5

The Conductivity of Metals

The most characteristic properties of metals are their high electrical and thermal conductivities. For example, the electrical conductivity at room temperature of silver is 0.6×10^8 ohm^{-1} m^{-1} as compared to $\approx 10^{-16}$ for a good insulator and 2×10^{-2} for a semiconductor such as germanium. There is a great deal of evidence to support the notion that the high conductivity in metals is associated with the presence of "free" or "conduction" electrons. These free electrons are able to move throughout the lattice and hence do not belong to particular atoms. The only electrons which have this high degree of freedom are those corresponding to the valence electrons in the atoms. Thus, one may think of a metal as consisting of a lattice of positive ion cores held together by means of a gas of electrons. The electron gas consists of the freely moving valence electrons; the ion cores constitute what is left of the atoms after they have parted with their valence electrons. Although the properties of the electron gas should be discussed on the basis of wave mechanics, the discussion in this chapter will be essentially classical. To this extent, the discussion must be considered as inaccurate. On the other hand, the main purpose here is to arrive at a reasonable qualitative picture concerning the mechanism of conductivity, and this purpose may be achieved by taking a less sophisticated point of view.

The first question we shall consider is this: in terms of the picture just outlined, how should Ohm's law be understood? Then we shall consider problems such as the following: why does the resistivity of metals and alloys increase with increasing temperature; why is the resistivity of

an impure metal higher than that of a pure metal; how can we understand Joule's law, which states that the heat developed in a wire is proportional to the square of the voltage across it?

5.1 Ohm's law and the relaxation time of electrons

There is probably no law more fundamental to the electrical engineer than Ohm's law. It is used every day in electrical engineering and consequently it seems proper to investigate how this law can be interpreted in terms of an atomic picture. In order to arrive at an atomic interpretation of Ohm's law, we shall formulate the law in the form

$$\mathbf{J} = \sigma \mathbf{E} \tag{5.1}$$

which states that the current density $\mathbf{J}$ is proportional to the field strength $\mathbf{E}$; the proportionality constant σ is the *electrical conductivity* of the metal. We shall also refer frequently to the *resistivity*

$$\rho = 1/\sigma \tag{5.2}$$

In connection with the macroscopic description of Ohm's law (5.1) the same remarks hold as were made in connection with the relationship $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$ in section 2.1. Thus, (5.1) holds only if the conductivity is independent of the direction in which it is measured. For polycrystalline materials it is therefore valid, but in the case of single crystals σ should in general be replaced by a tensor quantity. For simplicity we shall assume the material to be isotropic. Another remark that should be made here is that when an electric field $\mathbf{E}$ is applied at the instant $t = 0$, the current density requires a non-vanishing period to build up to the equilibrium value $\mathbf{J}$. However, the build-up times are very short (of the order of 10^{-14} seconds) and need not concern us at the moment.

One important feature of Ohm's law is the fact that the *current density remains constant in time* as long as E remains constant. Although this feature may seem trivial, it is not so when we consider the problem of conduction from the atomic point of view; in fact, it gives us a clue about the mechanism of conduction as we shall see presently.

Concerning the electron gas, we shall assume that there are n free electrons per m^3, where n is presumably of the same order of magnitude as the number of atoms per m^3 in the metal. In the absence of an external field, the velocity vectors associated with the electrons at a given instant add up to zero; hence, there is no resultant transport of charge. When an electric field E is applied along the x-direction, the electrons of charge

$-e$ are subjected to an acceleration given by

$$a_x = -(e/m)E \tag{5.3}$$

where m is the electronic mass. When v_x represents the velocity component of an electron along the x-direction, we may thus write for all electrons

$$\left(\frac{\partial}{\partial t} v_x\right)_{\text{field}} = -\frac{e}{m}E \tag{5.4}$$

The partials and the subscript "field" indicate that this represents the rate of change of v_x due only to the interaction of the electrons with the field; the purpose of introducing this notation will become clear later. Before drawing a conclusion from this argument, let us consider the relationship between the current density J along the x-direction and the velocity components v_x of the electrons. Considering a large group of electrons at a certain instant t, we may define an average velocity of the electrons in the group as

$$\langle v_x \rangle_t = \frac{1}{N} \sum_{i=1}^{N} v_{xi}(t) \tag{5.5}$$

where the subscript i refers to a particular electron in the group and where N is the number of electrons considered. If there are n electrons per m^3, the current density at the instant t may be written as

$$J(t) = -ne\langle v_x \rangle_t \tag{5.6}$$

This relationship follows immediately from Fig. 5.1, in which we have shown a block of the material with a cross section of 1 m^2 perpendicular to the x-axis. The total charge per second passing through a cross section

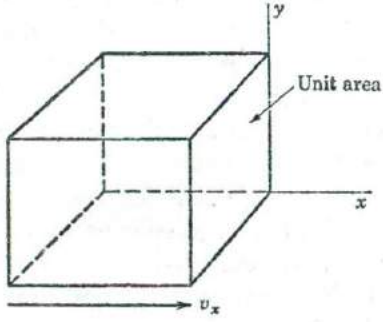


Fig. 5.1. Illustrating the statement that the number of particles crossing a unit area per second is equal to the number of particles contained in a volume equal to v_x. If the density of particles is n and their charge is $-e$, this gives a current density $J = -nev_x$.

is equal to the charge contained in a block of 1 m^2 in cross section and $\langle v_x \rangle$ meters long. (The average velocities in the y- and z-directions are zero, so for the present argument these may be taken as zero for each of the electrons.) Now, if (5.4) represented the *total* rate of change of v_x

for each of the electrons, it would also represent the total rate of change of $\langle v_x \rangle$. But then $\langle v_x \rangle_t$ in expression (5.6) would change at a constant rate, and hence $J(t)$ would change at a constant rate with time. Experiment tells us, however, that, apart from the transient during approximately 10^{-14} seconds, J is independent of time for constant E. What then is wrong with the atomic interpretation of Ohm's law given so far? It is evident from (5.6) that if J should be constant for a given E, the average velocity $\langle v_x \rangle$ must be constant for a given field. This can only mean that expression (5.4) does not give the total rate of change of $\langle v_x \rangle$, and that *there must be other processes which make the total rate of change* $d\langle v_x \rangle/dt$ *equal to zero.* In other words, what experiment requires is the existence of the following relationship

$$\left[\frac{\partial}{\partial t}\langle v_x \rangle\right]_{\text{field}} + \left[\frac{\partial}{\partial t}\langle v_x \rangle\right]_{\text{other}} = \frac{d}{dt}\langle v_x \rangle = 0 \tag{5.7}$$

which expresses the fact that under influence of an external field we observe a steady value for J. The subscript "other" refers to other processes which we have yet to account for. What clues do we have to find these other processes? First of all, if the electrons were in vacuum, there would be no other effect on the electrons but the one produced by the field. One clue then is provided by the fact that the *electrons move in a crystal lattice,* and that the presence of the lattice must account for the "other" processes. Another clue, which points in the same direction, may be derived from the fact that when a current passes through a material, the material is known to warm up. Now, the temperature of a material is a measure of the intensity with which the atoms in the lattice vibrate. Since only the free electrons absorb energy from the field, the extra energy of vibration must have been transferred to the lattice by the electrons. This is illustrated schematically in the block diagram of Fig. 5.2. The

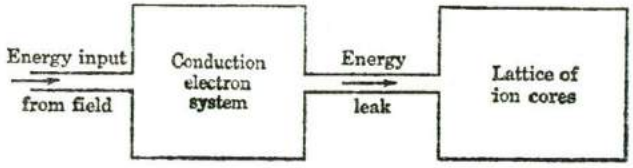


Fig. 5.2. Schematic diagram illustrating the system of conduction electrons absorbing energy from the external field, and transferring energy to the lattice of ion cores, where it appears as Joule heat.

small tube connecting the "electron gas" and the "lattice of ion cores" represents the leak through which an excess of energy of the electron gas

can be transferred to the lattice. From these arguments we arrive at the conclusion that the "free" electrons in a conductor must interact with the lattice of ion cores, and to this extent they are not completely free. We also conclude that the "other" processes referred to in equation (5.7) must probably be sought in the *electron-lattice interaction*. (It may be mentioned here that collisions between electrons would not solve the dilemma, because such collisions conserve momentum and hence would not change $\langle v_x \rangle$.)

How shall we describe the electron-lattice interaction mathematically? For the moment we shall follow a phenomenological approach; i.e., we shall make an assumption which we consider reasonable, without inquiring about the actual physical mechanism. This assumption is the following: suppose the electrons in a metal have an average velocity $\langle v_x \rangle_0$ at the instant $t = 0$; given that for $t > 0$ there is no external field, the electrons

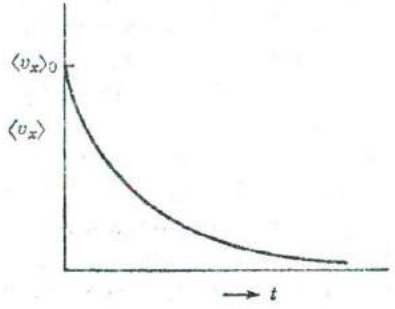


Fig. 5.3. Illustrating the influence of collisions alone on the average velocity of a group of particles with an average velocity $\langle v_x \rangle_0$ at the instant $t = 0$. The average velocity is assumed to decay exponentially with time, in accordance with equation (5.8).

interact only with the lattice and ultimately the average velocity will go to zero. We shall assume that under these circumstances the value of $\langle v_x \rangle$ at any instant $t(>0)$ is given by (see Fig. 5.3).

$$\langle v_x \rangle_t = \langle v_x \rangle_0 e^{-t/\tau} \tag{5.8}$$

where the quantity τ is called the *relaxation time* of the electrons. In other words, we assume that the electron-lattice interaction *by itself* will produce a rate of change of $\langle v_x \rangle$ given by

$$\left[\frac{\partial}{\partial t}\langle v_x \rangle\right]_{\text{coll}} = -\frac{1}{\tau}\langle v_x \rangle \tag{5.9}$$

The subscript "coll" refers to collisions carried out by the electrons with the lattice. These collision processes will be identified with the "other" processes appearing in equation (5.7).

On the basis of this picture, the conductivity can readily be found in terms of the atomic quantities. Substituting (5.4) and (5.9) into (5.7) we

obtain

$$-(e/m)E - (1/\tau)\langle v_x\rangle = 0 \quad \text{or} \quad \langle v_x\rangle = -(e\tau/m)E \tag{5.10}$$

The steady average velocity of the electrons in an applied field E is called the *drift velocity* of the carriers in the field. It is noted that the drift velocity is proportional to the field; the absolute magnitude of the proportionality factor

$$\mu_e = e\tau/m \tag{5.11}$$

is called the *mobility* of the electrons. The mobility may thus be defined as the magnitude of the average drift velocity per unit field. Substituting (5.10) into expression (5.6) for the current density, we find

$$J = (ne^2\tau/m)E \tag{5.12}$$

and it follows then from (5.1) that the measurable quantity σ is related to the atomic quantities by means of

$$1/\rho = \sigma = ne^2\tau/m \tag{5.13}$$

Thus, the experimental demonstration of a steady flow of current in a metal subjected to a constant field has compelled us to introduce an interaction between the electrons and the atomic lattice. This leaves us with the problem of the atomic interpretation of the relaxation time which describes this interaction. The question may also be raised as to whether a relaxation time can be defined at all, but this subject lies outside the scope of this book.

5.2 Relaxation time, collision time, and mean free path

In the preceding section we introduced the concept of a relaxation time on phenomenological grounds. In this section we shall see that such a relaxation time may be obtained from a picture in which one assumes the electrons to collide with "obstacles"; the nature of these obstacles will be discussed in subsequent sections.

Suppose an electron moves with a velocity v_x along the x-direction of a Cartesian coordinate system. Let us assume that the electron collides with an obstacle in such a way that the probability for it to be scattered over an angle between θ and $\theta + d\theta$ is given by $P(\theta)2\pi \sin\theta\, d\theta$, where $P(\theta)$ is an arbitrary function of the scattering angle θ [see Fig. 5.4(a)]. Evidently then, we should have

$$2\pi \int_{\theta=0}^{\pi} P(\theta) \sin\theta\, d\theta = 1 \tag{5.14}$$

We shall assume that the magnitudes of the velocities before and after

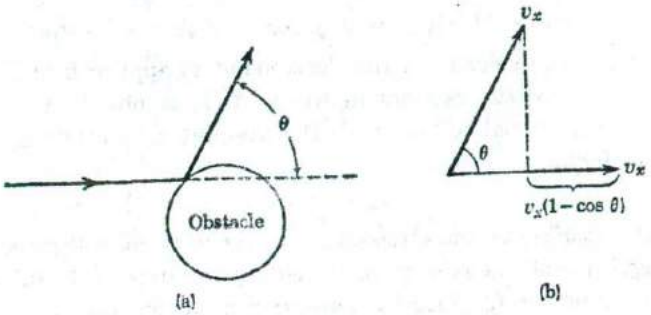


Fig. 5.4. Schematic representation of scattering of an electron by an obstacle over an angle θ is given in (a); (b) shows that as a result of this scattering process, the x-component of the velocity has been reduced by the amount $v_x(1 - \cos\theta)$, assuming that the magnitude of the velocity is the same after scattering as before.

scattering are the same. In connection with this assumption we should comment that actually some energy is transferred in the conduction process from the electrons to the lattice (represented in the present picture by the obstacles); this is, after all, the reason for the dissipation of Joule heat, as referred to in Fig. 5.2. However, for all cases of interest to the subject matter discussed in this book one can show that the percentage change in velocity resulting from the collisions is very small indeed. In discussing the collision process, therefore, we may assume equal velocities before and after the collision. Under these circumstances, an electron scattered over an angle θ has a velocity component along the x-direction equal to $v_x \cos\theta$. Hence, it has lost along the x-direction a velocity component amounting to (see Fig. 5.4b)

$$v_x(1 - \cos\theta) \tag{5.15}$$

Consider now a large group of electrons, all moving with the same velocity v_x along the x-direction. Assuming all these particles carry out one collision with an obstacle, what is the average velocity component lost by the particles? The answer is evidently

$$v_x(1 - \langle\cos\theta\rangle) \tag{5.16}$$

where $\langle\cos\theta\rangle$ is the average value of the cosine of the scattering angle. Since the probability for scattering over a given angle is described by $P(\theta)$, we must have

$$\langle\cos\theta\rangle = 2\pi \int_{\theta=0}^{\pi} P(\theta) \cos\theta \sin\theta \, d\theta \tag{5.17}$$

Note that if the scattering is spherically symmetric, i.e. if $P(\theta)$ is constant, $\langle\cos\theta\rangle = 0$.

Let us now consider the collisions from the point of view of time. Assume that the probability for an electron to collide with an obstacle during a short time interval dt is given by dt/τ_c, where τ_c is a constant with the dimensions of time. Let $F(t)$ represent the probability that an electron moves for a time t without suffering a collision, and let $F(t + dt)$ represent the same quantity for a period $t + dt$. We may then write

$$F(t + dt) = F(t) + \frac{dF}{dt}\,dt \tag{5.18}$$

on the other hand, we may also write

$$F(t + dt) = F(t)F(dt) = F(t)[1 - dt/\tau_c] \tag{5.19}$$

in view of the definition of τ_c. Hence, the last two equations give

$$dF/dt = -F/\tau_c \qquad \text{or} \qquad F(t) = e^{-t/\tau_c} \tag{5.20}$$

since $F(t) = 1$ for $t = 0$, which fixes the integration constant. What is the average time which elapses between collisions? The answer is

$$\langle t \rangle = -\int_0^\infty t\,\frac{dF}{dt}\,dt \tag{5.21}$$

because $-(dF/dt)\,dt$ gives the probability that the electron collides between t and $t + dt$. Substituting $F(t)$ from (5.20) into (5.21) we find

$$\langle t \rangle = \tau_c \tag{5.22}$$

Hence, the physical meaning of τ_c introduced above is that it represents the *average time between collisions*.

We are now prepared to answer the question: what is the rate of change of $\langle v_x \rangle$ for a group of electrons resulting from collisions with the obstacles described earlier? In accordance with (5.16) an electron loses per collision on the average

$$\langle v_x \rangle (1 - \langle \cos\theta \rangle)$$

Since the number of collisions carried out per second is on the average equal to $1/\tau_c$, we have

$$\left[\frac{\partial}{\partial t}\langle v_x \rangle\right]_{\text{coll}} = -\frac{\langle v_x \rangle}{\tau_c}(1 - \langle \cos\theta \rangle) \tag{5.23}$$

Comparing (5.23) with expression (5.9) which defines the relaxation time, we see that the relaxation time is related to the mean time between collisions as follows:

$$\tau = \frac{\tau_c}{1 - \langle \cos\theta \rangle} \tag{5.24}$$

When the scattering is isotropic, $\langle \cos\theta \rangle = 0$ and $\tau = \tau_c$. If the scattering takes place predominantly in the direction of incidence, $\langle \cos\theta \rangle$ will have

a positive value, and the relaxation time $\tau > \tau_c$. This is understandable qualitatively, because the electrons then will "remember" longer what their initial direction of motion was, and consequently the time required to make the velocity distribution random increases.

The mean free path of an electron, λ, will be defined as

$$\lambda = v\tau_c \tag{5.25}$$

where v is the total velocity of an electron. It will be evident that if the obstacles are hard spheres, the quantity λ would be determined by the concentration of obstacles. Thus, in such a case the collision time τ_c would be inversely proportional to the velocity v of the electrons for a given concentration of obstacles. Consequently, when $\langle \cos \theta \rangle$ is independent of the velocity of the electrons, the relaxation time will be inversely proportional to v. This raises an important question, because in the preceding section we introduced a single relaxation time τ, without any reference to the distribution of electron velocities. In the case of metals, however, it can be shown that the relaxation time τ which determines the conductivity always refers to a single total velocity, viz., the one corresponding to the *Fermi energy*. In order to explain this properly, one would have to carry through a procedure involving Fermi statistics, a subject which lies outside the scope of the present treatment. A few qualitative remarks may suffice here. The free electrons in a metal can accept a series of discrete energy levels. At absolute zero, all energy levels below a certain value W_F are filled, all those above W_F being empty; W_F is called the Fermi level of the electrons. The magnitude of W_F is of the order of several electron volts. At any temperature T, the probability of a state corresponding to the energy W being occupied by an electron is given by

$$f(W) = \frac{1}{\exp\left[(W - W_F)/kT\right] + 1} \tag{5.26}$$

For $T = 0$, this function is a step function, as indicated in Fig. 5.5; for $T > 0$, one obtains a curve which differs slightly from the one correspond-

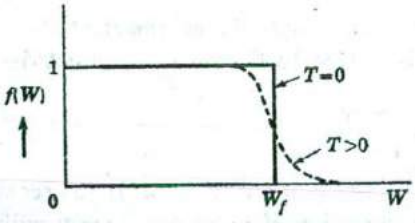


Fig. 5.5. Schematic representation of the Fermi distribution function (5.26) for $T = 0$ and for a temperature $T > 0$. Actually, the width of the energy region over which the dashed curve differs appreciably from the curve for $T = 0$ is very small compared to W_f for practical temperatures.

ing to $T = 0$, as indicated in an exaggerated fashion in Fig. 5.5. Since kT at room temperature is only 0.025 ev, kT is very small compared to W_F, and, consequently, the energy region over which temperature influences the energy distribution of the electrons is very narrow. The velocity of the electrons with the Fermi energy, v_F, is given by

$$\tfrac{1}{2}mv_F^2 = W_F \tag{5.27}$$

For $E_F = 5$ ev, which is a representative figure, one finds $v_F \cong 4 \times 10^6$ meters per second (1 ev = 1.6×10^{-19} joules). The relaxation time appearing in the theory of conductivity of metals refers to electrons with a velocity v_F.

We have seen then that it is possible to interpret the relaxation time in terms of collisions of the electrons with obstacles. Our next task will be to determine what these obstacles are, and if the model so obtained agrees with the observed dependence of the conductivity on temperature, impurity concentration, etc.

