(2x^3 - 4x^3) = 0.$$

The integrating factor $1/x^2$ reduces the equation to

$$(2xy + \frac{1}{x^2})dx + x^2dy + \tan z\,dz = 0 \quad \text{or} \quad (2xy\,dx + x^2dy) + \frac{1}{x^2}dx + \tan z\,dz = 0$$

whose solution is $x^2y - \frac{1}{x} + \ln \sec z = C$.

6. Solve $(2x^3 - z)z\,dx + 2x^2yz\,dy + x(z + x)dz = 0$.

The normal procedure here would be to show that the equation is integrable and then to seek an integrating factor. By examining for the preceding problems it will be found that, upon using the integrating factor, one variable appears only in an exact differential, for example, the variable z in $\tan z\,dz$ in Problem 5.

When the equation of this problem is divided by x^2z, the variable y appears only in the term $2y\,dy$ which is an exact differential. Thus, we shall use $1/x^2z$ as a *possible* integrating factor. The result is

$$2x\,dx + 2y\,dy + \frac{1}{z}dz + \frac{x\,dz - z\,dx}{x^2} = 0 \qquad \text{whose solution is} \qquad x^2 + y^2 + \ln z + \frac{z}{x} = C.$$

Of course, the separation of the variable here does not indicate that the equation is integrable; for example, $x\,dx + z\,dy + dz = 0$ is not integrable although x appears only in an exact differential.

7. Show that if $P\,dx + Q\,dy + R\,dz = 0$ is homogeneous (*i.e.*, if P, Q, R are homogeneous and of the same degree) then the substitution $x = uz$, $y = vz$ will separate the variable z from the variables u and v.

Let the coefficients P, Q, R be of degree n in the variables.

Substituting $x = uz$, $y = vz$, the given equation becomes

$$P(uz, vz, z)[u\,dz + z\,du] + Q(uz, vz, z)[v\,dz + z\,dv] + R(uz, vz, z)dz = 0.$$

Dividing out the common factor z^n and rearranging, we have

$$z[P(u, v, 1)du + Q(u, v, 1)dv] + [uP(u, v, 1) + vQ(u, v, 1) + R(u, v, 1)]dz = 0$$

or $$z(P_1\,du + Q_1\,dv) + (uP_1 + vQ_1 + R_1)dz = 0, \text{ where } P_1 = P(u, v, 1), \text{ etc.}$$

This may be written as $A)$ $\dfrac{P_1}{uP_1 + vQ_1 + R_1}du + \dfrac{Q_1}{uP_1 + vQ_1 + R_1}dv + \dfrac{1}{z}dz = 0$ in which the variable z occurs only in the last term.

Now the condition of integrability for $A)$ $\dfrac{1}{z}\left(\dfrac{\partial}{\partial u}\dfrac{Q_1}{uP_1 + vQ_1 + R_1} - \dfrac{\partial}{\partial v}\dfrac{P_1}{uP_1 + vQ_1 + R_1}\right) = 0,$

is satisfied provided the original equation is integrable and, when this occurs, the sum of the first two terms of $A)$ is an exact differential. Moreover, since the third term is an exact differential, $A)$ is an exact differential equation provided only that $P\,dx + Q\,dy + R\,dz = 0$ is integrable.

8. Solve the homogeneous equation $2(y + z)dx - (x + z)dy + (2y - x + z)dz = 0$.

The equation is integrable since $2(y + z)(-1 - 2) - (x + z)(-1 - 2) + (2y - x + z)(2 + 1) = 0$.

The transformation $x = uz$, $y = vz$ reduces the given equation to

$$2z(v + 1)(u\,dz + z\,du) - z(u + 1)(v\,dz + z\,dv) + z(2v - u + 1)dz = 0.$$

Dividing by z and rearranging, we have $2z(v+1)du - z(u+1)dv + (uv+u+v+1)dz = 0$ or dividing by $z(uv+u+v+1) = z(u+1)(v+1)$, $\dfrac{2\,du}{u+1} - \dfrac{dv}{v+1} + \dfrac{dz}{z} = 0.$

Then $2\ln(u+1) - \ln(v+1) + \ln z = \ln K$, $z(u+1)^2 = K(v+1)$,

$(x+z)^2 = K(y+z)$ or $y+z = C(x+z)^2$.

9. Solve the homogeneous equation $yz\,dx - z^2\,dy - xy\,dz = 0$.

The equation is integrable since $yz(-2z+x) - z^2(-y-y) - xy(z-0) = 0$.
The transformation $x = uz$, $y = vz$ reduces it to

$$vz^2(u\,dz + z\,du) - z^2(v\,dz + z\,dv) - uvz^2\,dz = 0.$$

Dividing by z^2 and rearranging $vz\,du - z\,dv - v\,dz = 0$ or $du - \dfrac{dv}{v} - \dfrac{dz}{z} = 0.$

Then $u - \ln v - \ln z = \ln K$, $vz = Ce^u$ or $y = Ce^{x/z}$.

10. Solve $(2y-z)dx + 2(x-z)dy - (x+2y)dz = 0$.

The equation is homogeneous and, by inspection, is seen to be exact since it may be written as $2(y\,dx + x\,dy) - (z\,dx + x\,dz) - 2(z\,dy + y\,dz) = 0.$

The solution is $2xy - xz - 2yz = C$.

11. Show that $xP + yQ + zR = C$ is the solution of $P\,dx + Q\,dy + R\,dz = 0$ when the equation is exact and homogeneous of degree $n \neq -1$.

First, we check the theorem using the equation of Problem 10. Here

$$xP + yQ + zR = x(2y-z) + 2y(x-z) - z(x+2y) = 2(2xy - xz - 2yz)$$

and we obtain the solution above.

From $xP + yQ + zR = C$, we obtain by differentiation

$$A)\quad (P + x\frac{\partial P}{\partial x} + y\frac{\partial Q}{\partial x} + z\frac{\partial R}{\partial x})dx + (Q + x\frac{\partial P}{\partial y} + y\frac{\partial Q}{\partial y} + z\frac{\partial R}{\partial y})dy + (R + x\frac{\partial P}{\partial z} + y\frac{\partial Q}{\partial z} + z\frac{\partial R}{\partial z})dz = 0.$$

Since the given equation is exact, $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, $\dfrac{\partial R}{\partial x} = \dfrac{\partial P}{\partial z}$, $\dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z}$.

Making these replacements, $A)$ becomes

$$B)\quad (P + x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z})dx + (Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z})dy + (R + x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z})dz = 0.$$

Since the given equation is homogeneous, $x\dfrac{\partial P}{\partial x} + y\dfrac{\partial P}{\partial y} + z\dfrac{\partial P}{\partial z} = nP$, etc., according to Euler's Formula on homogeneous functions.

Making these replacements, $B)$ becomes

$$(n+1)P\,dx + (n+1)Q\,dy + (n+1)R\,dz = 0$$

or, since $n \neq -1$,

$$P\,dx + Q\,dy + R\,dz = 0.$$

12. Solve $(y^2+z^2+2xy+2xz)dx + (x^2+z^2+2xy+2yz)dy + (x^2+y^2+2xz+2yz)dz = 0$.

The equation is homogeneous of degree 2 and is also exact since

$$\frac{\partial P}{\partial y} = 2(y+x) = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = 2(z+y) = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = 2(x+z) = \frac{\partial P}{\partial z}.$$

The solution is $x(y^2+z^2+2xy+2xz) + y(x^2+z^2+2xy+2yz) + z(x^2+y^2+2xz+2yz) = K$

or $$x(y^2+z^2) + y(x^2+z^2) + z(x^2+y^2) = C.$$

13. Solve the differential equation $P\,dx + Q\,dy + R\,dz = 0$ given only that the condition of integrability is satisfied.

Consider one of the variables, say z, as a constant for the moment and let the solution of the resulting equation

1) $$P\,dx + Q\,dy = 0$$

be

2) $$u(x,y,z) = \phi(z).$$

Differentiating 2) with respect to all the variables,

3) $$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = \phi'(z)\,dx = d\phi.$$

Now $\frac{\partial u}{\partial x} = \mu P$ and $\frac{\partial u}{\partial y} = \mu Q$, where $\mu = \mu(x,y,z)$ is an integrating factor of 1). Substituting in 3), we have $\mu P\,dx + \mu Q\,dy + \frac{\partial u}{\partial z}dz = d\phi$.

But from the given equation $\mu P\,dx + \mu Q\,dy + \mu R\,dz = 0$ so that

$$d\phi = \frac{\partial u}{\partial z}dz - \mu R\,dz = \left(\frac{\partial u}{\partial z} - \mu R\right)dz.$$

This relation is free of dx and dy and, using 2) if necessary, can be written as a differential equation in z and ϕ. Solving the integral for ϕ and substituting in 2), we have the required solution.