5.3 Electron scattering and the resistivity of metals

Formula (5.13) derived in section 5.1 permits us to calculate τ from experimental values of the conductivity, or at least to obtain an estimate of order of magnitude. If one assumes for monovalent metals that the number of free electrons per m^3 is equal to the number of atoms per m^3, one obtains at 0°C values for τ as given in Table 5.1. It is observed that

Table 5.1. CONDUCTIVITY (σ), FERMI LEVEL (W_F), MEAN FREE PATH (λ), AND RELAXATION TIME (τ) AT 0°C FOR SOME MONOVALENT METALS

Metal	σ (10^8 ohm^{-1} m^{-1})	W_F (ev)	λ (angstroms)	τ (10^{-14} sec)
Li	0.12	4.7	110	0.9
Na	0.23	3.1	350	3.1
K	0.19	2.1	370	4.4
Cu	0.64	7.0	420	2.7
Ag	0.68	5.5	570	4.1

the relaxation times are of the order of 10^{-14} seconds. Since this number may not convey to the reader any more than the fact that it is small, it is useful to consider the results in terms of the mean free path defined by (5.25). It was mentioned in the preceding section that τ in metals refers to electrons with the Fermi velocity v_F. Thus, if we assume that the scattering in these metals is isotropic (it can be shown that this is a good approximation for the temperatures of interest), we have $\tau = \tau_c$

and $\lambda = v_F \tau$. The values of λ so obtained are also indicated in Table 5.1. It is noted that at 0°C, the mean free path for scattering is several hundred angstroms. We may mention further that the resistivity of metals at low temperatures decreases to very small values, as may be seen from the example in Fig. 5.6 for sodium. Although some of the assumptions made

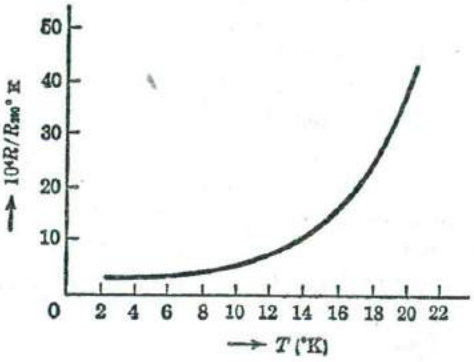


Fig. 5.6. The resistivity of sodium as a function of temperature in the low temperature region. The constant value approached at low temperatures is due to lattice imperfections and impurities, and corresponds to ρ_i in the text.

above are not applicable for liquid helium temperatures, we may state that the *resistivity of a perfect, pure single crystal of a metal goes to zero as T approaches zero.* In other words, at extremely low temperatures, λ reaches macroscopic values. From these arguments one can draw an important, though somewhat negative conclusion, viz., that the *obstacles* introduced earlier *are not the individual ion cores* in the metallic lattice. In fact, if we were to picture the conduction electrons as moving between the positive ion cores, one would expect, in terms of classical physics, mean free paths of the order of a few angstroms, as indicated in Fig. 5.7. This classical picture thus breaks down, and we have to look for another one. The reason for the breakdown of the classical picture lies in the wave aspect of

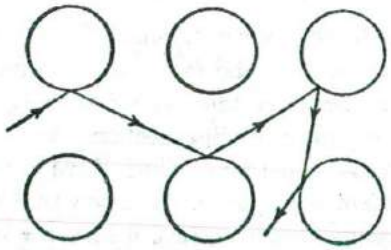


Fig. 5.7. Schematic representation of the classical picture of scattering of an electron by the ion cores; this picture predicts a mean free path of a few angstroms, in contradiction with experimental information.

elementary particles. In fact, it can be shown theoretically that when the wave nature of the electrons is taken into account properly, *an electron moving in a perfectly periodic lattice would not be scattered at all.* In other words, the waves representing a moving electron in a perfect crystal resemble the electromagnetic waves corresponding to visible light which passes undeviated through a crystal such as sodium chloride. Thus, the present interpretation of the electrical resistivity of metals is based on the notion that *scattering of the electrons is due to deviations from perfect periodicity of the lattice.* In pure metals, such deviations may be caused, for example, by the vibrations of the atoms in the lattice; they may also be caused by atoms which occupy "wrong" positions in the lattice, or by lattice sites which are unoccupied, or by impurity atoms. Thus, there are various kinds of obstacles which we have to consider in the scattering of electrons in metals, each of them giving rise separately to a certain relaxation time τ_j. Since the probability for scattering by a particular kind of obstacle per unit time is equal to $1/\tau_j$ (for isotropic scattering) the resultant relaxation time may be obtained from the relation

$$\frac{1}{\tau} = \sum_j \frac{1}{\tau_j} \tag{5.28}$$

since scattering probabilities are additive. Consider then a metal with a given concentration of imperfections, i.e. a given concentration of impurities and dislocated atoms. Let us assume that the relaxation time associated with all these imperfections is equal to τ_i. The resultant relaxation time may then be written as

$$1/\tau = 1/\tau_i + 1/\tau_T \tag{5.29}$$

where τ_T represents the relaxation time resulting from the thermal vibrations of the lattice. When the concentration of imperfections is independent of the temperature T, τ_i will presumably be independent of temperature (because λ_i is fixed and v_F is independent of T, for all practical purposes). The resistivity of the metal may then be represented in accordance with formula (5.13) by

$$\rho = \frac{m}{ne^2}\left(\frac{1}{\tau_i} + \frac{1}{\tau_T}\right) \tag{5.30}$$

The only temperature-dependent quantity appearing in this formula is τ_T (the number of electrons, n, in a metal is independent of T!). This formula then should explain the observed *positive temperature coefficient of the resistivity of metals* (we are ignoring some very special cases). As T increases, the amplitude of the atomic vibrations increases; hence the

deviations from periodicity become more severe and the scattering by lattice vibrations increases. Consequently, as T increases, λ decreases and since v_F is constant, τ_T decreases. Thus, as the temperature of a metal (or alloy) is raised, the electrical resistivity increases. At temperatures above the so-called Debye temperature of the material, the energy of the lattice vibrations increases linearly with T, and consequently, the resistivity increases linearly with T in this region, obeying the formula

$$\rho = \rho_i + aT \tag{5.30a}$$

where ρ_i and a are constants. The separation of the resistivity into a temperature-independent part and a temperature-dependent part is referred to as Matthiessen's rule. The Debye temperature may be considered as a measure for the maximum frequency of the lattice vibrations; hard materials have higher Debye temperatures than soft materials, as may be seen from the values given in Table 5.2 below. The Debye temperature plays an important role in the specific heat of a material in that above the Debye temperature the specific heat is approximately constant, whereas below it, the specific heat drops rapidly, approaching zero as $T \to 0$.

Table 5.2. DEBYE TEMPERATURE (θ_D) FOR SOME METALS, IN DEGREES ABSOLUTE

Metal	Ag	Al	Au	Cu	Fe	Pb
θ_D	215	398	180	315	420	88

As temperatures fall below the Debye temperature, τ_T increases rapidly and as T goes to zero, τ_T approaches infinity. There exists a close *relationship between the resistivity and the specific heat* of the lattice vibrations; in fact, the Debye temperature may be calculated from the resistivity versus temperature curve of a metal. Since the electrical resistivity at very low temperatures resulting from scattering of electrons by lattice vibrations becomes negligible, the total resistivity is essentially determined by the imperfections and impurities. Thus, with reference to Fig. 5.6, the constant resistivity near 4°K is a measure for the content of imperfections and impurities in the sodium sample.

The arguments given also make the following observation intelligible: when one adds to a metal like nickel a small amount of copper, the conductivity decreases, notwithstanding the fact that copper has a higher conductivity than nickel. The reason is, of course, that in replacing some of the nickel atoms in the lattice, the copper atoms destroy the lattice periodicity in their immediate neighborhood and thus act as new scattering centers for the conduction electrons.

elementary particles. In fact, it can be shown theoretically that when the wave nature of the electrons is taken into account properly, *an electron moving in a perfectly periodic lattice would not be scattered at all.* In other words, the waves representing a moving electron in a perfect crystal resemble the electromagnetic waves corresponding to visible light which passes undeviated through a crystal such as sodium chloride. Thus, the present interpretation of the electrical resistivity of metals is based on the notion that *scattering of the electrons is due to deviations from perfect periodicity of the lattice.* In pure metals, such deviations may be caused, for example, by the vibrations of the atoms in the lattice; they may also be caused by atoms which occupy "wrong" positions in the lattice, or by lattice sites which are unoccupied, or by impurity atoms. Thus, there are various kinds of obstacles which we have to consider in the scattering of electrons in metals, each of them giving rise separately to a certain relaxation time τ_j. Since the probability for scattering by a particular kind of obstacle per unit time is equal to $1/\tau_j$ (for isotropic scattering) the resultant relaxation time may be obtained from the relation

$$\frac{1}{\tau} = \sum_j \frac{1}{\tau_j} \tag{5.28}$$

since scattering probabilities are additive. Consider then a metal with a given concentration of imperfections, i.e. a given concentration of impurities and dislocated atoms. Let us assume that the relaxation time associated with all these imperfections is equal to τ_i. The resultant relaxation time may then be written as

$$1/\tau = 1/\tau_i + 1/\tau_T \tag{5.29}$$

where τ_T represents the relaxation time resulting from the thermal vibrations of the lattice. When the concentration of imperfections is independent of the temperature T, τ_i will presumably be independent of temperature (because λ_i is fixed and v_F is independent of T, for all practical purposes). The resistivity of the metal may then be represented in accordance with formula (5.13) by

$$\rho = \frac{m}{ne^2}\left(\frac{1}{\tau_i} + \frac{1}{\tau_T}\right) \tag{5.30}$$

The only temperature-dependent quantity appearing in this formula is τ_T (the number of electrons, n, in a metal is independent of T!). This formula then should explain the observed *positive temperature coefficient of the resistivity of metals* (we are ignoring some very special cases). As T increases, the amplitude of the atomic vibrations increases; hence the

deviations from periodicity become more severe and the scattering by lattice vibrations increases. Consequently, as T increases, λ decreases and since v_F is constant, τ_T decreases. Thus, as the temperature of a metal (or alloy) is raised, the electrical resistivity increases. At temperatures above the so-called Debye temperature of the material, the energy of the lattice vibrations increases linearly with T, and consequently, the resistivity increases linearly with T in this region, obeying the formula

$$\rho = \rho_i + aT \tag{5.30a}$$

where ρ_i and a are constants. The separation of the resistivity into a temperature-independent part and a temperature-dependent part is referred to as Matthiessen's rule. The Debye temperature may be considered as a measure for the maximum frequency of the lattice vibrations; hard materials have higher Debye temperatures than soft materials, as may be seen from the values given in Table 5.2 below. The Debye temperature plays an important role in the specific heat of a material in that above the Debye temperature the specific heat is approximately constant, whereas below it, the specific heat drops rapidly, approaching zero as $T \to 0$.

Table 5.2. DEBYE TEMPERATURE (θ_D) FOR SOME METALS, IN DEGREES ABSOLUTE

Metal	Ag	Al	Au	Cu	Fe	Pb
θ_D	215	398	180	315	420	88

As temperatures fall below the Debye temperature, τ_T increases rapidly and as T goes to zero, τ_T approaches infinity. There exists a close *relationship between the resistivity and the specific heat* of the lattice vibrations; in fact, the Debye temperature may be calculated from the resistivity versus temperature curve of a metal. Since the electrical resistivity at very low temperatures resulting from scattering of electrons by lattice vibrations becomes negligible, the total resistivity is essentially determined by the imperfections and impurities. Thus, with reference to Fig. 5.6, the constant resistivity near 4°K is a measure for the content of imperfections and impurities in the sodium sample.

The arguments given also make the following observation intelligible: when one adds to a metal like nickel a small amount of copper, the conductivity decreases, notwithstanding the fact that copper has a higher conductivity than nickel. The reason is, of course, that in replacing some of the nickel atoms in the lattice, the copper atoms destroy the lattice periodicity in their immediate neighborhood and thus act as new scattering centers for the conduction electrons.

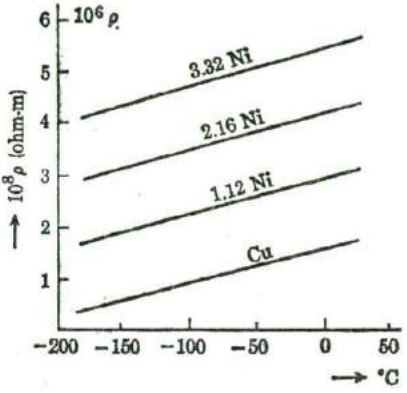


Fig. 5.8. The resistivity of alloys of nickel in copper as a function of temperature; the numbers indicate the atomic percentage of nickel in the alloy.

In Fig. 5.8 we give an example of the influence of temperature and of impurity content on the resistivity of copper. It is noted that ρ varies linearly with T in accordance with (5.30a), and that the resistivity of the alloys increases with increasing nickel content.

5.4 The heat developed in a current-carrying conductor

In this section we shall give an elementary derivation of the well-known experimental fact that the heat developed per m³ per second in a conductor carrying a current density J as a result of an applied field E is given by

$$W = JE = \sigma E^2 \tag{5.31}$$

If J is in amperes per m² and E is in volts per m, W is found in watts per m³. Consider a particular electron which at the instant $t = 0$ has carried out a collision with the lattice; at that instant let the velocity components of the electron be v_x, v_y, v_z. Assuming that at the instant $t(> 0)$ the electron has not yet collided again, and assuming the field E is applied along the *negative* x-direction, the velocity components of the electron at the time t will be given by

$$v_x + (e/m)Et, \qquad v_y, \qquad \text{and} \qquad v_z \tag{5.32}$$

The increment $(e/m)Et$ results from the acceleration along the x-direction produced by the field on the electron of charge $-e$. The increase in energy of the electron over the period t is therefore

$$(\Delta W)_t = \tfrac{1}{2}m[(2e/m)Ev_xt + (e/m)^2E^2t^2] \tag{5.33}$$

When this expression is averaged over a large number of electrons which have all lived through the period t without having suffered a collision,

but which presumably have a random distribution of their velocities, one finds

$$\langle \Delta W \rangle_t = \tfrac{1}{2}m[(e/m)^2E^2t^2] = e^2E^2t^2/2m \tag{5.34}$$

because $\langle v_x \rangle$ is zero for such a group.

Now, according to (5.20), the probability that an electron will live through the period t without making a collision is equal to $F(t) = \exp[-t/\tau_c]$. For simplicity we shall assume that the electrons are scattered isotropically, so that the average time between collisions, τ_c, is equal to the relaxation time τ. Also, the probability that an electron will suffer a collision during a time interval dt is given by $dt/\tau_c = dt/\tau$. Hence the product $e^{-t/\tau}(dt/\tau)$ gives us the probability that the electron makes a collision between t and $t + dt$. Consequently, the average energy increase of the electrons during the period between two collisions is equal to

$$\langle \Delta W \rangle = \int_{t=0}^{\infty} \langle \Delta W \rangle_t e^{-t/\tau} \frac{dt}{\tau} \tag{5.35}$$

where $\langle \Delta W \rangle_t$ is given by (5.34). Substituting (5.34) into (5.35) and carrying out the integration, one finds

$$\langle \Delta W \rangle = \tau^2 e^2 E^2/m \tag{5.36}$$

Assuming there are n electrons per m^3, and assuming that these electrons transfer their excess energy to the lattice, one obtains for the total energy dissipated per m^3 per second

$$W = \frac{n}{\tau}\langle \Delta W \rangle = \left(\frac{ne^2\tau}{m}\right) E^2 = \sigma E^2 \tag{5.37}$$

in agreement with the experimental relation (5.31).

5.5 The thermal conductivity of metals

When a homogeneous isotropic material is subjected to a temperature gradient, a flow of heat results in the direction opposite to the gradient. Thus, when dT/dx represents the temperature gradient and Q represents the heat flow density, one defines the thermal conductivity K by means of the equation

$$Q = -K(dT/dx) \tag{5.38}$$

When Q is expressed in watts per m^2, and dT/dx in degrees C per meter, one finds K in watt m^{-1} degree^{-1}. It is well known that, at least at normal temperatures, metals conduct heat much better than insulators. In Table 5.3 we have given values for K at room temperature for various materials to

Table 5.3. THERMAL CONDUCTIVITY K AT ROOM TEMPERATURE IN WATT METERS^{-1} DEGREES^{-1}, FOR VARIOUS MATERIALS

Material	K	Material	K
Al	209	Mica ($\perp$ cleavage plane)	0.75
Constantan (60 Cu, 40 Ni)	22.5	Quartz ($\parallel$ axis)	12.6
Cu	385	($\perp$ axis)	6.7
Fe	67	Carbon	4.2
Ni	54	Glass (window)	1.0

illustrate the range of thermal conductivities encountered. In insulating solids, the heat current is carried by the lattice vibrations. This, in part, is also the case in metals, but the thermal conductivity due to the conduction electrons predominates in these materials. We shall consider only the *thermal conductivity arising from the conduction electrons.*

Assuming a temperature gradient dT/dx along the x-direction, the average energy of the electrons will also be a function of x. Thus, let $W(0)$ be the average energy per electron in the plane $x = 0$. At a distance Δx from this plane, the energy of the electrons is then given by

$$W(\Delta x) = W(0) + \frac{dW}{dx}\Delta x = W(0) + \frac{dW}{dT}\frac{dT}{dx}\Delta x \tag{5.39}$$

The atomic picture of heat conduction by electrons is the following: One considers a unit area of the plane at $x = 0$, and calculates the net transport of energy through that plane resulting from electrons coming from the two regions $x > 0$ and $x < 0$. It is assumed that the energy of an electron which passes through the plane at $x = 0$ is equal to the average electron energy at Δx, where Δx gives the location along the x-direction where the electron last collided with the lattice. Thus, let v_{xi} represent the velocity component of a particular electron along the x-direction, and let $-\Delta x_i$ represent the location of the plane where it last collided If there were n_i such electrons per m^3, the number passing the plane $x = 0$ per second would be $n_i v_{xi}$. Since the energy carried by electron i is obtained by replacing Δx by $-\Delta x_i$ in (5.39), the total heat current density through the plane $x = 0$ in the positive x-direction may be written as

$$Q = \sum_i \left[W(0) - \frac{dW}{dT}\frac{dT}{dx}\Delta x_i \right] v_{xi} \tag{5.40}$$

where the summation extends over all electrons per m^3. Since the average velocity of the electrons per unit volume is zero, the sum containing the

quantity $W(0)$ vanishes. We are thus left with

$$Q = -\sum_i \frac{dW}{dT}\frac{dT}{dx} v_{xi}\,\Delta x_i = -n\frac{dW}{dT}\frac{dT}{dx}\langle v_x\,\Delta x\rangle \tag{5.41}$$

where n stands for the total number of conduction electrons per m³, and where $\langle v_x\,\Delta x\rangle$ is the average value of $v_x\,\Delta x$. Now, an electron moving in a direction making an angle θ with the x-axis has a velocity component $v_x = v\cos\theta$, where v is the total velocity. Similarly, if Δx measures the distance from the plane $x = 0$ to the plane where it collided last, the elec-

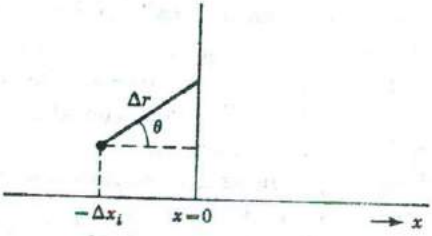


Fig. 5.9. Illustrating the geometry used in the derivation of the heat conductivity.

tron has actually covered a path length $\Delta r = \Delta x/\cos\theta$ between the instant it collided and the instant it arrives at the plane $x = 0$, as indicated in Fig. 5.9. We may thus write (5.41) in the form

$$Q = -n\frac{dW}{dT}\frac{dT}{dx}\langle v\,\Delta r\cos^2\theta\rangle = -\frac{1}{3}n\frac{dW}{dT}\frac{dT}{dx}\langle v\,\Delta r\rangle \tag{5.42}$$

where the last equality follows from the fact that

$$\langle\cos^2\theta\rangle = \frac{\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta}{\int_0^{\pi}\sin\theta\,d\theta} = \frac{1}{3} \tag{5.43}$$

At this point we shall make a statement that will not be proved here, but which bears relation to an earlier remark in connection with the electrical conductivity of metals: from a more rigorous treatment it can be shown that only the mean free path or the relaxation time corresponding to electrons with the Fermi energy is of importance. Thus v in (5.42) may be replaced by the velocity v_F (Fermi velocity), and Δr by λ; here, λ is the mean free path for scattering corresponding to electrons with the velocity v_F. Hence,

$$Q = -\frac{1}{3}n\frac{dW}{dT}\frac{dT}{dx}v_F\lambda \tag{5.44}$$

The quantity $n(dW/dT)$ represents the energy increase of the electron gas per m³ per degree; i.e., this quantity is simply the *specific heat of the*

electron gas at constant volume, C. Since $\lambda = v_F\tau$, where τ is the relaxation time, we have

$$Q = -\frac{1}{3} C v_F^2 \tau \frac{dT}{dx} \tag{5.45}$$

From Fermi-Dirac statistics it can be shown that the specific heat of the electron gas is equal to

$$C = \frac{n\pi^2 k^2 T}{m v_F^2} \tag{5.46}$$

The last two equations thus give

$$Q = -\frac{1}{3} \frac{n\pi^2 k^2 T \tau}{m} \frac{dT}{dx} \tag{5.47}$$

Comparing this expression with (5.38) we find for the thermal conductivity

$$K = \frac{1}{3} \frac{n\pi^2 k^2 T \tau}{m} \tag{5.48}$$

We can draw several conclusions from this result. First, we may estimate K from (5.48) and compare the result with the values given in Table 5.2. For $T = 300°\text{K}$, $n \cong 5 \times 10^{28}$ m^{-3}, $\tau \cong 10^{-14}$ seconds, $k = 1.38 \times 10^{-23}$ joule degree^{-1}, $m = 9 \times 10^{-31}$ kg, we find $K \cong 100$ watt m^{-1} degree^{-1}, which is of the correct order of magnitude. How does the thermal conductivity depend on temperature? All quantities appearing in expression (5.48) for K are independent of temperature, except T and τ. As mentioned in Section 5.3, τ varies as T^{-1} for temperatures above the Debye temperature. Hence, in this temperature region the *thermal conductivity is independent of temperature*. At low temperatures the behavior is more complicated, as illustrated in Fig. 5.10 for copper.