14. Solve $2(y+z)dx - (x+z)dy + (2y-x+z)dz = 0$. (See Problem 8.)

We treat z as a constant and solve $2(y+z)dx - (x+z)dy = 0$ or $\frac{dy}{dx} - \frac{2}{x+z}y = \frac{2z}{x+z}$, using the integrating factor $e^{-2\int dx/(x+z)} = \frac{1}{(x+z)^2}$, to obtain

A) $$\frac{y}{(x+z)^2} = \int \frac{2z}{(x+z)^3}dx = -\frac{z}{(x+z)^2} + \phi(z).$$

Differentiating *A)* with respect to all variables,

$$\frac{dy}{(x+z)^2} - \frac{2y}{(x+z)^3}(dx+dz) = -\frac{dz}{(x+z)^2} + \frac{2z}{(x+z)^3}(dx+dz) + d\phi$$

or $$2(y+z)dx - (x+z)dy + (2y-x+z)dz + (x+z)^3 d\phi = 0.$$

Comparing this with the given equation, it is seen that $(x+z)^3 d\phi = 0$ and $\phi = C$.
Since, from *A)*, $y+z = \phi(x+z)^2$, the solution is $y+z = C(x+z)^2$.

15. Solve $(e^x y + e^z)dx + (e^y z + e^x)dy + (e^y - e^x y - e^y z)dz = 0$.

The equation is integrable since

$$(e^x y + e^z)(e^y - e^y + e^x + e^y z) + (e^y z + e^x)(-e^x y - e^z) + (e^y - e^x y - e^y z)(e^x - e^x) = 0.$$

Considering z as a constant and solving the resulting equation

$$(e^x y\,dx + e^x dy) + e^y z\,dy + e^z dx = 0,$$

we have $$e^x y + e^y z + e^z x = \phi(z).$$

Differentiation with respect to all variables yields

$$(e^x y + e^z)dx + (e^y z + e^x)dy + (e^y + e^z x)dz = d\phi.$$

From the given equation, $(e^x y + e^z)dx + (e^y z + e^x)dy + (e^y + e^z x)dz = (e^x y + e^y z + e^z x)dz$.

Thus, $d\phi = (e^x y + e^y z + e^z x)dz = \phi dz$ and $\phi = Ce^z$. The required solution is

$$e^x y + e^y z + e^z x = Ce^z.$$

16. Solve $yz\,dx + (xz - yz^3)dy - 2xy\,dz = 0$.

The equation is integrable since $yz(x - 3yz^2 + 2x) + (xz - yz^3)(-2y - y) - 2xy(z - z) = 0$.
Considering y as a constant and solving the resulting equation

$$yz\,dx - 2xy\,dz = 0 \quad \text{or} \quad z\,dx - 2x\,dz = 0,$$

we obtain $\ln x - 2\ln z = \ln\phi(y)$ or $x = \phi z^2$.

Differentiating this result and making the replacement $\phi = x/z^2$, we have

$$dx - 2\phi z\,dz - z^2 d\phi = 0, \qquad dx - 2\frac{x}{z}dz - z^2 d\phi = 0, \quad \text{or} \quad yz\,dx - 2xy\,dz - yz^3 d\phi = 0.$$

Comparing this with the given differential equation, we have

$$(xz - yz^3)dy + yz^3 d\phi = (\phi z^3 - yz^3)dy + yz^3 d\phi = 0 \quad \text{or} \quad \phi dy + y\,d\phi - y\,dy = 0.$$

Then $\phi y - \frac{1}{2}y^2 = K$ or $\phi = \frac{1}{2}y + K/y$, so that the solution is

$$x = \phi z^2 = z^2(\tfrac{1}{2}y + K/y) \quad \text{or} \quad 2xy = y^2 z^2 + Cz^2.$$

17. Discuss geometrically the solution of the integrable total differential equation

$$P\,dx + Q\,dy + R\,dz = 0.$$

Let (x_0, y_0, z_0) be a general point in space for which not all of $P_0 = P(x_0, y_0, z_0)$, $Q_0 = Q(x_0, y_0, z_0)$, $R_0 = R(x_0, y_0, z_0)$ are zero.

Assuming that P, Q, R are single-valued, the set (P_0, Q_0, R_0) may be considered as direction numbers of a unique line through the point. Hence, the given differential equation may be thought of as defining at each point (x_0, y_0, z_0) a line

$$\frac{x - x_0}{P_0} = \frac{y - y_0}{Q_0} = \frac{z - z_0}{R_0}$$

and a plane $$P_0(x - x_0) + Q_0(y - y_0) + R_0(z - z_0) = 0$$ normal to the line.