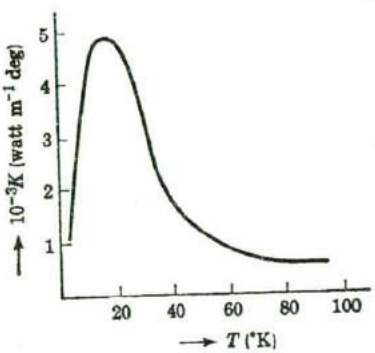


Fig. 5.10. The thermal conductivity of copper at low temperatures. [After R. Berman and D. K. C. MacDonald, *Proc. Roy. Soc.* (London) **A211**, 122 (1952)]

There exists an interesting relationship between the electrical conductivity and the thermal conductivity of a metal. According to ex-

pressions (5.13) and (5.48) we may write

$$\frac{K}{\sigma} = \frac{\pi^2}{3}\frac{k^2}{e^2}T \quad \text{or} \quad \frac{K}{\sigma T} = \frac{\pi^2}{3}\frac{k^2}{e^2} \equiv L \tag{5.49}$$

The theory thus predicts that $K/\sigma T$ for all metals should be equal to a universal constant; this law is known as the *Wiedemann-Franz law*. The constant represented by $L = 2.45 \times 10^{-8}$ watt ohm degree^2 is known as the *Lorenz number*. In Table 5.4 we give experimental values of the Lorenz number for various metals, at 0°C and at 100°C. It is observed that the agreement between theory and experiment is satisfactory. It should be remarked here that the contribution to K from the lattice vibrations has been neglected in our considerations. At low temperatures, the theory given does not apply, and a relaxation time for the thermal conductivity and electrical conductivity *cannot* be defined uniquely. Thus, the Lorenz number for copper near 15°K is an order of magnitude smaller than at room temperature. The Wiedemann-Franz law holds only above the Debye temperature.

Table 5.4. EXPERIMENTAL LORENZ NUMBER FOR VARIOUS METALS AT 0°C AND AT 100°C, EXPRESSED IN 10^{-8} WATT OHM DEGREE^2

Metal	0°C	100°C	Metal	0°C	100°C
Ag	2.31	2.37	Pb	2.47	2.56
Au	2.35	2.40	Pt	2.51	2.60
Cu	2.23	2.33	Mo	2.61	2.79

5.6 Superconductivity

A phenomenon which has defied a satisfactory atomic interpretation for many years is that of superconductivity. On the other hand, phenomenological theories based on thermodynamics (Gorter and Casimir) and on electrodynamics (London) have provided a certain amount of insight into the relationships between the various physical properties of superconductors. Also, new attempts to find an atomic interpretation made by Fröhlich and by Bardeen since 1950 appear to be promising, and it is not impossible that ultimately superconductivity can be understood within the framework of the modern theory of metals. A discussion of the various aspects of this subject falls outside the scope of this book and the interested reader is referred to the references provided at the end of this chapter. On the other hand, since superconductors have recently found applications in switching elements called *cryotrons*, it seems proper to provide at least a few experimental facts about superconductivity here.

Superconductivity was discovered in 1911 by Kamerlingh Onnes in Leiden when he observed that the electrical resistivity of mercury disappeared completely at temperatures below approximately 4.2 degrees Kelvin. As shown in Fig. 5.11, the resistance of the specimen measured

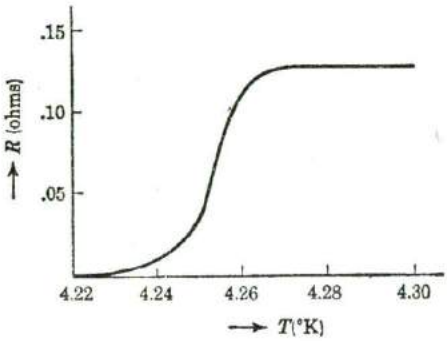


Fig. 5.11. The resistance of mercury as a function of temperature as observed by Kamerlingh Onnes in 1911.

by Onnes drops to zero within a very narrow temperature interval of only a few hundredths of a degree. Consequently, it is meaningful to introduce a *transition temperature*, T_c, at which the transition from the "normal" state to the superconducting state occurs. Since the discovery of the superconducting properties of mercury, many elements and some thirty compounds and alloys have been found which exhibit the same behavior. The superconducting elements and their transition temperatures are listed in Table 5.5. It is noted that the transition temperatures all lie

Table 5.5. SUPERCONDUCTING ELEMENTS AND THEIR TRANSITION TEMPERATURES, T_c (IN DEGREES ABSOLUTE)

Element	T_c	Element	T_c	Element	T_c
Al	1.14	Ru	0.47	Re	1.0
Ti	0.53	Cd	0.54	Os	0.71
V	5.1	In	3.37	Hg	4.12
Zn	0.79	Sn (white)	3.69	Tl	2.38
Ga	1.07	La	4.71	Pb	7.26
Zr	0.7	Hf	0.35	Th	1.32
Nb	9.22	Ta	4.38	U	0.8

below 10°K. It is, of course, possible that other elements become superconducting at sufficiently low temperatures, since measurements have not usually been carried out below about 0.07°K. It is interesting that the elements which at room temperature are good conductors (Cu, Ag,

Au and the alkali metals) are absent from the list of superconducting elements. In fact, the superconducting elements are relatively poor conductors at room temperature. One notices also the absence from Table 5.5 of the ferromagnetic elements Fe, Ni, Co. It may be mentioned here that the superconducting elements occur between rather well defined limits of the atomic volume (the average volume per atom).

The superconducting compounds and alloys do not necessarily have superconducting components. For example, Matthias and coworkers (1956) at Bell Telephone Laboratories recently prepared superconducting alloys of rhodium in molybdenum, neither of these elements having superconducting properties by themselves. Some superconducting compounds and alloys are listed in Table 5.6, together with their transition temperatures.

Table 5.6. SOME SUPERCONDUCTING COMPOUNDS AND ALLOYS AND THEIR TRANSITION TEMPERATURES, T_c (IN DEGREES ABSOLUTE)

Compound	T_c	Compound	T_c
Pb_2Au	7.0	NbN	14.7
$PbTl_3$	3.8	MoN	12.0
SnSb	3.9	NbB	6
CuS	1.6	ZrC	2.3

The question may be raised as to whether the resistivity of a superconductor is actually zero or just very small. In reply to this question one may state that in an unpublished experiment by Professor Collins at M.I.T., an induced current of several hundred amperes has been observed to flow in a superconducting lead ring for over a year, without a detectable change. It thus seems that $\rho = 0$ may be considered as one of the characteristic properties of the superconducting state.

The transition temperature of a superconductor can be reduced by

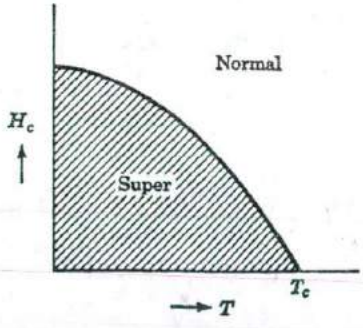


Fig. 5.12. Schematic representation of the critical magnetic field H_c as a function of temperature.

the application of a magnetic field. With reference to Fig. 5.12, suppose a superconductor has a temperature $T < T_c$. If a magnetic field H is applied, the material remains superconducting until a critical field H_c is reached such that for $H > H_c$, the material is in the normal state. The curve in Fig. 5.12 represents schematically the functional relationship between the critical field H_c and the temperature of the superconductor. The transition from the superconducting to the normal state under influence of a magnetic field is reversible. The function $H_c(T)$ follows with good accuracy a formula of the form

$$H_c(T) = H_0\left(1 - \frac{T^2}{T_c^2}\right) \tag{5.50}$$

where H_0 and T_c are constants characteristic of the material. Experimental $H_c(T)$ curves for a number of elements are given in Fig. 5.13.

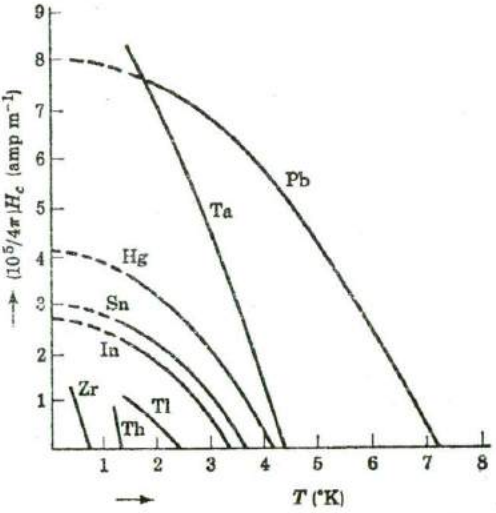


Fig. 5.13. The critical magnetic field H_c as a function of temperature for various elements.

The magnetic field which causes a superconductor to become normal is not necessarily an externally applied field; it may also arise as a result of electric current flow in the conductor. Thus, superconductivity in a long circular wire of radius r may be destroyed when the current I exceeds the value I_c, which at the surface of the wire would produce the critical magnetic field H_c. Hence, according to expression (4.8), the critical current $I_c = 2\pi r H_c$; this is referred to as *Silsbee's rule*. This rule prevents the use of superconductors as coils for the production of strong magnetic fields.

The disappearance of superconductivity for fields above the critical field is the principle on which a cryotron operates. With reference to Fig. 5.14, consider a wire made of a superconducting material A, inside a coil of a superconducting material B. Let the temperature of the system

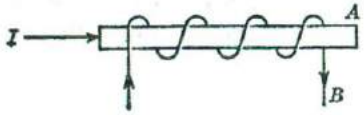


Fig. 5.14. A single cryotron in which the current I in A is controlled by the current in B.

be below the transition temperatures of the two materials, so that both are superconducting. The current in the central wire A can then be controlled by a current in the coil because the magnetic field produced by the latter may exceed the critical field of the core material at the operating temperature. The control current required to make the core "normal" depends on the d-c current flowing through the core because this current also produces a magnetic field. A metal such as tantalum is useful as a core material if the operating temperature is that of a liquid helium bath (4.2°K). The reason is, that for tantalum, $T_c = 4.38$ so that the critical field required to produce the normal state is relatively small (see Fig. 5.13). The coil material must be chosen so that it remains superconducting even if the control current flows; niobium or lead are thus suitable coil materials. The single cryotron represented in Fig. 5.14 may be used as an element in a more complicated device such as a flip-flop. Since the elements are superconductors, the power consumption is very low and it has been estimated that a large-scale digital computer could be built requiring no more than $\frac{1}{2}$ watt (not counting the helium cryostat and terminal equipment!). Also, since the elements can be made small, the essential parts of such a computer would occupy only one cubic foot.

So far, we have discussed only one aspect of superconductivity, viz., the fact that $\rho = 0$. An important experiment was carried out in 1933 by Meissner and Ochsenfeld which showed that the magnetic flux density in a superconductor also vanishes, i.e., $\mathbf{B} = 0$. This result does not follow from the fact that $\rho = 0$, as can be seen in the following way: if $\rho = 0$, the electric field $\mathbf{E}$ in a superconductor carrying a non-vanishing current must also vanish. In accordance with one of Maxwell's equations we have curl $\mathbf{E} = -d\mathbf{B}/dt$ so that, for $\mathbf{E} = 0$, this predicts $d\mathbf{B}/dt = 0$, but *not* $\mathbf{B} = 0$. In other words, the Meissner-Ochsenfeld experiment requires the characterization of a superconductor by *two* relations: $\rho = 0$ and $\mathbf{B} = 0$. For a detailed exposition of the consequences of these two relations for

the macroscopic behavior of superconductors the reader is referred to London's book cited below. Note that according to formula (4.24) the equation $\mathbf{B} = 0$ means that the magnetization in a superconductor $\mathbf{M} = -\mathbf{H}$. Consequently, since in general $M = (\mu_r - 1)H$ (see expression 4.23), we may also state that *for a superconductor the relative permeability is zero;* this is referred to as *perfect diamagnetism.*

With reference to the atomic interpretation of superconductivity, the results of experiments by Maxwell, and by Reynolds and coworkers, (1950) may be mentioned here. Making use of isotopes of mercury, they observed a decrease in the transition temperature T_c with increasing isotopic mass M such that

$$M^{1/2}T_c = \text{constant} \tag{5.51}$$

The same relationship has since been found for tin and lead isotopes. It is important to realize that a change in the isotopic mass leaves the electronic structure of the material unaltered; i.e., the relationship (5.51) indicates that the phenomenon of superconductivity is determined in some way by the interaction between the electrons and atoms in the superconductor. In the preceding sections we have seen that the normal resistivity of a metal is determined in part by the interaction between electrons and lattice vibrations. Since the vibrational frequency of an elastically bound particle is proportional to the square root of the mass of the particle, expression (5.51) suggests strongly that superconductivity may also result from the interaction of electrons and lattice vibrations. This idea was actually advanced before the experiments mentioned were carried out, and forms the basis of the theories of Frölich and Bardeen.

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Problems

5.1 A copper wire is 1 m long and has a uniform cross section of 0.1 mm^2; the resistance of the wire at room temperature is found to be 0.172 ohm. What is the resistivity of the material?

5.2 A uniform silver wire has a resistivity of 1.54×10^{-8} ohm m at room temperature. For an electric field along the wire of 1 volt cm^{-1} compute the average drift velocity of the electrons, assuming there are 5.8×10^{28} conduction electrons per m^3. Also calculate the mobility and relaxation time of the electrons.

5.3 Assuming the Fermi energy for silver is 5.5 ev, find the velocity of an electron with the Fermi energy. Compare the answer with that for the drift velocity in the preceding problem. What is the mean free path for scattering in the silver wire of the preceding problem?

5.4 A copper wire has a resistivity of 1.8×10^{-8} ohm m at room temperature (300°K). Assuming the copper is very pure, estimate the resistivity at 700°C and the percentage change in the resistivity from room temperature to 700°C.

5.5 Nichrome is a well-known alloy of nickel, iron, and chromium used for electric furnace windings. It has a resistivity of 1.0×10^{-6} ohm m at room temperature, and heating to a temperature of 700°C increases its resistivity by approximately 7 percent. Compare this information with that obtained in problem 5.4 and explain the difference in behavior of nichrome and copper. Assuming Matthiessen's rule holds for nichrome, find the resistivity due to impurity scattering alone.

5.6 If one plots the resistivity of disordered alloys of gold in copper as a function of the atomic concentration of gold, one obtains a dome-shaped curve with the following features: it starts at 1.7×10^{-8} ohm m for pure copper, then rises to a maximum value of 14×10^{-8} ohm m, occurring for equal atomic concentrations of the two elements, and then decreases to the value of 2.0×10^{-8} ohm m for pure gold. Explain the shape of this curve qualitatively.

5.7 Find the probability for an electronic state to be occupied at room temperature if the energy of this state lies 0.1 ev above the Fermi level. Do the same for a state which lies 0.1 ev below the Fermi level.

5.8 In the Fermi distribution function introduce a new variable $x = (W - W_F)/kT$; x measures the energy in units of kT relative to the Fermi energy W_F. Show that $-dF/dx$ is a symmetrical function with a maximum at $x = 0$. On the basis of the information so obtained, show that $\partial F/\partial W$ has an appreciable value only in an energy region of the order of kT on either side of the Fermi level.

5.9 A specimen of pure annealed copper has a resistivity of 1.56×10^{-8} ohm m at 300°K. If nickel is added to copper, the resistivity increases by 1.25×10^{-8} ohm m per added atomic percent nickel. Similarly, silver increases the resistivity of copper by 0.14×10^{-8} ohm m per added atomic percent. For an alloy of 0.2 atomic percent nickel and 0.4 atomic percent silver in copper, what is the theoretical resistivity of the alloy at 300°K and at 4°K? Also calculate the percentage increase in the resistivity per degree centigrade for the alloy and for pure copper at 300°K.

5.10 Given that the relaxation time of the conduction electrons in copper is equal to 2.7×10^{-14} seconds, compute the average increase in the x-component of the velocity between two collisions when an electric field of 1 volt cm^{-1} is applied in the negative x-direction. Compare this increase with the actual velocity of an electron with the Fermi energy ($W_F = 7.0$ ev for copper). What is the average increase in energy of the electrons between two collisions?

5.11 The temperature difference between the inside and outside of a glass window is 72° Fahrenheit. The glass has a thermal conductivity of 0.0025 calories sec^{-1} cm^{-1} degree^{-1} and is 1 mm thick. Find the energy loss in joules through the window per m^2 per hour.

5.12 The electrical resistivities of copper and kanthal (an alloy of iron, chromium and aluminum) at room temperature are respectively 1.7×10^{-8} and 1.4×10^{-6} ohm m. Assuming the Wiedemann-Franz law holds for these materials, find the *electronic* contributions to the thermal conductivities of these materials. (In alloys such as kanthal, the contribution to K from the lattice vibrations is comparable to that from the electrons.)

5.13 For pure metals, the theory predicts a thermal conductivity which is practically independent of temperature above the Debye temperature. Why is this so? In contrast, consider an alloy such as nichrome or kanthal in which the scattering of electrons due to "impurities" predominates over scattering by lattice vibrations. What temperature dependence do you predict for the electronic thermal conductivity in these alloys?

5.14 Suppose the conduction electrons in a metal are subjected to an alternating electric field $F = E_0 \cos \omega t$ applied along the x-direction. Let v_x represent the average x-component of the velocity of the electrons and let τ be their relaxation time. Set up the equation of motion for an electron and show that

$$v_x(t) = -\frac{e\tau E_0}{m}\left[\frac{\cos \omega t}{1 + \omega^2\tau^2} + \frac{\omega\tau \sin \omega t}{1 + \omega^2\tau^2}\right]$$

where m is the electronic mass. Note that the first term in brackets is in phase with the field, whereas the second lags 90 degrees behind the field.

From this result, show that the frequency dependence of the conductivity is given by $\sigma(\omega) = \sigma_{\text{static}}/(1 + \omega^2\tau^2)$.

5.15 Consider the conduction electrons in a metal subjected to an alternating electric field produced by light waves. Since $\tau \approx 10^{-14}$ seconds, $\omega\tau \gg 1$ for such fields, and the equation of motion for an electron is simply $m(d^2x/dt^2) = -eE_0 \cos \omega t$, where x represents the displacement resulting from a field $E_0 \cos \omega t$ along the x-direction. What is the polarization P in the metal if there are N free electrons per unit volume? From this, show that the relative dielectric constant is given by

$$\epsilon_r = 1 - (Ne^2/\epsilon_0 m\omega^2)$$

(If $\epsilon_r > 0$, the metal is transparent to light at normal incidence; if $\epsilon_r < 0$ one can show that total reflection occurs. When applied to sodium, one finds that it is transparent in the ultraviolet for wavelengths $\lambda \lesssim 2100$ angstroms, in agreement with experiment.)

6

The Mechanism of Conduction in Semiconductors

6.1 Classifying materials as semiconductors

In principle, the interpretation of the electrical conductivity of semiconductors is similar to that for metals. However, as a result of the difference in the nature of the chemical bond in these materials as compared to the metallic bond, the details are more complicated. The first question which arises is: which materials are classified as semiconductors, and on what basis are they classified as such? In order to answer this question, let us first consider Table 6.1 in which the electrical resistivity of a number of materials is given at 20°C.