The solution $f(x, y, z) = C$ of the given differential equation represents a family of surfaces such that through a general point (x_0, y_0, z_0) of space there passes a single surface S_0 of the family. The equation of the tangent plane π_0 to this surface at the point is

$$(x - x_0)\frac{\partial f}{\partial x_0} + (y - y_0)\frac{\partial f}{\partial y_0} + (z - z_0)\frac{\partial f}{\partial z_0} = 0$$

and the equations of the normal line L_0 are $\dfrac{x-x_0}{\frac{\partial f}{\partial x_0}} = \dfrac{y-y_0}{\frac{\partial f}{\partial y_0}} = \dfrac{z-z_0}{\frac{\partial f}{\partial z_0}}$.

From Problem 1, $\dfrac{\partial f}{\partial x} = \lambda P$, $\dfrac{\partial f}{\partial y} = \lambda Q$, $\dfrac{\partial f}{\partial z} = \lambda R$. Hence, the solution of an integrable total differential equation in three variables is a family of surfaces whose tangent plane and normal at each point are respectively the plane and line associated with the point by the differential equation.

PAIRS OF TOTAL DIFFERENTIAL EQUATIONS IN THREE VARIABLES.

18. Solve the system: $(y+z)dx + (z+x)dy + (x+y)dz = 0$

$(x+z)dx + y\,dy + x\,dz = 0$.

Both equations are integrable. The first may be written as

$$(y\,dx + x\,dy) + (z\,dy + y\,dz) + (x\,dz + z\,dx) = 0$$

and the solution is $xy + yz + zx = C_1$.

The second may be written as $x\,dx + y\,dy + (z\,dx + x\,dz) = 0$ and the solution is

$$x^2 + y^2 + 2xz = C_2.$$

Thus, $xy + yz + zx = C_1$, $x^2 + y^2 + 2xz = C_2$ constitute the general solution.

Through each point in space there passes a single surface of each of the two families. Since the two surfaces on a point have a curve in common, the solution of the pair of differential equations is a family of curves. This family of curves may be given by the equations of any two families of surfaces passing through the family of curves. For example,

$$xy + yz + zx = C_1, \quad x^2 + y^2 + 2(C_1 - xy - yz) = C_2$$

also constitute the general solution.

19. Solve the system: 1) $yz\,dx + xz\,dy + xy\,dz = 0$

2) $z^2(dx+dy) + (xz + yz - xy)dz = 0$.

The first equation is integrable, with solution 3) $xyz = C_1$, but the second is not.

Multiply 1) by z, multiply 2) by y, and subtract to obtain $z^2(y-x)dy + y^2(z-x)dz = 0$. Multiply this by yz and substitute $xyz = C_1$ from 3). The result is

$$z^2(y^2z - C_1)dy + y^2(yz^2 - C_1)dz = 0 \quad \text{or} \quad z\,dy + y\,dz - C_1\left(\frac{dy}{y^2} + \frac{dz}{z^2}\right) = 0$$

whose solution is 4) $yz + C_1\left(\dfrac{y+z}{yz}\right) = C_2$.

Equations 3) and 4) constitute a general solution. However, 4) may be replaced by the simpler form 4') $xy + yz + xz = C_2$, obtained from 4) by substituting for C_1.

20. Solve the system: $dx + 2dy - (x+2y)dz = 0$

$2dx + dy + (x-y)dz = 0$.

Here $X = \lambda\begin{vmatrix} 2 & -(x+2y) \\ 1 & x-y \end{vmatrix} = 3\lambda x$, $Y = \lambda\begin{vmatrix} -(x+2y) & 1 \\ x-y & 2 \end{vmatrix} = -3\lambda(x+y)$, $Z = \lambda\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3\lambda$.

For the choice $\lambda = -1/3$, $X = -x$, $Y = x+y$, $Z = 1$, and we write the system in the symmetric form

$$\frac{dx}{-x} = \frac{dy}{x+y} = \frac{dz}{1}.$$

From the integrable equation $\frac{dx}{-x} = \frac{dz}{1}$, we obtain $z + \ln x = C_1$.

From the integrable equation $\frac{dx}{-x} = \frac{dy}{x+y}$, we obtain $x^2 + 2xy = C_2$.

Thus, $z + \ln x = C_1$, $x^2 + 2xy = C_2$ constitute the general solution.

21. Solve the system $\frac{dx}{x} = \frac{dy}{x+z} = \frac{dz}{-z}$. Find the equations of the integral curves through the points *a)* $(1,1,1)$ and *b)* $(2,1,1)$.

Consider the equations $\frac{dx}{x} = \frac{dz}{-z}$ and $\frac{dx}{x} = \frac{dy}{x+z}$. The first is integrable and yields $xz = C_1$. The second is not integrable but is reduced to $dy = (1 + C_1/x^2)dx$ by the substitution $z = C_1/x$. Integrating, we have $y = x - C_1/x + C_2$ or, substituting $C_1 = xz$, $y - x + z = C_2$. Thus, $xz = C_1$, $y - x + z = C_2$ constitute the general solution.

The integral curve through the point $(1,1,1)$ is the intersection of the hyperbolic cylinder $xz = 1$ and the plane $y - x + z = 1$. The integral curve through $(2,1,1)$ is the intersection of the cylinder $xz = 2$ and the plane $y - x + z = 0$.

22. Solve $\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{y-x}$.

No equation is integrable. By means of the multipliers $l = m = 1$, $n = 0$, we obtain

$$\frac{dz}{y-x} = \frac{l\,dx + m\,dy + n\,dz}{l(y-z) + m(z-x) + n(y-x)} = \frac{dx+dy}{y-x} \quad \text{or} \quad dx + dy - dz = 0. \quad \text{Then}$$

A) $$x + y - z = C_1.$$

Using *A)* to eliminate z in $\frac{dx}{y-z} = \frac{dy}{z-x}$, we obtain $\frac{dx}{C_1 - x} = \frac{dy}{y - C_1}$. Then

$\ln(x - C_1) + \ln(y - C_1) = \ln C_2$, or $(x - C_1)(y - C_1) = C_2$, or, eliminating C_1 by means of *A)*,

B) $$(z-y)(z-x) = C_2.$$

A) and *B)* constitute the general solution.

23. Solve $\frac{x^2dx}{y^3} = \frac{y^2dy}{x^3} = \frac{dz}{z}$.

From the integrable equation $\frac{x^2dx}{y^3} = \frac{y^2dy}{x^3}$ or $x^5dx - y^5dy = 0$, we obtain

A) $$x^6 - y^6 = C_1.$$

We may then use *A)* to eliminate x in the non-integrable equation $\frac{dz}{z} = \frac{y^2dy}{x^3}$. However, it is simpler to use the multipliers $l = m = 1$, $n = 0$ to obtain $\frac{dz}{z} = \frac{x^2dx + y^2dy}{x^3 + y^3}$. Then $x^3 + y^3 = C_2z^3$.

24. Solve the system $\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)^3z}$.

Using $l = m = 1$, $n = 0$, we obtain $\frac{dz}{(x+y)^3z} = \frac{dx+dy}{(x+y)^2}$ or $\frac{dz}{z} = (x+y)(dx+dy)$. Then

$$(x+y)^2 - 2\ln z = C_1.$$

Using $l_1 = m_1 = 1,\ n_1 = 0$ and $l_2 = 1,\ m_2 = -1,\ n_2 = 0,$ we obtain $\dfrac{dx+dy}{(x+y)^2} = \dfrac{dx-dy}{(x-y)^2}.$ Then

$$\frac{1}{x+y} = \frac{1}{x-y} + K \quad \text{or} \quad y = C_2(x^2 - y^2).$$

25. Solve the system $\dfrac{dx}{y} = \dfrac{dy}{-x} = \dfrac{dz}{2x-3y}.$

The equation $\dfrac{dx}{y} = \dfrac{dy}{-x}$ or $x\,dx + y\,dy = 0$ is integrable and we obtain $x^2 + y^2 = C_1.$

Using $l = 3,\ m = 2,\ n = 1,$ we find $3(y) + 2(-x) + (2x - 3y) = 0.$ Hence, $3\,dx + 2\,dy + dz = 0$ and $3x + 2y + z = C_2.$

26. Solve the system $\dfrac{dx}{4y-3z} = \dfrac{dy}{4x-2z} = \dfrac{dz}{2y-3x}.$

We seek multipliers l, m, n such that *A)* $l(4y-3z) + m(4x-2z) + n(2y-3x) = 0.$

Rearranging *A)* in the form $(4m-3n)x + (4l+2n)y + (-3l-2m)z = 0,$ we see that it will be satisfied when