Table 6.1. ELECTRICAL RESISTIVITY OF VARIOUS MATERIALS AT 20°C, IN OHM METER

Metals		Semiconductors		Insulators	
Ag.....	1.6×10^{-8}	Ge ("pure")	0.47	glass	$10^{10}-10^{11}$
Cu.....	1.7×10^{-8}	Si ("pure")	3000	mica	9×10^{14}
Al.....	2.8×10^{-8}	Fe_3O_4	~ 0.01	diamond	10^{14}
Fe.....	10×10^{-8}	InSb	$\sim 2 \times 10^{4}$		
Constantan..	49×10^{-8}	Sn (gray)	$\sim 2 \times 10^{-6}$		
Nichrome....	100×10^{-8}				

It is observed that at this temperature, the materials classified as semiconductors have a resitivity somewhere between that of a typical metal and that of a typical insulator. However, since the resistivity of semi-

conductors in general depends strongly on temperature, this kind of classification is not altogether satisfactory. In fact, at very low temperatures, say near liquid helium temperatures, a semiconductor may become indistinguishable from an insulator as far as the magnitude of the resistivity is concerned. A better criterion for characterizing semiconductors may be established by considering Fig. 6.1 which gives a schematic plot of resis-

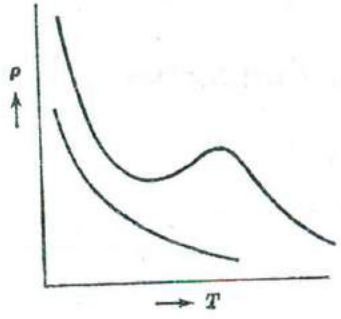


Fig. 6.1. Two typical forms of the resistivity of a semiconductor as function of temperature.

tivity versus temperature for two typical cases encountered in practice. It is observed that these materials exhibit a *negative temperature coefficient of the resistivity*, at least over part of the temperature range. In the preceding chapter we have seen that metals, on the other hand, have a positive $d\rho/dT$ for all temperatures (except in some very special cases). The occurrence of a negative $d\rho/dT$ is therefore a generally accepted basis for classifying a material as a semiconductor. At this point it should be remarked that insulators also have a negative $d\rho/dT$, and that in this respect they behave as semiconductors. The difference between semiconductors and insulators is therefore only of a *quantitative* nature, as will be seen more clearly later; insulators and semiconductors differ only in the magnitudes of their resistivity.

Although silicon and germanium are the most popular semiconductors, there are, of course, many others. In Table 6.2 we have indicated the semiconducting elements as they appear in the periodic system; these materials are called *elemental semiconductors*. There are also a large number of compound semiconductors, such as metallic-oxides and sulfides. Many of these are of great practical importance. For example, PbS is used in photoconductive devices, BaO in oxide coated cathodes, cesium antimonide in photomultipliers, etc. Within the scope of the present volume it is impossible to discuss the properties of all these materials, and we shall be interested in general principles rather than in specific materials. Because of the importance of germanium and silicon in present-day electrical engi-

Table 6.2. THE A SUBGROUPS OF THE 3RD, 4TH, 5TH, 6TH, AND 7TH COLUMNS OF THE PERIODIC SYSTEM OF ELEMENTS; THE ELEMENTS ENCLOSED BY THE LINE ARE SEMICONDUCTORS

IIIA	IVA	VA	VIA	VIIA
B	C	N	O	F
Al	Si	P	S	Cl
Ga	Ge	As	Se	Br
In	Sn	Sb	Te	I
Tl	Pb	Bi	Po	At

neering, we shall use these materials to illustrate the behavior of elemental semiconductors; some general remarks on compound semiconductors will be made as the discussion warrants them.

6.2 The chemical bond in Si and Ge and its consequences

Silicon and germanium are elements of the fourth column in the periodic table. In the solid state they crystallize into what is known as the *diamond*

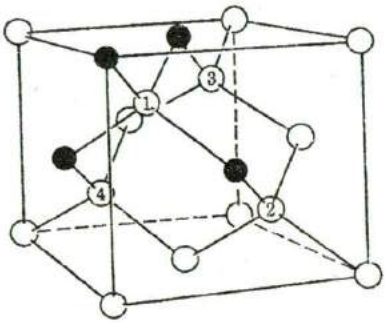


Fig. 6.2. The crystal structure of diamond, showing each atom surrounded by four others at the corners of a regular tetrahedron. The lattice consists of a face-centered cubic arrangement of atoms plus four atoms inside the cube; the latter are numbered 1, 2, 3, 4. The four atoms which are bound to atom No. 1 are shown in black.

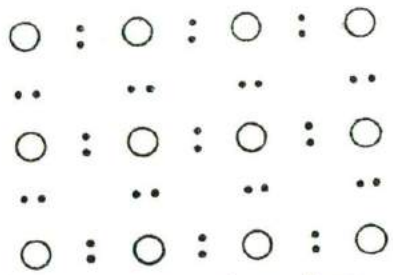


Fig. 6.3. Schematic two-dimensional representation of the electron-pair bonds in the diamond structure; the dots represent electrons. In the actual three-dimensional structure, the electron-pair bonds emerge from a given atom to the corners of a regular tetrahedron, the angles between the bonds being approximately 109°.

structure, represented in Fig. 6.2. In this structure, a given atom is surrounded by four others occupying the corners of a regular tetrahedron. The bonds between a given atom in this structure and its neighbors are called *electron-pair bonds* because bonds of this type are accomplished by pairs of electrons, each partner contributing one electron to each bond. Thus, since each atom has four valence electrons, it has just enough to provide for electron-pair bonds with four other atoms. A two-dimensional schematic picture of the electron-pair bonds is given in Fig. 6.3. In the actual three-dimensional lattice, the bonds make angles of approximately 109 degrees with each other, in a fashion similar to the four bonds in a molecule such as CH_4. Bonds of this kind, in which the atoms share electrons with each other are referred to as *homopolar*, in contrast with *heteropolar* or ionic bonds.

In a pure silicon or germanium crystal at absolute zero, all valence electrons take part in the homopolar bonds mentioned; i.e., *there are no electrons which have the freedom to move through the crystal.* This situation may be contrasted with that encountered in metals, in which there is a constant number of "free" electrons at all temperatures. Thus, at absolute zero, silicon and germanium are insulators, simply because there are no mobile charge carriers available. Suppose now that the temperature is raised; i.e., the vibrations carried out by the atoms in the lattice are made more violent. Some of the valence electrons may then absorb a sufficient amount of energy from the lattice vibrations to be released from the bonds, and once set free they can move through the crystal and contribute to the conductivity in an applied electric field. It is worthwhile at this point to investigate the consequences of the presence of a broken bond. With reference to Fig. 6.4 suppose an electron is taken away from a particular

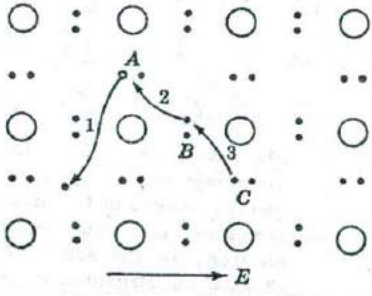


Fig. 6.4. Schematic illustration of the consequences of breaking an electron-pair bond A; the open circle at A represents a hole. By the successive electron jumps 1, 2, 3, the hole is created and moves in the direction of the field; the released electron drifts in a direction opposite to the field.

bond A, as indicated. We are then left with a bond which contains only one electron instead of two. Suppose now that the material is subjected to an electric field as indicated. An electron from a bond such as B may then,

under influence of the field, tend to move into the vacant electronic state in A; this leaves a vacant electronic state in bond B and an electron from another bond such as C may fill this vacant state. By repeating this process we see that charge is transported under influence of an electric field by virtue of the *absence* of an electron in an electron-pair bond. Note that *the vacant electronic state moves in the same direction as would a positive charge carrier*. For this reason, these vacant electronic states are called *positive holes*, or simply *holes*. We have thus come to the qualitative conclusion that as the temperature of the material is raised from zero to some temperature $T > 0$, a number of the bonds will be broken and conduction may be observed as a result of the motion of electrons and holes under influence of an external field.

For the pure elements Ge and Si under discussion, it is obvious from what has been said above that the number of "free" electrons must equal the number of mobile holes. Semiconductors of this kind are called *intrinsic semiconductors*. They must be distinguished from *extrinsic semiconductors*, in which the charge carriers (electrons or holes) are present as a result of *impurities* built into the crystal.

When a "free" electron moves about in the crystal it may encounter one of the broken bonds mentioned earlier. It is then possible that the electron recombines with the hole, and consequently two charge carriers are lost for the conduction process. On the other hand, any of the normal bonds may generate a free electron and a hole by absorption of energy from the lattice vibrations. Thus, at a given temperature T one will ultimately establish an equilibrium situation in which the number of electron-hole pairs generated by thermal motion per second equals the number of electron-hole pairs lost per second due to recombination. At a given temperature then, there will be a certain average number of electrons and an equal average number of holes in the material. As the temperature is raised, the average number of free charge carriers increases. This process of generation and recombination may be written in the form of a chemical reaction as follows:

$$\text{bound electron} \leftrightarrows \text{free electron} + \text{hole} \tag{6.1}$$

The extent to which this reaction proceeds to the right at a given temperature depends on the chemical bond. If the electrons are bound strongly, the number of charge carriers will be relatively small.

6.3 The density of carriers in intrinsic semiconductors; the energy gap

In this section we shall put the qualitative picture outlined in the preceding section on a more quantitative basis. This is usually done in terms of a so-called *energy band picture*, which arises from the following arguments. We have seen that at absolute zero, all valence electrons in a material such as germanium take part in the bonds between the atoms. We have also seen that such a valence electron may become free upon absorption of a sufficient amount of energy. This implies that the energy of a free electron is higher than that of a bound valence electron, and schematically we may thus consider the simple energy diagram represented in Fig. 6.5.

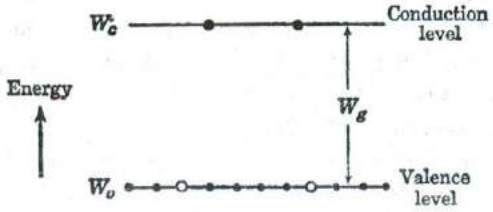


Fig. 6.5. Schematic illustration of the energy of an electron taking part in the valence bonds (dots in the valence level W_v) and the energy of electrons moving freely through the crystal (W_c). At absolute zero, all states in the W_v level are occupied by electrons and all those in the W_c level are empty. At temperatures different from zero, a fraction of the valence electrons is thermally excited into the conduction level (dots in W_c level), leaving holes in the valence level indicated by the open circles.

In this diagram it has been assumed that all valence electrons have the same energy W_v, whereas all free electrons have been assumed to have an energy W_c; the subscript "c" refers to "conduction" electrons, which is the usual name for the free electrons in semiconductors. We shall refer to the W_c level as the "conduction level," and to W_v as the "valence level." The black dots in Fig. 6.5 represent electrons, the open circles represent holes. At absolute zero, let there be N electrons per m^3 in the valence level and, of course, no electrons in the conduction level. Given the energy difference

$$W_g = W_c - W_v \tag{6.2}$$

how does the number of conduction electrons vary with temperature? In equilibrium at a temperature T, let there be n conduction electrons per m^3 and an equal density of holes in the intrinsic semiconductor under dis-

cussion. In the preceding section we have seen that conduction electrons may recombine with holes. The probability that such a recombination process will occur is proportional to the number of electron-hole collisions; thus, the number of recombination processes per m^3 per second will be proportional to the density of electrons (n) and to the density of holes (n). Hence, we may write

$$\left(\frac{\partial n}{\partial t}\right)_{\text{recombination}} = -rn^2 \tag{6.3}$$

where r is a constant; the minus sign indicates a decrease in n resulting from recombination. The rate at which electron-hole pairs are created by absorption of thermal energy is presumably proportional to the density of electrons which are available for excitation, $(N - n)$, and to a Boltzmann factor exp $(-W_g/kT)$, where W_g is the energy gap between the valence and conduction levels. Thus,

$$\left(\frac{\partial n}{\partial t}\right)_{\text{thermal excitation}} = G(N - n)e^{-W_g/kT} \tag{6.4}$$

where G is a constant. Since in equilibrium the sum of (6.3) and (6.4) must vanish ($\delta n/\delta t = 0$), we conclude that

$$rn^2 = G(N - n)e^{-W_g/kT} \tag{6.5}$$

As long as $n \ll N$, which is usually the case, one finds for the density of conduction electrons

$$n = \text{constant} \times e^{-W_g/2kT} \tag{6.6}$$

Note that the density of conduction electrons (which in our case equals the hole density) is determined by the ratio of *half* the gap width and kT; the factor of $\frac{1}{2}$ enters as a result of the n^2 in the recombination rate.

The picture given here is inaccurate in the sense that in a solid, the valence electrons do not all have the same energy; nor do the conduction electrons all have the same energy. In fact, if atoms are brought together to form a solid, the discrete atomic energy levels broaden into a band of very closely spaced energy levels; the broadening is a result of the interaction between the atoms. The same is true for the energy levels which are available to the conduction electrons. A more accurate electron energy level picture for intrinsic semiconductors and insulators is represented in Fig. 6.6. On the basis of certain simplifying assumptions concerning the relation between the energy and wavelength of the electrons in the valence band and in the conduction band, one again finds for n an expression of the same form as (6.6). The constant can be evaluated from statistical mechanics under these circumstances, and as shown in monographs on

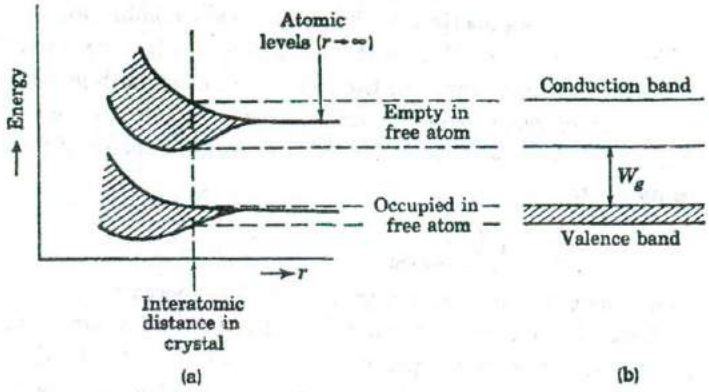


Fig. 6.6. In (a) we have represented schematically how the electronic energy levels broaden into bands when the atoms are brought close together. In the crystalline state, the energy band picture presented in (b) is derived from (a) as indicated, leading to a band of energies for the valence electrons (valence band), and a band of energies corresponding to electrons moving about in the crystal (conduction band).

semiconductors one finds

$$n = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \left(\frac{m_e m_h}{m^2}\right)^{3/4} T^{3/2}\, e^{-W_g/2kT} \tag{6.7}$$

Here, h is Planck's constant ($= 6.62 \times 10^{-34}$ joule sec); m is the mass of an electron in free space. The quantities m_e and m_h represent respectively the effective mass of a conduction electron and the effective mass of a hole. In general, m_e and m_h are not equal to m. The reason for their appearance is that in the modern theory of solids the electrons are described in terms of wave mechanics. As a result of the wave character of the electrons, the motion of an electron in a crystal under influence of an electric field is different from that of an electron in free space. The theory shows that this difference can be expressed formally by introducing an effective mass m_e for an electron in a crystal so that the acceleration produced by an electric field E is still given by the classical Newtonian equation $a = -eE/m_e$.

Assuming for the moment as an approximation that $m_e = m_h = m$, one finds from (6.7) upon substituting the numerical values

$$n \cong 5 \times 10^{21} T^{3/2}\, e^{-W_g/2kT} \text{ per m}^3 \tag{6.8}$$

where T is expressed in degrees absolute. As an example, consider a semiconductor with an energy gap $W_g = 1$ ev, which is typical for semiconduc-

tors such as Ge and Si. At room temperature, $kT \cong 0.025$ ev, so that $W_g/2kT \cong 20$. This gives $n \cong 5 \times 10^{16}$ per m^3, whereas the number of atoms is approximately 5×10^{28} per m^3. Hence, only a very small fraction of the valence electrons is actually excited into the conduction band at this temperature.

The exponent in (6.7) provides the predominant part of the temperature dependence of n, the $T^{3/2}$ factor varying relatively slowly. The functional relationship between n and T may be given in the form of a plot of $\log (n/T^{3/2})$ versus $1/T$; this yields a straight line in those cases for which the assumptions involved in the derivation of (6.7) are valid. A schematic plot of this kind is given in Fig. 6.7.

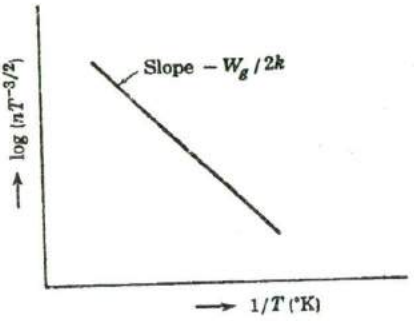


Fig. 6.7. Schematic illustration of the variation of the carrier density with temperature for an intrinsic semiconductor, according to equation (6.8).

Since the conductivity of a material is proportional to the density of charge carriers (see formula 5.13 for metals), the conductivity of a semiconductor depends strongly on the magnitude of the energy gap W_g. To illustrate this relationship, we consider in Table 6.3 the resistivity at room temperature and the energy gap for the elements in the fourth group of the periodic table. All these elements crystallize in the diamond structure. It is observed that ρ increases very rapidly with increasing W_g as a result

Table 6.3. ROOM TEMPERATURE RESISTIVITY AND ENERGY GAP OF ELEMENTS IN THE FOURTH GROUP

	C (diamond)	Si	Ge	Sn (gray)	Pb
ρ (ohm m).....	10^{14}	3000	0.47	2×10^{-6}	2×10^{-7}
W_g (ev).....	5.2	1.21	0.75	0.08	No gap

of the Boltzmann factor appearing in expression (6.7). Note the gradual transition from an insulator to a metal as one moves from diamond to lead; in lead, there is no energy gap and the density of electrons in the conduction band is essentially independent of temperature.

The considerations above apply not only to intrinsic elemental semiconductors, but also to intrinsic compound semiconductors, such as ZnO, PbS, etc.

6.4 The conductivity of intrinsic semiconductors

In Chapter 5 we saw that the conductivity of a metal is determined by the density of conduction electrons, and by their relaxation time; this led to the formula (5.13) $\sigma = ne^2\tau/m$. The conductivity of semiconductors can be treated on a similar basis, the difference being that, in general, one deals with *two* types of charge carriers, viz. electrons and holes. Thus, if there are n_e conduction electrons and n_h holes per m^3, we may write the conductivity of a semiconductor in the form

$$\sigma = \frac{n_e e^2 \tau_e}{m_e} + \frac{n_h e^2 \tau_h}{m_h} \tag{6.9}$$

Here, τ_e and τ_h are the *relaxation times* of the electrons and the holes respectively; m_e and m_h are the effective masses of electrons and holes. According to the results obtained in the preceding section, we have for an intrinsic semiconductor $n_e = n_h$. We can also define for a semiconductor two mobilities, one for electrons (μ_e) and one for holes (μ_h); these quantities are defined in analogy with (5.11) as follows:

$$\mu_e = e\tau_e/m_e \qquad \text{and} \qquad \mu_h = e\tau_h/m_h \tag{6.10}$$

They represent the average drift velocities produced by a field of 1 volt per m. Thus, for an intrinsic semiconductor with $n_e = n_h = n$ we may write

$$\sigma_{\text{intrinsic}} = ne(\mu_e + \mu_h) = ne^2\left(\frac{\tau_e}{m_e} + \frac{\tau_h}{m_h}\right) \tag{6.11}$$

where n is given by expression (6.7). In an intrinsic semiconductor the charge carriers are scattered mainly by the lattice vibrations. On the basis of certain simplifying assumptions, the theory leads for this type of scattering to relaxation times which are proportional to $T^{-3/2}$. Assuming for the moment that this is correct, the theory thus predicts for the conductivity of an intrinsic semiconductor

$$\sigma_{\text{intrinsic}} = \text{constant} \times e^{-W_e/2kT} \tag{6.11a}$$

where we have used expression (6.7) for $n_e = n_h = n$. Although the relaxation times determined from experiment are not exactly proportional to $T^{-3/2}$, it is noted that the temperature dependence predicted by the theory is essentially determined by the Boltzmann factor, even if the constant in

(6.11a) varies somewhat with T. The form of (6.11a) is in agreement with experiment, as may be seen from Fig. 6.8, where the logarithm of σ versus $1/T$ has been plotted for intrinsic Ge and Si. Note that such a curve permits one to evaluate the energy gap W_g from the slope. By way of illustra-

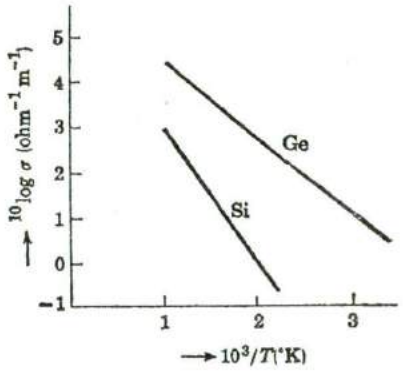


Fig. 6.8. The logarithm of the conductivity of germanium and silicon in the intrinsic region as a function of $1/T$.

tion we give here numerical values for the mobilities and carrier density in germanium, as derived from an analysis of the electrical properties of this material:

$$n_e = n_h = n = 2 \times 10^{22} T^{3/2} \times 10^{-3700/T}$$
$$\mu_e(T) = 3.5 \times 10^3 T^{-1.67} \text{ m}^2 \text{ volt}^{-1} \text{ sec}^{-1}$$
$$\mu_h(T) = 9.1 \times 10^4 T^{-2.3} \text{ m}^2 \text{ volt}^{-1} \text{ sec}^{-1}$$
$$\mu_e \text{ at room temperature} = 0.38 \text{ m}^2 \text{ volt}^{-1} \text{ sec}^{-1}$$
$$\mu_h \text{ at room temperature} = 0.18 \text{ m}^2 \text{ volt}^{-1} \text{ sec}^{-1}$$

For silicon, the mobilities as function of temperature are given by

$$\mu_e(T) = 4.0 \times 10^3 T^{-2.6} \text{ m}^2 \text{ volt}^{-1} \text{ sec}^{-1}$$
$$\mu_h(T) = 2.5 \times 10^4 T^{-2.3} \text{ m}^2 \text{ volt}^{-1} \text{ sec}^{-1}$$
$$\mu_e \text{ at room temperature} = 0.17 \text{ m}^2 \text{ volt}^{-1} \text{ sec}^{-1}$$
$$\mu_h \text{ at room temperature} = 0.035 \text{ m}^2 \text{ volt}^{-1} \text{ sec}^{-1}$$

All these values should be considered as approximate, since they have been derived on the basis of an analysis which involves models which are known not to be exact.

6.5 Carrier densities in *n*-type semiconductors

Most of the engineering applications of semiconductors involve semiconductors which have been "doped" intentionally with specific impurities, rather than intrinsic material. Thus, by adding a small amount of a particular impurity to the melt of germanium, crystals may be grown in which these impurities are incorporated. From X-ray diffraction data it has been

established that in most cases the impurity atoms occupy lattice positions which in the pure material are occupied by germanium atoms; the same is true when the pure material is silicon. Of particular importance in the case of Si and Ge is doping with elements from the third and fifth columns of the periodic table; the most frequently used elements from the third column are B, Al, Ga, and In, and from the fifth: P, As, and Sb. Addition of a fraction of a percent of such elements may increase the conductivity by several powers of ten. Also, the temperature dependence of the conductivity is strongly affected by the impurity content. The reader is reminded here that these features are altogether different from those obtained by adding small amounts of impurities to metals. The strong dependence of the electrical properties of semiconductors on impurity content may be understood from the arguments given presently.

Consider a crystal of germanium in which a small fraction of the Ge atoms is replaced by, say, Sb atoms. An antimony atom contains five valence electrons, and since it occupies a normal Ge-lattice site, it requires only four valence electrons to form four electron-pair bonds with its neighbors. Thus, the Sb atom has one valence electron which is not being used in the chemical bonds with its neighbors. It is therefore not surprising that this "extra" electron is not as strongly bound as the other valence electrons. How strongly is the extra electron bound; i.e., how much energy would be required to set it free? An approximate numerical answer to this question may be obtained as follows. Suppose the electron is taken away from the antimony site; we are then left with an Sb^+ ion, carrying a charge $+e$. Consequently, if the electron remains near the antimony site, it does so because it is attracted by the field of the Sb^+ ion. The extra electron may thus be pictured as moving around a positive ion of charge $+e$, in a way similar to the motion of an electron in a hydrogen atom around a proton of charge $+e$ (see Fig. 6.9). There is, however, an important difference

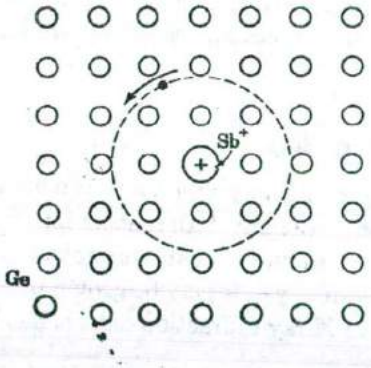


Fig. 6.9. Schematic representation of an antimony donor atom incorporated in a germanium lattice; the extra electron is pictured as revolving in a Bohr orbit around the Sb^+ ion.

between these two cases: the extra electron moving in the field of the Sb^+ ion sees the ion embedded in a dielectric medium, viz., germanium, which has a relative dielectric constant $\epsilon_r = 16$; the electron in a hydrogen atom, on the other hand, moves in free space. This difference is important, because the presence of the dielectric between the electron and the Sb^+ ion reduces the field strength and hence reduces the binding energy of the electron. This may be seen readily by following through the simple Bohr model for the present case. We assume the electron to move in a circular orbit of radius r; if v represents the velocity of the electron in the orbit, we obtain the equilibrium condition

$$\frac{e^2}{4\pi\epsilon_0\epsilon_r r^2} = \frac{mv^2}{r} \tag{6.12}$$

The left-hand side represents the Coulomb force between the electron and the ion; the right-hand side is the centrifugal force. The total energy of the electron is then equal to

$$\begin{aligned} W &= W_{\text{kin}} + W_{\text{pot}} = \tfrac{1}{2}mv^2 + W_{\text{pot}} \\ &= \frac{e^2}{8\pi\epsilon_0\epsilon_r r} - \frac{e^2}{4\pi\epsilon_0\epsilon_r r} = -\frac{e^2}{8\pi\epsilon_0\epsilon_r r} \end{aligned} \tag{6.13}$$

Now, according to one of Bohr's quantum postulates, 2π times the angular momentum of the electron should be equal to an integer times Planck's constant h. If we consider the electron to be in its lowest energy state (the ground state), the integer is 1, so that

$$mvr = h/2\pi \tag{6.14}$$

Squaring both sides of this equation, and eliminating v^2 by means of (6.12), one obtains

$$r = \epsilon_0\epsilon_r h^2/me^2\pi \tag{6.15}$$

The energy of the electron may be obtained by substituting r from (6.15) into (6.13), which gives

$$W = -me^4/8\epsilon_0^2\epsilon_r^2 h^2 \tag{6.16}$$

From these results we may draw the following conclusions. According to (6.15), the radius of the electron orbit is proportional to ϵ_r. Thus, according to this model the radius of the orbit of an electron moving in the field of an Sb^+ ion embedded in germanium is $\cong 16 \times 0.53 \cong 8.5$ angstroms. Also, according to (6.16), the energy of the electron is proportional to $1/\epsilon_r^2$. Since $-W$ represents the energy required to take the electron away from the Sb atom, and since the energy required to ionize a hydrogen atom is equal to $me^4/8\epsilon_0^2h^2 = 13.6$ ev, the ionization energy of the Sb atom is only approximately 0.05 ev. The model employed here would, of course, apply

equally well to other elements of the fifth column, since only the dielectric constant ϵ_r appears as a parameter in the formulas. The model itself, however, is rather crude and it is not surprising, therefore, that the estimated value is not exactly equal to values obtained from an analysis of experimental data. Thus, in germanium, the ionization energies of P, As, and Sb are found to be approximately 0.01 ev, and differ somewhat from element to element.

The same reasoning applies as well to impurities from the fifth column in silicon. Since silicon has a somewhat lower dielectric constant ($\epsilon_r = 12$), one expects the ionization energies of these elements in Si to be somewhat higher than in germanium. This is indeed the case, as may be seen from the experimental values given here:

Impurity.....	P	As	Sb
Ionization energy (ev)...	0.045	0.049	0.039

We should remark here that as the impurity content increases, the ionization energies decreases; the values given apply to low concentrations (say $\lesssim 10^{23}$ per m^3).

What are the consequences of the results obtained above? Let us first consider a crystal of germanium containing a small amount of Sb impurities at absolute zero. Since at $T = 0$, the electronic system is in its lowest energy state, all valence electrons will be in the valence band, and all the Sb impurities will be un-ionized. The energy band picture corresponding to this situation is depicted in Fig. 6.10(a). The picture is similar to that

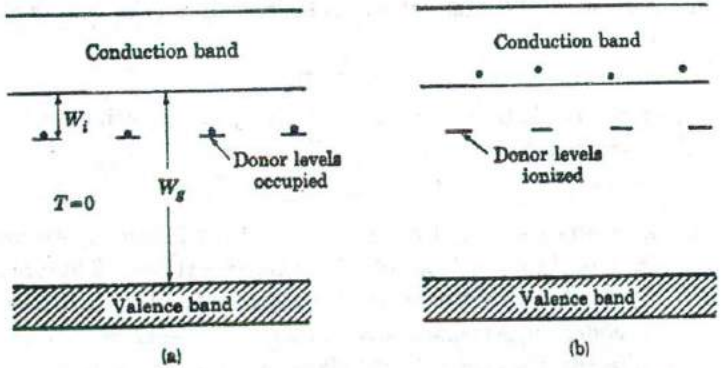


Fig. 6.10. In (a) the energy band scheme is drawn for a semiconductor containing donor atoms; at absolute zero, the donor atoms are not ionized and the donor electrons occupy levels W_i below the bottom of the conduction band. In (b) the donor atoms are ionized by thermal excitation, leading to conduction by electrons.

for an intrinsic semiconductor in that the valence band is filled and the conduction band is empty. However, the "extra" electrons associated with the Sb atoms occupy states which lie only a fraction of a volt below the conduction band; in the case of Ge, the extra electron associated with an Sb atom lies 0.01 ev below the ionization continuum, which in the crystal corresponds to the conduction band. Thus, the short bars in Fig. 6.10(a) represent the energy levels for the extra electrons of the Sb atoms; the black dots represent the electrons themselves. As the temperature is raised, the lattice vibrations become more intense, and, by absorbing the proper amount of energy, some of the Sb atoms may become ionized; i.e., they release their electrons to the conduction band as indicated in Fig. 6.10(b). Since the ionization energy of the Sb atoms is much smaller than the energy W_g required to break one of the valence bonds, the Sb atoms will donate electrons to the conduction band at much lower temperatures than the valence band will. The Sb atoms are called *donor levels*, because of their property of being able to donate electrons to the conduction band. The statistical mechanics involved in calculating the number of electrons donated to the conduction band at a certain temperature T by a given number of impurity atoms is similar to that discussed in section 6.3. Thus, assuming for the moment that the temperature is still sufficiently low to neglect excitation from the valence band into the conduction band, we may proceed as follows: let there be N_d donors, of which n are ionized at a temperature T; there are then $(N_d - n)$ un-ionized donors. The rate at which electrons in the conduction band recombine with ionized donors is proportional to the number of conduction electrons and proportional to the number of ionized donors. Hence

$$(\partial n/\partial t)_{\text{recombination}} = -rn^2 \tag{6.17}$$

where r is a constant. The rate at which electrons are excited from unionized donors into the conduction band is equal to

$$(\partial n/\partial t)_{\text{ionization}} = g(N_d - n) \tag{6.18}$$

where g is a function of temperature. In equilibrium, the sum of (6.17) and (6.18) must be equal to zero. Hence,

$$n^2/(N_d - n) = \text{constant depending on } T \text{ alone} \tag{6.19}$$

It can be shown from statistical mechanics that the constant appearing in (6.19) is proportional to the Boltzmann factor $\exp(-W_i/kT)$, where W_i represents the ionization energy of the impurities. Thus,

$$n^2/(N_d - n) \propto e^{-W_i/kT} \tag{6.20}$$

Note that as T increases, n increases exponentially. When kT becomes

larger than W_i, nearly all donor levels are ionized, and n approaches N_d. This is the case, for example, for Sb impurities ($W_i \cong 0.01$) at room temperature ($kT \cong 0.025$ ev). When the temperature is raised to sufficiently high values, there will also be excitation of electrons from the valence band to the conduction band. These results may be illustrated with reference to Fig. 6.11, where we have plotted schematically the density of electrons in

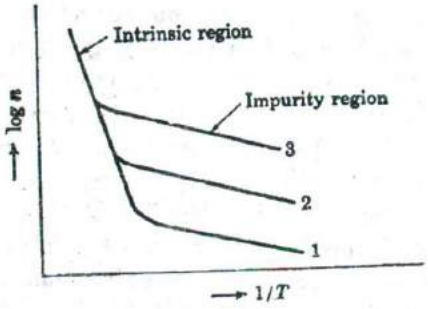


Fig. 6.11. Schematic representation of the logarithm of the electron concentration for a donor-doped semiconductor as a function of $1/T$. In the order, 1, 2, 3 the curves correspond to samples with increasing donor concentration.

a semiconductor containing various amounts of donor levels. At low temperatures, the density of conduction electrons is determined essentially by the donor concentration. The slope of the log n versus $1/T$ curve is in that region determined by half the ionization energy [$n \ll N_d$ in (6.20)]. At high temperatures, one enters the intrinsic region, and because there are many more valence electrons than donor levels, the density of conduction electrons is ultimately determined only by excitation from the valence band. To give an example: for germanium nearly all impurities are ionized above liquid air temperatures; a sample doped so that its room temperature resistivity is 0.001 ohm m becomes intrinsic at approximately 200°C.

A semiconductor containing donor levels is referred to as *n-type* material, to indicate that the conduction below the intrinsic region is due to negative charge carriers. The carrier density is of course reflected in the conductivity of the material. The temperature dependence of the conductivity may be rather complicated as a result of the temperature dependence of the relaxation time and because the impurities themselves introduce new scattering processes. The details of these processes will not be discussed here.

6.6 p-type semiconductors

Returning to the elemental semiconductors, silicon and germanium, let us now consider the consequences of doping with elements from the

third group of the periodic table; as an example, consider boron in germanium. The B atoms again occupy places in the lattice which in the pure material are occupied by Ge atoms. However, the boron atom has only three valence electrons, i.e. it is one electron short of being able to complete the four electron-pair bonds with its neighbors. The absence of an electron in one of these bonds is again called a hole, but in the present case the hole is "bound" to the boron atom, at least at absolute zero. If, as indicated in Fig. 6.12, an electron such as A from a normal electron-pair bond would

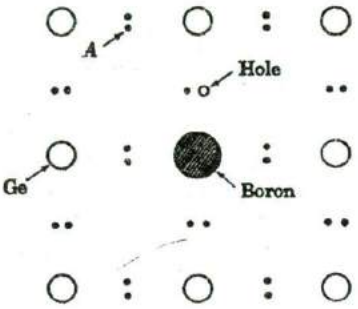


Fig. 6.12. Schematic illustration of a hole associated with incorporating a boron atom in a germanium lattice; the boron atom lacks one electron to form four electron-pair bonds. Actually, the hole may be pictured as revolving around the negative B^- ion, in a way similar to the electron revolving around the Sb^+ ion in Fig. 6.9.

jump into the hole, subsequent jumps from other electrons into the hole would lead to hole conduction, as in the case of intrinsic semiconductors. However, some *energy is required for the first step in this process*, and it can be shown that the energy required for a valence electron to be excited into the hole is approximately equal to the ionization energy of donor levels in the same material; i.e., the bound holes of the boron atoms may be represented in the energy level diagram as occupying levels slightly above the top of the valence band, as indicated in Fig. 6.13(a). At absolute zero, the holes remain bound to the impurities, but as the temperature is raised, valence electrons may be excited into the bound holes, with the result that holes are created in the valence band, as indicated in Fig. 6.13(b). The bound hole levels are called *acceptor levels*, since they can accept electrons from the valence band. Materials of this kind are called *p-type*, because the conduction current is carried by holes. Much of the discussion of the preceding section with reference to electrons can be applied to the present case and it may therefore suffice to give an example of the conductivity as a function of temperature for a p-type semiconductor, viz., cuprous oxide. In Fig. 6.14 we have represented $\log \sigma$ as a function of $1/T$ for p-type cuprous oxide, the concentration of acceptor levels being varied. At low temperatures, the conductivity results predominantly from holes produced by excitation of valence electrons into acceptor levels. The slope of the curve

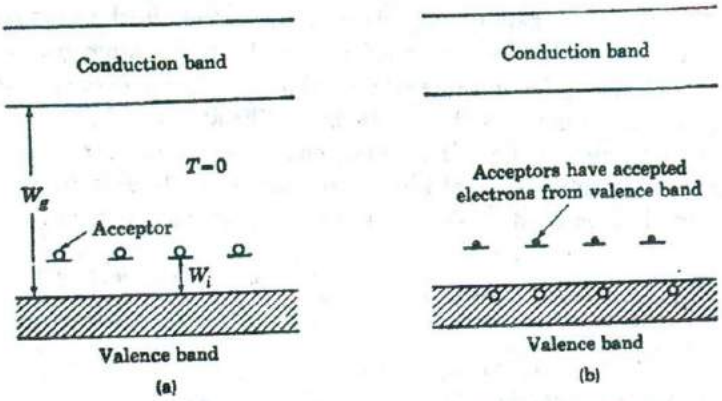


Fig. 6.13. At absolute zero, a hole such as the one indicated in Fig. 6.12 is bound to the acceptor atom, as indicated in (a) by the open circles above the valence band. At the expense of an energy W_i electrons from the valence band may be raised into the acceptor levels; this is indicated in (b).

in that region is determined by the energy required to excite a valence electron into an acceptor level. As the temperature is raised, one ultimately enters the intrinsic region, where the slope is determined by the energy gap between the valence band and the conduction band. In the present material, the acceptor levels are formed in the following manner: In pure Cu_2O

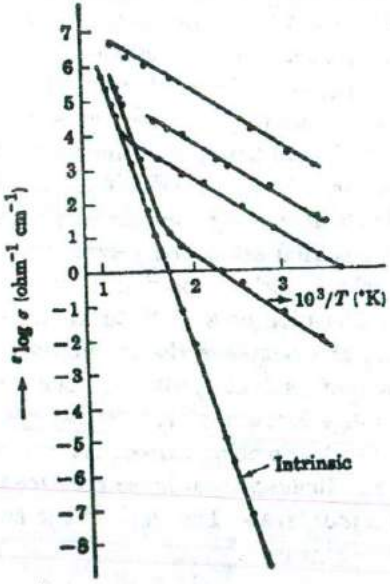


Fig. 6.14. The natural logarithm of the conductivity of various samples of cuprous oxide as function of $1/T$. The acceptor concentration increases as one goes to the higher lying curves. [After W. Juse and B. Kurtschatow, *Physik. Z. Sowjetunion* **2**, 453 (1933)]

there is one oxygen O^{2-} ion for every two Cu^{+} ions; one speaks in this case of a *stoichiometric composition*. If the material is heated in oxygen, the chemical formula is no longer exactly given by Cu_2O, because some "extra" oxygen is incorporated in the lattice. The extra oxygen atoms are apt to accept one or two electrons, because they have the tendency to become O^{2-} rather than remain neutral. The oxygen atoms satisfy this need by accepting electrons from the valence band, which leads to mobile holes in the valence band. The acceptor levels in this case are thus the extra oxygen atoms incorporated in the lattice by heat treatment in oxygen. By varying the pressure of the oxygen, one may vary the acceptor concentration. As the acceptor concentration is increased, the conductivity of the material at low temperatures increases, as illustrated in Fig. 6.14.

6.7 Hall effect and carrier density

Since the conductivity of a material is determined by the product of the density and the mobility of charge carriers, a measurement of the conductivity alone does not permit one to evaluate these two factors separately. However, in case there is only one type of charge carrier, the density of the charge carriers can be found from a measurement of the so-called *Hall coefficient* R_H of the material. In such cases then, measurement of σ and of R_H allows one to determine both the carrier density and mobility. It is from such measurements that the mobility data given in preceding sections were obtained. We shall now discuss the Hall effect and its interpretation, assuming the carriers are positive holes of charge e. In Fig. 6.15 consider a slab of p-type material in which there is a current density J re-

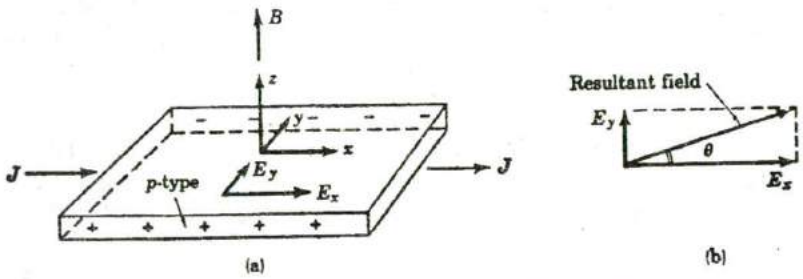


Fig. 6.15. Illustrating the geometry with reference to the discussion of the Hall effect for a p-type semiconductor. The voltage difference between the front and back faces of the slab in (a) gives the Hall voltage. In (b) the resultant field and the Hall angle are represented in the xy-plane.

sulting from an applied electric field E_x. Since the carriers are assumed to be positive, they will drift with an average velocity $\langle v_x \rangle$ in the x-direction. When a magnetic field of flux density B_z (webers per m^2) is applied along the z-direction, the carriers will experience a Lorentz force perpendicular to $\langle v_x \rangle$ and to B_z; for the configuration of Fig. 6.15, the force will be directed along the negative y-axis. The magnitude of this force is given by $e\langle v_x \rangle B_z$. Thus, the positive carriers are driven toward the front face in the sample of Fig. 6.15, resulting in an excess of holes near the front face and a deficiency of holes near the back face. These charges will in turn create an electric field along the positive y-direction. In this way, then, an equilibrium situation is established in which the Lorentz force is just compensated by the force produced by the electric field along the y-axis. We may therefore write

$$eE_y = e\langle v_x \rangle B_z \tag{6.21}$$

On the other hand, the current density in the sample is given by

$$J_x = n_h e \langle v_x \rangle \tag{6.22}$$

where n_h is the density of holes. Eliminating $\langle v_x \rangle$ from the last two equations we may define the Hall coefficient R_H as

$$R_H = \frac{E_y}{B_z J_x} = \frac{1}{en_h} \tag{6.23}$$

The current density J_x can be calculated from the total current and the cross section of the sample. The field E_y may be found by attaching voltage probes at the front and back faces of the sample, and by dividing the measured Hall voltage by the width of the sample in the y-direction. Thus, R_H can be obtained from measurements, and n_h may be calculated. It should be noted here that proper averaging over the velocity distribution of the carriers gives $R_H = A/en_h$, where A is a constant of the order of unity. For a simple semiconductor model, for example, one finds instead of (6.23)

$$R_H = \frac{3\pi}{8} \frac{1}{en_h} \tag{6.24}$$

The reader may verify for himself that if the carriers were electrons instead of holes, the polarity of the voltage measured across the Hall probes would be reversed. Hence, for electrons one finds in analogy with (6.24)

$$R_H = \frac{E_y}{B_z J_x} = -\frac{1}{en_e} \tag{6.25}$$

where n_e represents the density of conduction electrons. We thus conclude that a measurement of the Hall coefficient of a semiconductor with one kind of charge carriers provides us with:

(a) the *sign of the charge carriers,*

(b) the *density of charge carriers.*

From the considerations above it follows that the total electric field in the sample of Fig. 6.15 makes a non-vanishing angle θ with the x-axis, as illustrated in Fig. 6.15(b). The angle θ is called the *Hall angle.*

When two types of charge carriers are present at the same time, the Hall effect is more complicated and a measurement of the Hall coefficient does not permit one immediately to calculate the carrier densities. Thus, under the same assumptions as were involved in the derivation of (6.23) one finds

$$R_H = \frac{E_y}{B_z J_x} = \frac{1}{e} \frac{(n_h \mu_h^2 - n_e \mu_e^2)}{(n_h \mu_h + n_e \mu_e)^2} \tag{6.26}$$

where n_e and n_h are the electron and hole densities, and μ_e and μ_h the carrier mobilities. Note that the Hall coefficient is positive if $n_h \mu_h^2 > n_e \mu_e^2$ and negative if $n_h \mu_h^2 < n_e \mu_e^2$.