$4m - 3n = 0,\ 4l + 2n = 0,\ -3l - 2m = 0$ or $l : m : n = 2 : -3 : -4.$ Then

$$2\,dx - 3\,dy - 4\,dz = 0 \quad \text{and} \quad 2x - 3y - 4z = C_1.$$

Using the arrangement $4(ly + mx) + 3(-lz - nx) + 2(ny - mz) = 0$ and setting $ly + mx = 0,$ $-lz - nx = 0,\ ny - mz = 0,$ we obtain $l : m : n = x : -y : -z.$ Then

$$x\,dx - y\,dy - z\,dz = 0 \quad \text{and} \quad x^2 - y^2 - z^2 = C_2.$$

27. Solve the system $\dfrac{p\,dx}{(q-r)yz} = \dfrac{q\,dy}{(r-p)xz} = \dfrac{r\,dz}{(p-q)xy}.$

Consider $l(q-r)yz + m(r-p)xz + n(p-q)xy = 0.$

From $q(lyz - nxy) + r(mxz - lyz) + p(nxy - mxz) = 0$ we obtain $l : m : n = x : y : z.$ Then

$$px\,dx + qy\,dy + rz\,dz = 0 \quad \text{and} \quad px^2 + qy^2 + rz^2 = C_1.$$

From $z(lqy - mpx) + y(npx - lrz) + x(mrz - nqy) = 0$ we obtain $l : m : n = px : qy : rz.$ Then

$$p^2x\,dx + q^2y\,dy + r^2z\,dz = 0 \quad \text{and} \quad p^2x^2 + q^2y^2 + r^2z^2 = C_2.$$

28. Solve the system $\dfrac{dx}{x^2+y^2-yz} = \dfrac{dy}{-x^2-y^2+xz} = \dfrac{dz}{(x-y)z}.$

Using $l = m = 1,\ n = -1,$ we obtain $(x^2 + y^2 - yz) + (-x^2 - y^2 + xz) - (x-y)z = 0.$ Then

$$dx + dy - dz = 0 \quad \text{and} \quad x + y - z = C_1.$$

Using $l = xz,\ m = yz,\ n = -(x^2 + y^2),$ we find

$$xz(x^2 + y^2 - yz) + yz(-x^2 - y^2 + xz) - (x^2 + y^2)(x-y)z = 0.$$

Then $$xz\,dx + yz\,dy - (x^2 + y^2)dz = 0 \quad \text{or} \quad \frac{x\,dx + y\,dy}{x^2 + y^2} - \frac{dz}{z} = 0$$

and $$\ln(x^2 + y^2) - 2\ln z = \ln C_2$$

or $$x^2 + y^2 = C_2z^2.$$

29. Solve the system $\dfrac{dx}{2x} = \dfrac{dy}{-y} = \dfrac{dz}{4xy^2 - 2z}$.

From $\dfrac{dx}{2x} = \dfrac{dy}{-y}$ we have $xy^2 = C_1$. By inspection, $2y^2(2x) + 2yz(-y) - y^2(4xy^2 - 2z) = 0$;

then $2y^2\,dx + 2yz\,dy - y^2\,dz = 0$ or $2dx - \dfrac{y^2\,dz - 2yz\,dy}{y^4} = 0$, and $2x - \dfrac{z}{y^2} = C_2$.

30. Discuss geometrically the general solution of $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$.

For convenience, let us assume that in solving the given system we have obtained a pair of integrable equations

$$P_1\,dx + Q_1\,dy + R_1\,dz = 0 \quad \text{and} \quad P_2\,dx + Q_2\,dy + R_2\,dz = 0$$

whose integrals are respectively

$$g(x,y,z) = C_1 \quad \text{and} \quad h(x,y,z) = C_2.$$

Through a general point (x_0, y_0, z_0) of space there pass two surfaces (one of each of the above families) whose curve of intersection C_0 is the integral curve of the given system through the point. The tangent planes to the two surfaces at (x_0, y_0, z_0) are normal to the directions (P_1, Q_1, R_1) and (P_2, Q_2, R_2) evaluated at the point, and the line of intersection L_0 of these planes is normal to the rwo directions. Let (X, Y, Z) be a set of direction numbers for L_0; then

$$X = \lambda\begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix}, \qquad Y = \lambda\begin{vmatrix} R_1 & P_1 \\ R_2 & P_2 \end{vmatrix}, \qquad Z = \lambda\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}$$

are proportional to P, Q, R (all evaluated at the point).