As an example of measurements of the Hall coefficient of semiconductors, we reproduce in Fig. 6.16 a set of curves for n-type silicon, plotted as

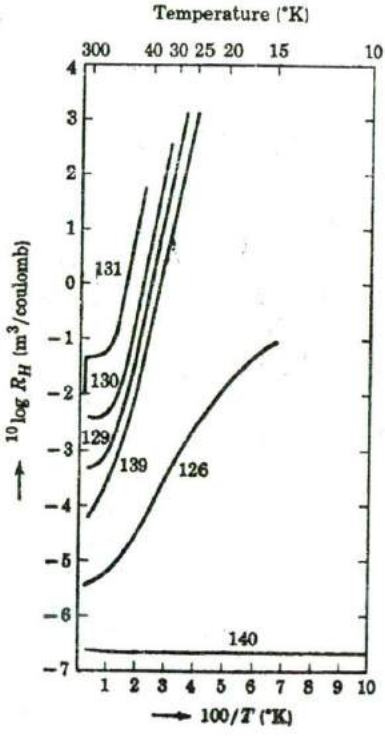


Fig. 6.16. Hall coefficient as function of temperature for single crystals of n-type silicon. The donor densities for these samples are approximately given by $7.4 \times 10^{18}/R_H$ per m³, where R_H is the Hall coefficient at room temperature.

log R_H versus $1/T$. Large Hall coefficients correspond to low carrier densities, and hence to purer samples. Sample 140 contains a considerable donor density, and, since the ionization energy of the donors decreases with increasing donor concentration, essentially all donors in this sample are ionized in the temperature region covered by the measurements; hence the constant value of R_H for this sample. The other samples exhibit clearly the increase in carrier concentration with increasing temperature. From the straight portions of the upper curves, the donor ionization energy can be determined.

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Problems

6.1 Silicon and germanium have the diamond structure represented in Fig. 5.2. How many atoms are there on the average in a volume equal to a^3 where a is the edge of the elementary cube in Fig. 5.2?

6.2 Let a represent the edge of the elementary cube shown in Fig. 5.2. For Si, $a = 5.43$ angstrom and for Ge, $a = 5.62$ angstrom. Calculate the number of atoms, N, per m^3 in these materials.

6.3 Let n_e and n_h represent the densities of free electrons and holes in a semiconductor. At a certain temperature T let g be the number of electrons excited thermally per unit volume from the valence band into the conduction band. Let the rate of recombination of electrons and holes be proportional to the product $n_e n_h$. If n_i represents the carrier density for intrinsic material, show that in general $n_e n_h = n_i^2$ in thermal equilibrium.

6.4 The resistivity of intrinsic germanium at 27°C is equal to 0.47 ohm m. Assuming electron and hole mobilities of respectively 0.38 and 0.18 m^2 $volt^{-1}$ sec^{-1}, calculate the intrinsic carrier density n_i at 27°C.

6.5 The resistivity of intrinsic silicon at 27°C is 3000 ohm m. Assuming electron and hole mobilities of respectively 0.17 and 0.035 m^2 $volt^{-1}$ sec^{-1}, calculate the intrinsic carrier density n_i at 27°C.

6.6 Germanium is doped with 1.00×10^{-2} atomic percent of antimony. Assuming that at room temperature all antimony atoms are ionized, compute the electron and hole densities n_e and n_h; you may assume that the electron density is determined only by the donors. (Hint: Make use of the result in problem 6.3.) From this information, calculate the resistivity of this material at room temperature if the electron and hole mobilities are 0.38 and 0.18 m^2 $volt^{-1}$ sec^{-1}, respectively.

6.7 The Hall coefficient of a specimen of doped silicon is found to be 3.66×10^{-4} m^3 $coulomb^{-1}$; the resistivity of the specimen is 8.93×10^{-3} ohm m. Find the mobility and density of the charge carriers, assuming single carrier conduction.

6.8 A specimen of a semiconductor is 1 mm thick and 1 cm wide. A magnetic flux density of 0.5 weber m^{-2} is applied parallel to the 1 mm edge, and Hall voltage contacts are attached to measure the voltage across the width of the sample. The current flowing lengthwise through the sample is 10 milliamperes. If the Hall coefficient of the material is 3.66×10^{-4} m^3 $coulomb^{-1}$, compute the voltage measured between the Hall contacts.

6.9 A specimen of a semiconductor has a Hall coefficient of 3.66×10^{-4} m^3 $coulomb^{-1}$ and a resistivity of 8.93×10^{-3} ohm m. In a Hall effect experiment a magnetic flux density of 0.5 weber m^{-2} is used. Find the Hall angle.

6.10 Give a derivation of formula (6.26) for the Hall coefficient of a semiconductor with two types of carriers.

7

Junction Rectifiers and Transistors

In this chapter we apply the information obtained in the preceding section to the processes occurring in *junction rectifiers* and *transistors*. Rectifying contacts occur not only between n- and p-type semiconductors, but also between metals and between metals and semiconductors. The main purpose of this chapter is to provide the reader with some insight into the physical processes which are responsible for the behavior of such devices, rather than to give a more or less complete review of the various kinds of rectifiers and transistors. For this reason, only the junction rectifier and transistor are discussed. The first three sections deal with preparatory material which is essential for an understanding of the physics of these devices.

7.1 Minority and majority carrier densities in semiconductors

In section 6.3 we discussed the density of conduction electrons and holes in an intrinsic semiconductor on the basis of the reaction

$$\text{valence electron} \leftrightarrows \text{conduction electron} + \text{hole} \tag{7.1}$$

Denoting the number of valence electrons per m³ per second excited thermally into the conduction band by $g(T)$, the recombination coefficient by r, and the *intrinsic carrier concentration* by n_i, we have according to (7.1) for thermal equilibrium

$$g(T) = rn_i^2 \qquad \text{or} \qquad n_i^2 = g(T)/r \tag{7.2}$$

Thus, n_i is determined only by the temperature of the material. We have

also seen in sections 6.5 and 6.6 that a semiconductor may be doped with donors or acceptors, and in that case the density of one type of carrier predominates. Thus, in the presence of donor levels, the electrons are the *majority carriers* and the holes the *minority carriers*. In thermal equilibrium, let an impurity semiconductor contain n_e electrons and n_h holes per m³ at a given temperature. It will be evident that even in this case, where $n_e \neq n_h \neq n_i$, reaction (7.1) must be in thermal equilibrium. Since neither the rate of generation $g(T)$ nor the recombination rate r depend on the concentration of donors or acceptors for concentrations used in practical semiconductors, we must require that in thermal equilibrium

$$g(T) = r n_e n_h \qquad \text{or} \qquad n_e n_h = n_i^2 \tag{7.3}$$

Hence, n_e and n_h are related to the intrinsic carrier density. This provides a means for calculating the minority and majority carrier densities of a semiconductor, provided the conductivity of the material is known as well as n_i and the mobility of the majority carriers. For example, let the conductivity of an n-type semiconductor be σ. Since the hole density is very small compared to the electron density (assuming the temperature is low enough that we are not in the intrinsic region), the conductivity is essentially determined by the electrons, i.e.

$$\sigma \cong n_e e \mu_e \tag{7.4}$$

where μ_e is the electron mobility. If the mobility is known, n_e can be calculated from σ, and n_h can be found from equation (7.3), provided n_i is known.

In Table 7.1 we give values for n_e and n_h for samples of germanium of a given room-temperature conductivity, calculated in the manner outlined above. The conductivity range given here covers that of germanium crystals used in semiconductor rectifiers and transistors. For these calculations, the following values were used:

$$\mu_e = 0.36 \text{ m}^2 \text{ volt}^{-1} \text{ sec}^{-1}$$
$$\mu_p = 0.17 \text{ m}^2 \text{ volt}^{-1} \text{ sec}^{-1}$$
$$n_i = 2.5 \times 10^{19} \text{ m}^{-3}$$

Table 7.1. MAJORITY AND MINORITY CARRIER CONCENTRATIONS FOR n- AND p-TYPE SAMPLES OF GERMANIUM FOR A GIVEN CONDUCTIVITY σ AT ROOM TEMPERATURE

p-Type Samples			n-Type Samples		
σ (ohm^{-1} m^{-1})	n_e (m^{-3})	n_h (m^{-3})	σ (ohm^{-1} m^{-1})	n_e (m^{-3})	n_h (m^{-3})
10^4	1.70×10^{16}	3.68×10^{22}	10^4	1.75×10^{22}	3.57×10^{16}
10^2	1.70×10^{17}	3.68×10^{21}	10^2	1.75×10^{21}	3.57×10^{17}
10	1.70×10^{18}	3.68×10^{20}	10	1.75×10^{20}	3.57×10^{18}

7.2 Drift currents and diffusion currents; the Einstein relation

In the preceding chapter we discussed the conduction mechanism in a semiconductor subjected to an electric field. As shown in section 6.4, the conductivity associated with the conduction electrons could be written in the form $\sigma_e = n_e e\mu_e$, where μ_e is the mobility of the electrons. The current density associated with the drift of the electrons due to the applied field would then be given by $J_e = n_e e\mu_e E$. However, an electron current may flow in a semiconductor even in the absence of an electric field, viz., if there exists a *gradient of the electron density*. Consequently, one deals in semiconductors frequently with two kinds of current: a *drift current* (due to the electric field) and a *diffusion current* (due to a gradient of the carrier concentration); these concepts apply, of course, to electrons as well as to holes. For our purpose it is sufficient to consider the case in which there exists a gradient of the carrier density in one direction only; we shall take this direction to be the x-direction. According to diffusion theory, the net flow of carriers through a plane of unit area perpendicular to the x-axis is proportional to $-dn/dx$, where n is the density of carriers; the minus sign indicates the well-known fact that when the concentration increases along the positive x-direction, the flow of particles due to the diffusion process is in the negative x-direction. The proportionality constant relating the particle flow density and the concentration gradient is the *diffusion constant*, D. Thus, the hole current in any point x may be written as the sum of two contributions, one from the field and one from the diffusion process:

$$J_h = n_h e\mu_h E - eD_h \frac{dn_h}{dx} \tag{7.5}$$

where D_h is the diffusion constant of the holes. The charge appears in the diffusion contribution because the electric current is equal to the *particle* current times the charge e per particle. For electrons, we may write down a similar expression for the current density:

$$J_e = n_e e\mu_e E + eD_e \frac{dn_e}{dx} \tag{7.6}$$

In this case the diffusion contribution is positive because the charge per electron is $-e$. The total current is given by the sum of equations (7.5) and (7.6).

There exists an important relationship between the diffusion coefficient and the mobility of the carriers, which is known as the *Einstein relation*. This relationship may be derived by considering a situation in a semiconductor in which there exists an electric field E and a concentration gradient

such that the resultant current is zero. Under these conditions of no current flow, the system is in thermal equilibrium and Boltzmann statistics applies. Consider, for example, a potential $V(x)$, producing in the point x an electric field given by

$$E(x) = -\frac{dV}{dx} \tag{7.7}$$

In thermal equilibrium, the density of holes as function of x is given by the Boltzmann expression

$$n_h(x) = Ce^{-eV/kT} \tag{7.8}$$

where C is a constant. Hence, there will be a gradient of the hole density given by

$$\frac{dn_h}{dx} = -\frac{e}{kT} n_h \frac{dV}{dx} = \frac{e}{kT} n_h E(x) \tag{7.9}$$

Since the hole current vanishes in thermal equilibrium, we have in accordance with (7.5) and (7.9)

$$0 = n_h e\mu_h E - \frac{e^2}{kT} D_h n_h E$$

from which we obtain the Einstein relation

$$D_h = (kT/e)\mu_h \tag{7.10}$$

Hence, the *diffusion constant is proportional to the mobility* and for a given temperature may be calculated from the latter. A similar relationship holds, of course, for electrons:

$$D_e = (kT/e)\mu_e \tag{7.11}$$

For germanium we find, for example, from the room temperature mobilities given in connection with Table 7.1

$$D_h = 0.0044 \text{ m}^2 \text{ sec}^{-2}$$

$$D_e = 0.0093 \text{ m}^2 \text{ sec}^{-2}$$

In subsequent sections we shall see that diffusion currents play an important role in rectifier and transistor action.

7.3 The continuity equation for minority carriers

An important equation which governs the behavior of minority carriers in rectifying junctions and transistors is the *continuity equation;* the quantitative theory of junctions and transistors is, in fact, based on this equation. This equation will be derived in this section for the minority carriers (electrons) in a p-type semiconductor. Consider then a p-type semiconductor

which for the moment will be assumed to carry no electron current or hole current. Let n_{e0} and n_{h0} represent the thermal *equilibrium densities* of minority and majority carriers respectively. In accordance with (7.3) we may then write

$$n_{e0}n_{h0} = n_i^2 = g/r \tag{7.12}$$

Suppose now that in some way or other the electron concentration at a given instant $t = 0$ is different from n_{e0}, say

$$n_e = n_{e0} + \delta n_e \tag{7.13}$$

It can be shown that for semiconductors with conductivities of the kind used in transistors and rectifiers, the hole density must also be different from the equilibrium concentration n_{h0}, so as to maintain *charge neutrality*. Thus, the hole density may be written as

$$n_h = n_{h0} + \delta n_h \tag{7.14}$$

where $\delta n_h \cong \delta n_e$. If the semiconductor has an excess density of electrons δn_e at the instant $t = 0$, what will the electron density be for times when $t > 0$, assuming the semiconductor is left to itself? This question may be answered in words in the following manner:

$$\begin{bmatrix}\text{time rate}\\ \text{of increase}\\ \text{of electrons}\end{bmatrix} = \begin{bmatrix}\text{rate of}\\ \text{thermal generation}\\ \text{of electrons}\end{bmatrix} + \begin{bmatrix}\text{rate of}\\ \text{increase due to}\\ \text{recombination}\end{bmatrix} \tag{7.15}$$

Mathematically, this may be written as

$$\left(\frac{dn_e}{dt}\right)_{J_e=0} = g - r(n_{e0} + \delta n_e)(n_{h0} + \delta n_h) \tag{7.16}$$

We have included the condition $J_e = 0$ to remind us of the fact that to this point we have assumed that there is no current in the semiconductor. Writing out the last term in (7.16) we find

$$r(n_{e0} + \delta n_e)(n_{h0} + \delta n_h) = r[n_{e0}n_{h0} + n_{h0}\delta n_e + n_{e0}\delta n_h + \delta n_e\delta n_h] \tag{7.17}$$

At this point we make use of the fact that we are discussing the *minority carriers;* i.e., since we are dealing with a p-type semiconductor we may take $n_{h0} \gg n_{e0}$. Since $\delta n_e \cong \delta n_p$ we may then, to a first approximation, neglect the last two terms in (7.17). Realizing further that $rn_{e0}n_{h0} = g$, in accordance with (7.12), we find from (7.16)

$$\left(\frac{dn_e}{dt}\right)_{J_e=0} = -rn_{h0}\delta n_e \tag{7.18}$$

Now, since $n_e = n_{e0} + \delta n_e$, where n_{e0} is the time-independent equilibrium

density, it follows that

$$\frac{d}{dt}(\delta n_e) = -rn_{h0}\delta n_e = -\frac{\delta n_e}{\tau_e} \tag{7.19}$$

Here, τ_e is called the *lifetime* of the electrons; note that τ_e is determined only by the recombination coefficient r and by the equilibrium hole density; i.e., τ_e is a constant if the approximations made above are justified. The physical meaning of τ_e becomes clearer by integrating (7.19):

$$(\delta n_e)_t = (\delta n_e)_{t=0}\, e^{-t/\tau_e} \tag{7.20}$$

This equation states that the *excess electron density will disappear exponentially with time*, the time constant being the lifetime τ_e of the electrons.

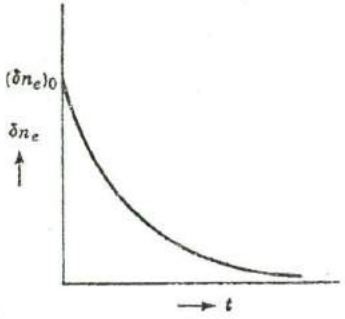


Fig. 7.1. Schematic representation of the decay of the excess of minority carriers according to formula (7.20).

This result is illustrated in Fig. 7.1. For later use, it is convenient to write our result in the form

$$\left(\frac{dn_e}{dt}\right)_{J_e=0} = \frac{n_{e0} - n_e}{\tau_e} \tag{7.21}$$

which follows immediately from (7.18) and (7.19).

What equation will the minority carriers satisfy in case there is an electron current flowing as well? In words we may write the answer to this question in the following form:

$$\begin{bmatrix}\text{total rate of}\\ \text{change of } n_e\end{bmatrix} = \begin{bmatrix}\text{rate of change}\\ \text{of } n_e \text{ for } J_e = 0\end{bmatrix} + \begin{bmatrix}\text{rate of change}\\ \text{of } n_e \text{ due to } J_e\end{bmatrix} \tag{7.22}$$

The first term on the right-hand side is given by (7.21), which leaves us with the problem of finding a mathematical expression for the last term. With reference to Fig. 7.2 consider two planes of unit area at x and $x + dx$. If the current density J_e is independent of x, there are presumably just as many particles entering the volume element between x and $x + dx$ through the plane at x, as there are particles leaving through the plane at $x + dx$. Hence, if J_e is independent of x, the last term in (7.22) is zero. Consider

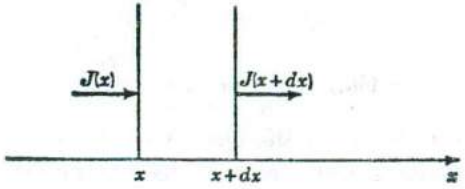


Fig. 7.2. Illustrating two planes perpendicular to the x-axis at the points x and $x + dx$. The current density at x is $J(x)$; the current density at $x + dx$ is $J(x + dx) = J(x) + (dJ/dx)\,dx$.

now the case where $J_e = J_e(x)$, so that the derivative dJ_e/dx is *not* equal to zero. The *particle* current density J_p is equal to $-(1/e)J_e$, where $-e$ is the electronic charge. The number of particles per m² per second entering the volume element at x is equal to $J_p(x)$; the number of particles per m² per second leaving the volume element at $x + dx$ is equal to $J_p(x + dx)$. Hence, the *net* increase of the number of particles in the volume element dx due to the current is

$$\left(\frac{\partial n_e}{\partial t}\right)_{\text{current}} \cdot dx = J_p(x) - J_p(x + dx) = -\frac{dJ_p}{dx}\,dx = \frac{1}{e}\frac{dJ_e}{dx}\,dx \tag{7.23}$$

Hence, the last term of (7.22) is equal to $-(1/e)(dJ_e/dx)$; i.e., this term depends on the *derivative of the current* with respect to x. Substituting (7.23) and (7.21) into (7.22) we then obtain the continuity equation which must be satisfied by the minority carriers:

$$\frac{dn_e}{dt} = \frac{n_{e0} - n_e}{\tau_e} + \frac{1}{e}\frac{dJ_e}{dx} \tag{7.24}$$

For the holes as minority carriers in an n-type semiconductor one finds the corresponding equation to be

$$\frac{dn_h}{dt} = \frac{n_{h0} - n_h}{\tau_h} - \frac{1}{e}\frac{dJ_h}{dx} \tag{7.25}$$

The current densities J_e and J_h are given by equations (7.6) and (7.5); *these four equations are the basic equations underlying the theory of junction rectifiers and transistors.*

7.4 Semi-quantitative discussion of the *n-p* junction rectifier

In this section we shall consider the current-voltage relationship for a junction consisting of an n-type semiconductor and a p-type semiconduc-

tor, as indicated schematically in Fig. 7.3(a). The n-type region contains donor levels and thus has a large concentration of electrons and a low concentration of holes; the p-type region contains acceptor levels, giving it a large concentration of holes and a low concentration of electrons, as indi-

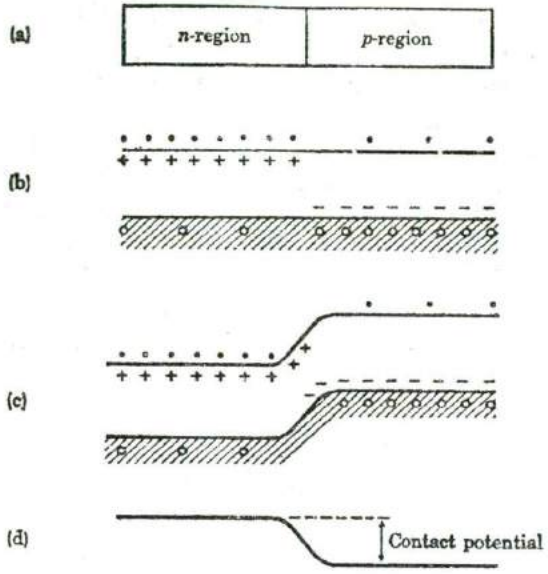


Fig. 7.3. (a) represents the n- and p-regions of the junction. (b) represents the electrons as black dots in the conduction band and the holes as open circles in the valence band for the two regions; this situation does *not* correspond to equilibrium. The ionized donors are represented by positive signs, the ionized acceptors by negative signs. In (c) the junction is in thermal equilibrium; note the positive space charge on the n-side and the negative space charge on the p-side of the junction. The potential distribution in (c) corresponds to electrons. In (d) the potential distribution is drawn for holes.

cated in Fig. 7.3(b). For simplicity we shall assume for the moment that the change from n- to p-type is abrupt. In practice this corresponds to a *fused junction* (*alloy-junction*). In so-called *grown junctions*, there is a more gradual transition from the n-type to the p-type semiconductor.

The situation drawn in Fig. 7.3(b) is unstable, and it will be evident that under influence of the concentration gradients near the junction, electrons will diffuse from n to p and holes will diffuse from p to n. Let us consider the consequences of these processes. As we take electrons away from the n-region, we are left with positive ionized donors which are no

compensated any longer by the negative electronic charge; thus, in the n-region a positive *space charge* will be set up. Consequently, the potential energy of an electron will become a function of the coordinate x. Similarly, the p-region will acquire a negative space charge near the junction, because negative acceptor levels are no longer compensated by holes. This will give rise to an energy band picture as depicted in Fig. 7.3(c); the reader is reminded of the fact that the potentials refer to electrons, i.e. to negative particles. In equilibrium, the diffusion of electrons uphill is just compensated by the electrons coming downhill as a result of the field set up by the space charge. The same holds for the holes, but for the moment we shall concentrate our attention on the electrons. It is evident that in the equilibrium state, the potential difference between the bulk n-region and between the bulk p-region must be of the order of the gap width divided by e. The potential difference set up in this manner is called the *contact potential* of the junction. For the usual germanium junctions this contact potential amounts to approximately half a volt. The potential as seen by the holes in the configuration corresponding to Fig. 7.3(c) is represented in Fig. 7.3(d); i.e., holes wishing to go from p to n must climb the same potential hill as electrons in going from n to p.

In the equilibrium situation depicted in Fig. 7.3(c) let J_{so} represent the

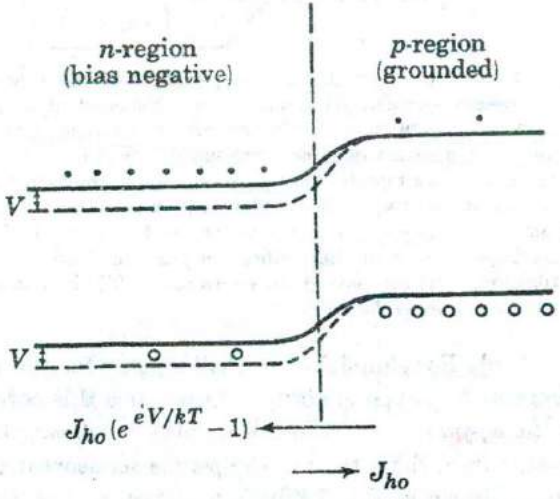


Fig. 7.4. Energy band picture for a forward biased n-p junction; the n-region is made V volts negative relative to the p-region. The dashed curves indicate the bands in the absence of bias. The hole currents are indicated.

magnitude of the two compensating electron currents, one associated with electrons moving uphill, the other with electrons moving downhill. Suppose now that the junction is *biased* so as to make the n-region negative with respect to the p-region. Assuming the latter is connected to ground, the potential seen by the electrons is then as indicated in Fig. 7.4. The electron current associated with the motion of electrons from p to n is not affected by the bias. However, the electrons tending to move from n to p now see a smaller potential hill than before; in fact, the probability that an electron will climb the hill is increased by a factor of $e^{eV/kT}$, where V is the applied potential between the p-region and the n-region. Consequently, for the biased junction there will be a net electron current given by

$$J_e = J_{e0}(e^{eV/kT} - 1) \tag{7.26}$$

A similar reasoning may be applied to the hole current; the holes moving from p to n also see a lower potential barrier than in the unbiased situation. Thus, the biased junction carries a net hole current

$$J_h = J_{h0}(e^{eV/kT} - 1) \tag{7.27}$$

Consequently, the total current through the junction as given by the sum of (7.26) and (7.27) is equal to

$$J = (J_{e0} + J_{h0})(e^{eV/kT} - 1) \tag{7.28}$$

Since $(J_{e0} + J_{h0})$ is a constant, the current density increases exponentially with the voltage when applied so as to make the n-region negative with respect to the p-region; this type of bias is called *forward bias*, since it gives a much larger total current than a bias of the same magnitude applied in the opposite direction. In fact, if the n-region is biased positively relative to the p-region, we find from similar reasoning as above

$$J = (J_{e0} + J_{h0})(1 - e^{-eV/kT}) \tag{7.29}$$

where V refers now to the *reverse bias*. Note that the reverse current approaches a saturation value $(J_{e0} + J_{h0})$ as the reverse bias is increased.

In Fig. 7.5 we have represented the forward and reverse currents characteristics for a p-n junction; there is evidently good agreement between the theoretical curves (7.28) and (7.29) and measured points. It is observed that the system under discussion has a *non-linear* current-voltage characteristic and that it acts as a *rectifier*.

The simplified treatment presented here, of course, gives no information concerning the magnitude of the equilibrium currents J_{e0} and J_{h0}. In order to calculate these currents in terms of the properties of the material, it is

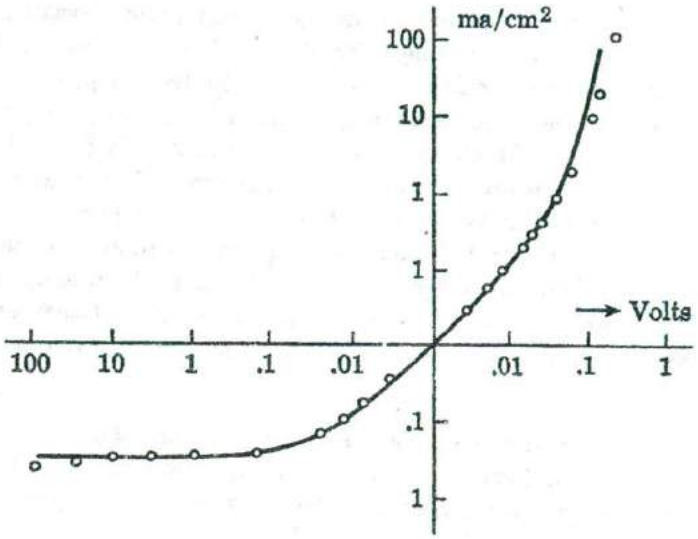


Fig. 7.5. Current-voltage characteristic of a p-n junction. The dots are measured points, the curve is theoretical. [After W. Shockley, *Proc. IRE* 40, 1289 (1952)]

necessary to make a study of the continuity equation derived in the preceding section; this will be done in the next section.

7.5 Quantitative treatment of the *n-p* junction rectifier

Although the form of the equations for the forward and reverse currents derived in the preceding section are essentially correct, the derivation does not give much insight into the actual mechanism. In order to see in more detail what is actually happening in and near the junction, we refer to Fig. 7.6, where we have represented the energy band picture corresponding to a forward biased n-p junction, the potentials referring to electrons rather than to holes. We shall assume that for $x < 0$, and for $x > x_p$, the bottom of the conduction band is essentially horizontal. Actually, because of the current flow, there will be a slight gradient of the potential in the bulk n- and p-regions, but since the conductivity of the material is relatively high, the potential drop across these regions will be small compared to that across the barrier between them.

We shall see below that in order to discuss the mechanism of conduction in detail, it is necessary to know the electron and hole concentrations at the points $x = 0$ and $x = x_p$. These concentrations constitute the *boundary*

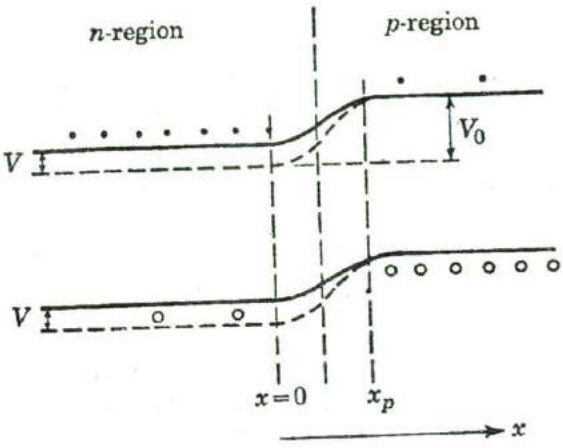


Fig. 7.6. Forward biased n-p junction. The dashed curves give the energy bands in thermal equilibrium, corresponding to the contact potential V_0. The forward bias V is indicated.

conditions which will be used to solve the continuity equations for minority carriers in the bulk n- and p-regions. We shall express these concentrations in terms of the following quantities:

n_{e0} = density of electrons in *p-type* material in thermal equilibrium
n_{h0} = density of holes in *n-type* material in thermal equilibrium
V_0 = voltage difference in absence of current between p-region and n-region
V = applied forward bias

First consider the case when $V = 0$; i.e., the junction is in thermal equilibrium. The electron concentration for the region $x \leqslant 0$ is then given by

$$n_e = n_{e0} e^{eV_0/kT} \tag{7.30}$$

This follows from Boltzmann statistics and from the fact that the concentration of electrons for $x \geqslant x_p$ is equal to n_{e0}. What happens now when the forward voltage V is applied? Holes are then injected from the p-region into the n-region, so that the *hole concentration near* $x = 0$ *will be larger than* n_{h0}. Since the material should remain neutral, the electron concentration at $x = 0$ must increase somewhat over that given by (7.30). However, the increase in the electron density at $x = 0$ required to satisfy the neutrality condition is very small compared to (7.30). Thus, we may assume that (7.30) gives the electron concentration in the region $x \leqslant 0$ even if current flows. If that is so, we can find the concentration of electrons in the point $x = x_p$ in the case of a forward bias from (7.30) by applying

Boltzmann statistics.* We thus find

$$n_e(x_p) = n_e(0)e^{-e(V_0-V)/kT} = n_{e0}e^{eV/kT} \tag{7.31}$$

Note that as a result of the forward bias, the electron density at x_p is higher than it was in the absence of the bias, by a factor $e^{eV/kT}$. This illustrates clearly the *injection* of electrons into the p-region by the n-region. For large distances into the p-region, the electron density is, of course, equal to the thermal equilibrium value n_{e0}.

Without repeating the arguments here, the reader may readily verify for himself that the hole density at the point $x = 0$ is given by

$$n_h(0) = n_{h0}e^{eV/kT} \tag{7.32}$$

which expresses the injection of holes into the n-region.

We shall now proceed to calculate the total current flowing across the junction. For the moment, let us consider the electronic current in the region $x > x_p$, i.e. the minority carrier current in the p-region. In this region, the continuity equation (7.24) applies:

$$\frac{dn_e}{dt} = \frac{n_{e0} - n_e}{\tau_e} + \frac{1}{e}\frac{dJ_e}{dx} \tag{7.33}$$

where n_e is the electron density as function of x in the region $x > x_p$. The electron current density J_e is in general given by expression (7.6). However, since the field strength in this region is negligible, we only have to consider the diffusion current, i.e.

$$J_e = eD_e\frac{dn_e}{dx} \tag{7.34}$$

In the state of steady flow of current, $dn_e/dt = 0$ anywhere. Thus, the equation describing the behavior of the electrons for purposes of the present problem is obtained by substituting (7.34) into (7.33) and by putting $dn_e/dt = 0$, i.e.

$$n_e(x) = n_{e0} + D_e\tau_e\frac{d^2n_e}{dx^2} = n_{e0} + L_e^2\frac{d^2n_e}{dx^2} \tag{7.35}$$

The quantity $L_e = (D_e\tau_e)^{1/2}$ has the dimensions of length, and is called the *diffusion length:* the reason for this name will become evident later. Since

* Strictly speaking, Boltzmann statistics is valid only *in thermal equilibrium;* i.e., if in expression (7.6) $J_e = 0$. However, for the material used in transistors, it t rns out that each of the two terms on the right-hand side of (7.6) is of the order of 10^8 ampere m^{-2}; this estimate is based on a barrier thickness $x_p \approx 10^{-6}$ m. The actual currents flowing under forward bias in the junction, on the other hand are only of the order of 100 ampere m^{-2}, which is negligibly small compared to the two terms on the right-hand side of (7.6). Consequently, Boltzmann statistics may be used even if current flows.

n_{e0} is a constant, (7.35) may be written as

$$n_e(x) - n_{e0} = L_e^2 \frac{d^2(n_e - n_{e0})}{dx^2} \tag{7.36}$$

The general solution of this equation is of the form

$$n_e(x) - n_{e0} = Ae^{-x/L_e} + Be^{x/L_e} \tag{7.37}$$

where A and B are two constants which are determined by the boundary conditions. One of these boundary conditions is that for very large values of x, $n_e(x)$ should become equal to the equilibrium minority carrier density n_{e0}. Consequently, $B = 0$, because otherwise n_e would go to infinity for $x \to \infty$. The constant A is determined by the boundary condition (7.31); this gives

$$Ae^{-x_p/L_e} = n_{e0}(e^{eV/kT} - 1)$$

Calculating A, we then find from (7.37) the solution

$$n_e(x) - n_{e0} = n_{e0}e^{(x_p - x)/L_e}(e^{eV/kT} - 1) \quad \text{for } x \geqslant x_p \tag{7.38}$$

For a given bias voltage V, we see that the electron density decreases exponentially to n_{e0} for increasing values of x beyond x_p, as illustrated in Fig. 7.7. Note that L_e is a measure for the length into the p-type material

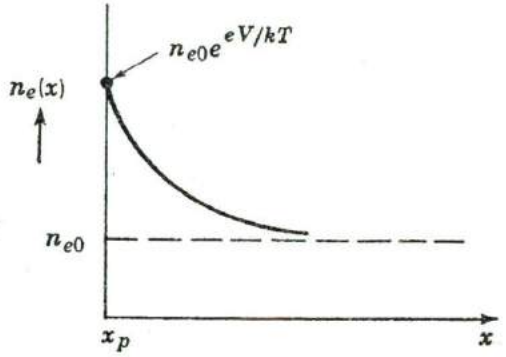


Fig. 7.7. The electron concentration in the p-region decreases with increasing distance from the value $n_{e0}e^{(eV/kT)}$ at x_p to n_{e0} for large values of x, in accordance with equation (7.38).

over which the excess density has been reduced by a factor $1/e$ (where $e = 2.7$). The reason for this decrease is that at x_p the electron density is higher than the equilibrium density n_{e0} as a result of electron injection from the n-region. This produces a diffusion of excess electrons into the p-region, but at the same time the density of excess electrons gradually disappears as one moves into the p-region because of recombination with

holes. The depth over which the excess electrons are "noticeable" is of the order of L_e, hence the name "diffusion length" for L_e.

Since we have now an expression for $n_e(x)$, we may calculate the electron current from (7.34). Differentiating (7.38) with respect to x, substituting the result into (7.34) and finally putting $x = x_p$, we find for the electron current density at x_p

$$J_e(x_p) = -\frac{eD_e n_{e0}}{L_e}(e^{eV/kT} - 1) \tag{7.39}$$

The minus sign indicates that the conventional positive current flows from p to n.

The hole current density in the n-region at $x = 0$ may be obtained in a similar way. It may suffice to give the result:

$$J_h(0) = -\frac{eD_h n_{h0}}{L_h}(e^{eV/kT} - 1) \tag{7.40}$$

Here, $L_h = (D_h\tau_h)^{1/2}$ is the diffusion length of holes in the n-region, and again the minus sign indicates that the current flows from p to n. Thus, the total current density flowing across the junction is given by the sum of (7.39) and (7.40); the magnitude of the total current density flowing from p to n is then

$$J = e\left(\frac{D_e n_{e0}}{L_e} + \frac{D_h n_{h0}}{L_h}\right)(e^{eV/kT} - 1) \tag{7.41}$$

We note that the form of this equation is the same as that of expression (7.28) given in the preceding section.

It is of importance for the discussion of transistors later, to say a few words about the ratio of the electron and hole currents. It follows from (7.40) and (7.41) that $J_e/J_h = (D_e n_{e0} L_h)/(D_h n_{h0} L_e)$. As a consequence of the Einstein relation between the diffusion coefficient and mobility of carriers, we may write $D_e/D_h = \mu_e/\mu_h$. Furthermore, if n_{hp} represents the hole density in the p-region *in thermal equilibrium*, it must follow according to (7.3) that $n_{e0}n_{hp} = n_i^2$. Similarly, if n_{en} represents the equilibrium density of electrons in the n-region we must have $n_{en}n_{h0} = n_i^2$. Hence

$$\frac{J_e}{J_h} = \frac{\mu_e n_{en} L_h}{\mu_h n_{hp} L_e} = \left(\frac{\sigma_n}{\sigma_p}\right)\left(\frac{L_h}{L_e}\right) \tag{7.42}$$

Apart from the ratio L_h/L_e which is of the order of unity, the ratio of the electron to hole current is determined by the conductivity σ_n of the n-region over the conductivity σ_p of the p-region. Thus, if the conductivity of the n-region is 100 times as large as the conductivity of the p-region, the current across the junction is carried for 99 percent by electrons and for 1 percent by holes. Typical values for L_e and L_h are of the order of 10^{-3} m.

For reverse bias, one obtains a current through the junction which is of the same form as equation (7.29), the identification of J_{e0} and J_{h0} being the same as above.

The procedure followed here in the calculation of the junction current may be summarized as follows:

(a) set up the boundary conditions for the minority carriers in the n- and p-regions for $x = x_p$ and $x = 0$;

(b) write down the continuity equation for the minority carriers for the region $x \geqslant x_p$. Solving the equation for the proper boundary conditions gives $n_e(x)$ in the p-region;

(c) from $n_e(x)$ find dn_e/dx and employ this information to calculate the electron diffusion current at x_p;

(d) repeating steps (b) and (c) for the hole minority carriers in the n-region ($x \leqslant 0$) find the hole current at $x = 0$;

(e) the total junction current is obtained by adding the electron current at $x = x_p$ and the hole current at $x = 0$.