Now L_0 is the tangent to C_0 at (x_0, y_0, z_0), since the tangent to a space curve at one of its points lies in the tangent plane at the point of any surface containing the curve, Hence, the integral curves of the system $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ consist of a doubly infinite system of curves characterized by the fact that at any point (x_0, y_0, z_0) the tangent to the curve through the point has (P_0, Q_0, R_0) as direction numbers.

31. Show that the family of integral surfaces of 1) $P\,dx + Q\,dy + R\,dz = 0$ and the family of integral curves of 2) $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ are orthogonal.

This follows from the fact that at any general point (x_0, y_0, z_0) the direction (P_0, Q_0, R_0) is:
a) normal to the integral surface of 1) through the point (see Problem 17) and
b) the direction of the integral curve of 2) through the point (see Problem 30).

32. Solve 1) $y\,dx + x\,dy - (x + y + 2z)\,dz = 0$ consistent with *a)* $z = a$, *b)* $x + y + 2z = 0$, *c)* $x + y = 0$, *d)* $xy = a$.

Equation 1) is not integrable. From each given surface we may obtain an integrable total differential equation. Our problem then is to solve this differential equation simultaneously with 1) using the particular solution of the former rather than the general solution as in *f)* of the introduction of this chapter.

a) Here $z = a$, $dz = 0$. Substituting in 1), we obtain $y\,dx + x\,dy = 0$; then $xy = C$.
Equations $z = a$, $xy = C$ are said to constitute a solution of 1).

b) Substituting $x+y+2z=0$ in 1), we obtain $y\,dx+x\,dy=0$ and $xy=C$.
The solution is $xy=C,\quad x+y+2z=0$.

c) Here $y=-x,\quad dy=-dx$. Substituting in 1), we obtain $x\,dx+z\,dz=0$ and $x^2+z^2=C$.
The solution is $x^2+z^2=C,\quad x+y=0$.

d) Here $xy=a,\quad x\,dy+y\,dx=0$. Equation 1) reduces to $(x+y+2z)dz=0$.
Then, either $x+2y+2z=0$ or $dz=0$ and $z=C$.
$xy=a,\ x+y+2z=0$ and $z=C,\ xy=a$ constitute the solution.

33. Discuss geometrically the problem of solving $P\,dx+Q\,dy+R\,dz=0$ consistent with the given relation $g(x,y,z)=0$.

From the relation $g(x,y,z)=0$, we obtain $\dfrac{\partial g}{\partial x}dx+\dfrac{\partial g}{\partial y}dy+\dfrac{\partial g}{\partial z}dz=0$.

We solve the system $P\,dx+Q\,dy+R\,dz=0,\quad \dfrac{\partial g}{\partial x}dx+\dfrac{\partial g}{\partial y}dy+\dfrac{\partial g}{\partial z}dz=0$ using the particular solution $g(x,y,z)=0$ of the latter. Let

$$f(x,y,z)=C,\quad g(x,y,z)=0$$

constitute the solution. The integral curves are those cut out on the surface $g(x,y,z)=0$ by the system of surfaces $f(x,y,z)=C$. Thus, Problem 32*c* may be stated as: Find all curves lying on the surface (plane) $x+y=0$ which satisfy the differential equation

$$y\,dx+x\,dy-(x+y+2z)dz=0.$$

At a general point (x_0,y_0,z_0) on the surface $g(x,y,z)=0$, the line of intersection L_0 of the tangent planes to $g(x,y,z)=0$ and the surface of the system $f(x,y,z)=C$, through the point, is tangent to the curve of intersection of the two surfaces. Thus, we have found the *family of curves* on the given surface $g(x,y,z)=0$ whose tangent at any point lies in the plane, through this point, determined by the differential equation. (See Problem 17.)

For example, consider Problem 32*c*. On the prescribed surface $x+y=0$, choose any point $(a,-a,b)$. At this point, the tangent plane $x+y=0$ (here, the plane itself) is normal to the direction $(1,1,0)$ and the tangent plane to the surface (of the family) $x^2+z^2=a^2+b^2$ is normal to the direction $(a,0,b)$. A set of direction numbers for the line of intersection L of these planes [the tangent to the curve through $(a,-a,b)$] is $(-b,b,a)$.

Now the plane through $(a,-a,b)$ determined by the given differential equation is normal to the direction $[y,x,-(x+y+2z)]_{(a,-a,b)}=(-a,a,-2b)$. Since $(-b,b,a)$ and $(-a,a,-2b)$ are normal directions, the line L lies in the plane determined by the differential equation.

34. Solve 1) $2z\,dx+dy+y\,dz=0$ consistent with 2) $x+y+z=0$.

From 2), $y=-x-z$ and $dy=-dx-dz$. Substituting for y and dy in 1), we obtain

3) $\quad (2z-1)dx-(x+z+1)dz=0.$

The transformation $z=z_1+1/2,\quad x=x_1-3/2$ reduces 3) to

4) $\quad 2z_1\,dx_1-(x_1+z_1)dz_1=0$, a homogeneous equation.

The transformation $x_1=uz_1$ reduces 4) to $(u-1)dz_1+2z_1\,du=0$ or $\dfrac{dz_1}{z_1}+\dfrac{2\,du}{u-1}=0$.

Then $$\ln z_1+2\ln(u-1)=\ln K$$

or $$z_1(u-1)^2=K.$$

Replacing u by x_1/z_1, x_1 by $x+3/2$ and z_1 by $z-1/2$, this becomes $(x-z+2)^2=C(2z-1)$.

SUPPLEMENTARY PROBLEMS

Test for integrability and solve when possible.

35. $(y+3z)dx + (x+2z)dy + (3x+2y)dz = 0$ *Ans.* $xy + 2yz + 3xz = C$

36. $(\cos x + e^x y)dx + (e^x + e^y z)dy + e^y dz = 0$ $e^x y + e^y z + \sin x = C$

37. $dx + (x+z)dy + dz = 0$ $y + \ln(x+z) = C$

38. $z^3 dx + z\,dy - 2y\,dz = 0$ $xz^2 + y = Cz^2$

39. $x^2 dx - z^2 dy - xy\,dz = 0$ Not integrable.

40. $(x+z)^2 dy + y^2(dx+dz) = 0$ $y(x+z) = C(x+y+z)$

41. $2x(y+z)dx + (2yz - x^2 + y^2 - z^2)dy + (2yz - x^2 - y^2 + z^2)dz = 0$ $x^2 + y^2 + z^2 = C(y+z)$

42. $yz\,dx - 2xz\,dy + xy\,dz = 0$ $y^2 = Cxz$

43. $x\,dx + y\,dy + (x^2 + y^2 + z^2 + 1)z\,dz = 0$ $(x^2 + y^2 + z^2)e^{z^2} = C$

44. $z(x^2 - yz - z^2)dx + xz(x+z)dy + x(z^2 - x^2 - xy)dz = 0$ $(x+y)/z + (y+z)/x = C$

Solve the following pairs of equations.

45. $dx + dy + (x+y)dz = 0$, $z(dx+dy) + (x+y)dz = 0$ *Ans.* $x + y = C_1 e^{-z}$, $x + y = C_2/z$

46. $2yz\,dx + x(z\,dy + y\,dz) = 0$, $y\,dx - x^2 z\,dy + y\,dz = 0$ $x^2yz = C_1$, $x^2z + x + z = C_2$

47. $\dfrac{x\,dx}{y^5 z} = \dfrac{dy}{x^2 z} = \dfrac{dz}{y^5}$ $x^4 - y^4 = C_1$, $x^2 - z^2 = C_2$

48. $\dfrac{3\,dx}{yz} = \dfrac{dy}{xz} = \dfrac{dz}{xy}$ $3x^2 - y^2 = C_1$, $y^2 - z^2 = C_2$

49. $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{(x+y)(1+2xy+3x^2y^2)}$ $x - y = C_1$, $z = xy + x^2y^2 + x^3y^3 + C_2$

50. $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$ $y = C_1 z$, $x^2 + y^2 + z^2 = C_2 z$

51. $\dfrac{dx}{3y - 2z} = \dfrac{dy}{z - 3x} = \dfrac{dz}{2x - y}$ $x + 2y + 3z = C_1$, $x^2 + y^2 + z^2 = C_2$

52. $\dfrac{dx}{x(2y^4 - z^4)} = \dfrac{dy}{y(z^4 - 2x^4)} = \dfrac{dz}{z(x^4 - y^4)}$ $xyz^2 = C_1$, $x^4 + y^4 + z^4 = C_2$

53. $\dfrac{dx}{x(z^2 - y^2)} = \dfrac{dy}{y(x^2 - z^2)} = \dfrac{dz}{z(y^2 - x^2)}$ $xyz = C_1$, $x^2 + y^2 + z^2 = C_2$