7.6 Thickness and capacitance of the junction barrier

In the preceding sections we were concerned with calculating the current-voltage relationship for an n-p junction. In this discussion the potential barrier between the n- and p-regions played an important role. So far, however, nothing has been said about the thickness of the barrier layer. In the present section we shall discuss this problem on the basis of a simplified picture in which it is assumed that the transition from the n-region to the p-region takes place abruptly in a plane perpendicular to the x-axis going through the point $x = x_1$ (see Fig. 7.8). We shall assume that everywhere in the n-region the donor concentration is constant and equal to N_d per m^3. Similarly we shall assume the acceptor concentration N_a to be constant everywhere in the p-region. When the junction is in thermal equilibrium, the conventional potential (referring to a positive unit charge) across the junction varies as indicated in Fig. 7.8(b). We shall measure x to the left, as indicated, and denote by $x = 0$ the point such that for $x \leqslant 0$ the potential may be considered as constant. The potential in the region $x \leqslant 0$ will be taken as zero. The potential in the region left of the point $x = x_2$ is also constant and presumably equal to the contact potential V_0 in thermal equilibrium. Given N_d, N_a and V_0, what is the thickness of the barrier, x_2? In order to answer this question, we first remind the reader that in thermal equilibrium, electrons have been transferred from part of the n-region into the p-region, thus producing a positive space charge in

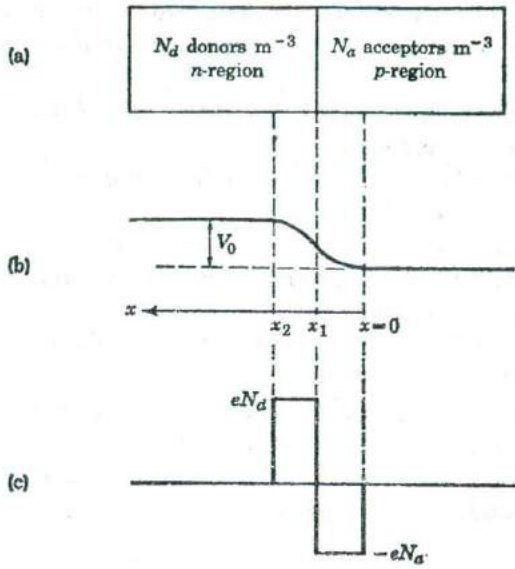


Fig. 7.8. (a) represents an n-p junction which is assumed to change abruptly from n to p for $x = x_1$. In (b) the conventional potential has been indicated when the junction is in thermal equilibrium, corresponding to a contact potential V_0. In (c) the assumed space charge is represented.

the n-region resulting from uncompensated positive ionized donors. Similarly, holes have been transferred from the p-region into the n-region, leaving a net negative space charge in the p-region. Since the electron density in the region between x_1 and x_2 is small, we shall assume that the space charge density in that region is simply eN_d as indicated in Fig. 7.8(c). Similarly, in the region between $x = 0$ and $x = x_1$ we shall assume a space charge density $-eN_a$. The potential is determined by the space charge through Poisson's equation. For the two regions we thus have the Poisson equations

$$\frac{d^2V_1}{dx^2} = \frac{eN_a}{\epsilon} \qquad \text{for the region } 0 \leqslant x \leqslant x_1 \tag{7.43}$$

$$\frac{d^2V_2}{dx^2} = -\frac{eN_d}{\epsilon} \qquad \text{for the region } x_1 \leqslant x \leqslant x_2 \tag{7.44}$$

Here, $\epsilon = \epsilon_0\epsilon_r$ where $\epsilon_0 = 8.854 \times 10^{-12}$ farad m^{-1} and ϵ_r is the relative dielectric constant of the medium. Integrating expression (7.43) under the condition that in the point $x = 0$ (dV_1/dx) is zero, we find

$$\frac{dV_1}{dx} = \frac{eN_a}{\epsilon}x \qquad 0 \leqslant x \leqslant x_1 \tag{7.45}$$

Integrating this equation, and remembering that we normalized the potential such that it is zero for $x = 0$, we find

$$V_1(x) = \frac{eN_a}{2\epsilon} x^2 \qquad 0 \leqslant x \leqslant x_1 \tag{7.46}$$

Hence, as we move to the left from the point $x = 0$, the potential rises quadratically with distance [see Fig. 7.8(b)].

The rest of the procedure runs as follows: when expression (7.44) is integrated twice, there will occur two integration constants in the general solution for $V_2(x)$. However, in the point $x = x_1$ the potential and the field strength obtained from the solution (7.46) must be the same as the corresponding quantities obtained by using the solution of (7.44) for the point $x = x_1$. This gives two equations from which the two integration constants can be found, so that $V_2(x)$ will then be known also. Finally, $V_2(x_2)$ can be put equal to the value V_0 which is assumed to be known, and from the resulting equation we shall be able to find x_2. The algebra involved in this procedure is shown below.

From (7.46) we obtain

$$V_1(x_1) = \frac{eN_a}{2\epsilon} x_1^2 \qquad \text{and} \qquad \left(\frac{dV_1}{dx}\right)_{x_1} = \frac{eN_a}{\epsilon} x_1 \tag{7.47}$$

Integrating (7.44) once and applying this to the point $x = x_1$ we obtain

$$\left(\frac{dV_2}{dx}\right)_{x_1} = -\frac{eN_d}{\epsilon} x_1 + A_1 \tag{7.48}$$

where A_1 is an integration constant which can be found from (7.48) and the second equation in (7.47) by equating the two derivatives. Hence, $A_1 = (ex_1/\epsilon)(N_d + N_a)$. Integrating (7.44) twice, we obtain the general solution

$$V_2(x) = -\frac{eN_d}{2\epsilon} x^2 + A_1 x + A_2 \tag{7.49}$$

where A_1 has been determined before and A_2 is a second integration constant. Applying (7.49) to the point $x = x_1$, substituting A_1, and making use of the fact that $V_2(x_1) = V_1(x_1)$ we obtain $A_2 = -(x_1^2 e/2\epsilon)(N_a + N_d)$. The potential V_2 in the point x_2 obtained by substituting for A_1 and A_2 in (7.49) is thus given by

$$V_2(x_2) = V_0 = -\frac{eN_d}{2\epsilon} x_2^2 + \frac{e}{\epsilon}(N_d + N_a) x_1 x_2 - \frac{e}{2\epsilon}(N_a + N_d) x_1^2 \tag{7.50}$$

Note that we cannot calculate x_2 from this equation unless x_1 can be expressed in terms of x_2. This can be done by realizing that at the point x_2,

the field strength vanishes. Hence, in accordance with (7.49),

$$\left(\frac{dV_2}{dx}\right)_{x_2} = 0 = -\frac{eN_d}{\epsilon} x_2 + \frac{ex_1}{\epsilon}(N_d + N_a)$$

so that

$$x_1 = \frac{N_d}{N_d + N_a} x_2 \tag{7.51}$$

Substitution of x_1 into (7.50) finally leads to the equation

$$V_0 = \frac{e}{2\epsilon}\left(\frac{N_d N_a}{N_d + N_a}\right) x_2^2 \tag{7.52}$$

and consequently

$$x_2 = \left[\frac{2\epsilon V_0(N_d + N_a)}{eN_d N_a}\right]^{1/2} \tag{7.53}$$

Consider a typical example with $N_d \approx N_a \approx 10^{21}$ per m^3 (see Table 7.1), $V_0 \cong 0.5$ volt, $\epsilon_0 = 8.85 \times 10^{-12}$ farad m^{-1}, $\epsilon_r \cong 16$ (for germanium) and $e = 1.6 \times 10^{-19}$ coulomb. We then find $x_2 \approx 10^{-6}$ m.

It is evident that this procedure may also be applied to the case of a biased junction. Thus, for a forward bias of V volts, we would simply have to replace V_0 in equation (7.53) by $V_0 - V$, illustrating that the *thickness of the barrier decreases with increasing forward bias*. Similarly, *the thickness of the barrier increases with increasing reverse bias*.

From the electrical engineering point of view, the barrier layer is equivalent with a charged condenser. In fact, the total positive charge in the n-type region is equal to $Q = eN_d(x_2 - x_1)$ coulombs per m^2, and the tota negative charge in the p-type region is $-eN_a x_1$ coulombs per m^2; because of relation (7.51) the positive and negative charges per m^2 have equal magnitudes. The effective capacitance C per m^2 of the barrier layer follows from the equation

$$C = \frac{dQ}{dV_0} = \frac{d}{dV_0}[eN_d(x_2 - x_1)] = \frac{d}{dV_0}\left[ex_2 \frac{N_d N_a}{N_d + N_a}\right] \tag{7.54}$$

where the last equality follows from (7.51). Substituting for x_2 from (7.53) in (7.54) and performing the differentiation one finds

$$C = \left[\frac{e\epsilon N_d N_a}{2V_0(N_d + N_a)}\right]^{1/2} = \frac{\epsilon}{x_2} \tag{7.55}$$

where the last equality follows from (7.53). Hence, the capacitance of the layer is the same as that of a parallel plate condenser with a distance x_2 between the plates and filled with a medium of dielectric constant $\epsilon = \epsilon_0\epsilon_r$. Since x_2 is of the order of 10^{-6} m in a typical case, the capacitance of the barrier layer is of the order of 150 microfarads per m^2.

7.7 The *n-p-n* junction transistor

In this section we shall discuss the physical processes which determine the action of a junction transistor. The structure of an *n-p-n* junction transistor is represented in Fig. 7.9; it consists of two *n*-type regions with a *p*-type region sandwiched between them. One of the *n-p* junctions is

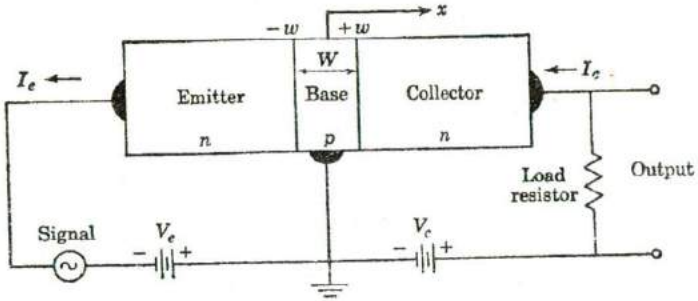


Fig. 7.9. Diagram of an *n-p-n* junction transistor with a base-to-ground connection. The emitter junction is biased in the forward direction, the collector junction in the reverse direction.

biased in the forward direction (emitter junction), the other in the reverse direction (collector junction). The *p*-region is called the base region; the two *n*-regions are referred to as the emitter and collector regions. A junction transistor may also be of the *p-n-p* type and the arguments given below apply equally well to it as to the *n-p-n* transistor under discussion. For a typical junction transistor, the characteristics describing the various regions are of the following orders of magnitude:

conductivity of the emitter region, $\sigma_e \approx 10^4$ ohm^{-1} m^{-1},
conductivity of the base region, $\sigma_b \approx 10^2$ ohm^{-1} m^{-1},
conductivity of the collector region, $\sigma_c \approx 10$ ohm^{-1} m^{-1},
diffusion length of minority carriers, $L \approx 10^{-3}$ m,
width of the base region, $W \approx 10^{-5}$ m ($\ll L$).

In Fig. 7.10 we have represented the electron potential in the three regions, as determined by the bottom of the conduction band, for the biased transistor. The magnitudes of the emitter bias, V_e, and of the collector bias, V_c, are indicated. Let us consider the currents flowing in the device in the absence of an a-c signal, but with the bias voltages applied as indi-

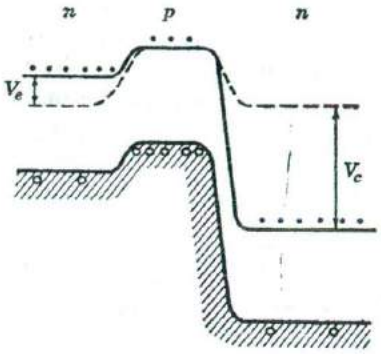


Fig. 7.10. Illustrating the energy band picture for the biased transistor.

cated. Since the emitter is biased negatively with respect to the base region, electrons are injected by the emitter region into the base region. Also, holes are injected by the base region into the emitter region. The total emitter current I_e then consists of an electronic current, I_{ee}, and a hole current, I_{eh}, so that

$$I_e = I_{ee} + I_{eh} \tag{7.56}$$

In Fig. 7.9 the emitter current has been indicated as a conventional positive current leaving the external ohmic contact of the emitter region. Two conditions must be satisfied for good transistor action:

(a) nearly the whole emitter current should consist of electrons injected from the emitter n-region into the base p-region;

(b) nearly all electrons emitted by the emitter should travel through the base and be collected by the collector.

The reason for these requirements will become clear later, but we may state at this point that if these requirements are satisfied, one collects a current across the high resistance of the reverse-biased collector junction, whereas it is injected through the low resistance of the forward-biased emitter junction; this provides *power amplification*.

In connection with (a) the question can be raised as to what determines the ratio I_{ee}/I_{eh}? At first glance one may be tempted to employ expression (7.42), which would give $I_{ee}/I_{eh} = (\sigma_e/\sigma_b)(L_h/L_e) \cong \sigma_e/\sigma_b$. This answer is *incorrect*, however, because the base region has a width which is small compared to the diffusion length of minority carriers. This has the effect of increasing the ratio I_{ee}/I_{eh} and one can in fact show that

$$I_{ee}/I_{eh} = (\sigma_e/\sigma_b)(L_h/W) \tag{7.57}$$

The reason for replacing the diffusion length of electrons in the base material L_e by the width of the base region W is the following: due to the electron injection by the emitter into the base region, the density of electrons

in the base region at $x = -w$ (see Fig. 7.9) is raised to the value

$$n_e(-w) = n_{e0b}e^{eV_e/kT} \tag{7.58}$$

in accordance with (7.38); n_{e0b} is the equilibrium density of electrons in the base region and V_e is the magnitude of the emitter bias. Similarly, the density of electrons in the base region at the position $x = +w$ is reduced to the value

$$n_e(+w) = n_{e0b}e^{-eV_c/kT} \cong 0 \tag{7.59}$$

where V_c is the magnitude of the collector voltage; since $eV_c \gg kT$, we have $n_e(+w) \cong 0$. Now, if we assume that the base region is so narrow that we may neglect recombination of electrons in the base region for the moment, the electron current in the base region due to diffusion from $x = -w$ to $x = +w$ is independent of x, and consequently dn_e/dx must be constant in the base region. This is indicated in Fig. 7.11. The electron

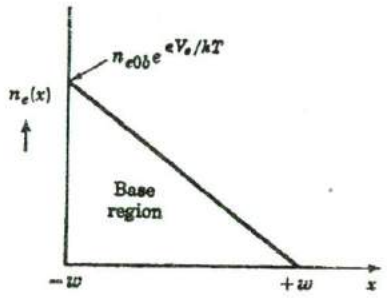


Fig. 7.11. The density of electrons in the base region of the n-p-n transistor varies linearly with x.

current resulting from the diffusion in the base region is then given approximately by

$$I_{ee} \cong -AeD_e\frac{dn_e}{dx} \cong A\frac{eD_e}{W}n_{e0b}e^{eV_e/kT} \tag{7.60}$$

where A is the cross section and $W = 2w$ is the width of the base region. Comparing this expression with (7.39) for $eV_e \gg kT$ we see that L_e has been replaced by W. The hole current from the base to the emitter region, on the other hand, is not affected by the width of the base; these arguments thus show that (7.57) is correct.

The *emitter efficiency*, γ, may be defined as the fraction of the emitter current which is carried by electrons, i.e.,

$$\gamma = I_{ee}/I_e = I_{ee}/(I_{ee} + I_{eh}) \tag{7.61}$$

For the numerical values given in the beginning of this section we find from (7.57) that $I_{ee}/I_{eh} \cong 10^4$, so that γ is nearly unity.

With regard to the requirement (b) mentioned before, we should con-

sider what fraction of the electrons injected by the emitter into the base region actually arrives at the collector. If the base region has a width $W \ll L_e$, nearly all electrons injected into the base region will be collected by the collector. By solving the diffusion equation for electrons in the base region one can show that a fraction $\frac{1}{2}(W/L_e)^2$ of the electrons disappears by recombination as the electrons travel through the base region. We may thus introduce the *transport factor*, β, defined by

$$\beta = \frac{I_{ec}}{I_{ee}} \cong 1 - \frac{1}{2}\left(\frac{W}{L_e}\right)^2 \tag{7.62}$$

where I_{ec} is the electron current collected at the collector.

The currents I_{ee}, I_{eh}, and I_{ec} discussed so far are indicated schematically in Fig. (7.12). There is, of course, current flow between the base and col-

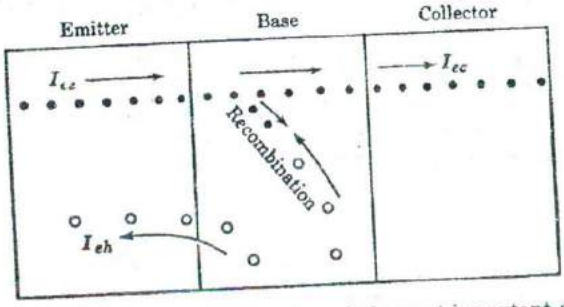


Fig. 7.12. Schematic representation of the most important electron and hole currents flowing in an n-p-n junction transistor. The dots represent electrons; the circles holes. The arrows indicate the direction of motion of the carriers, not the direction of conventional current flow, except for holes.

lector regions, also. Since the collector junction is biased in the reverse direction, these currents are very small compared to the currents I_{ee} and I_{ec}, and we shall not discuss them here. In this approximation, then, we find from (7.61) and (7.62) for the ratio of the total collector current to the total emitter current

$$\alpha = \frac{I_c}{I_e} \cong \frac{I_{ec}}{I_e} = \beta\gamma \tag{7.63}$$

From these considerations we find the *approximate equivalent circuit* for the grounded-base junction transistor as indicated in Fig. 7.13. The resistance r_c represents the collector junction resistance; it is large compared to r_e because of the reverse bias of the collector. The capacitance C_c is of importance only for a-c signals; it represents the capacitance of the collector junction barrier and, because it tends to short-circuit the

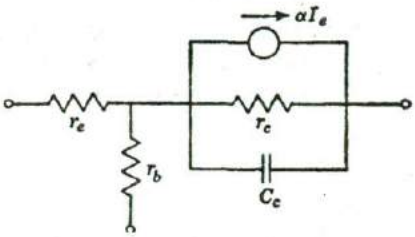


Fig. 7.13. Approximate equivalent circuit for the junction transistor.

collector resistance, it is one of the factors which limits high frequency response. The current generator of strength αI_e forms the active element of the transistor. The resistance r_b gives the resistance produced by the base material.

For a typical transistor under operating bias conditions the values of these parameters are as follows:

$r_e \cong 50$ ohms $\qquad \alpha \cong 0.995$

$r_c \cong 5 \times 10^6$ ohms $\qquad C_c \cong 20$ micromicrofarads

$r_b \cong 200$ ohms

For a more detailed discussion of transistors and transistor circuits the reader is referred to the literature on the subject.

References

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Problems

7.1 Indium antimonide (InSb) has an electron mobility of 6 m^2 $volt^{-1}$ sec^{-1} and a hole mobility of 0.2 m^2 $volt^{-1}$ sec^{-1}. The highest room temperature (300°K) resistivity found to date is 2×10^{-4} ohm m. Assuming the material is intrinsic, determine the intrinsic carrier density n_i at room temperature.

7.2 Gallium antimonide (GaSb) has an electron mobility of 0.3 m^2 $volt^{-1}$ sec^{-1} and a hole mobility of 0.1 m^2 $volt^{-1}$ sec^{-1}. The highest resistivity of this material at 300°K found to date is 2×10^{-3} ohm m. Assuming the material is intrinsic, calculate the intrinsic carrier density n_i at 300°K. Compare the answer with that for indium antimonide in the preceding problem, and comment on whether this agrees with the fact that the for-

bidden energy gaps of GaSb and InSb are respectively 0.80 and 0.20 ev; what do you conclude?

7.3 The electron and hole mobilities in a silicon single crystal at room temperature (300°K) are, respectively, 0.17 and 0.025 m^2 $volt^{-1}$ sec^{-1}. Find the diffusion coefficient of electrons and holes at this temperature.

7.4 A p-type silicon crystal has a room temperature (300°K) resistivity of 9×10^{-4} ohm m. What would be the Hall coefficient of the sample. Mobilities are given in the preceding problem.

7.5 The lifetimes of carriers in a semiconductor in part are strongly determined by the condition of the surface. In one particular experiment, the lifetime of electrons at room temperature in an n-type germanium crystal with a ground surface is found to be 78 microseconds, whereas with an acid-etched surface it is 340 microseconds. Assuming an electron mobility of 0.36 m^2 $volt^{-1}$ sec^{-1}, find the diffusion length of the electrons in both specimens.

7.6 The bulk n-region of an n-p germanium junction has a resistivity at 300°K of 10^{-4} ohm m; the resistivity of the bulk p-region is 10^{-2} ohm m. Find the carrier concentrations from Table 7.1. Assuming the junction is in thermal equilibrium, find the potential drop across the junction, applying Boltzmann statistics to the carrier densities.

7.7 The junction referred to in the preceding problem is subjected to a forward bias of 0.25 volts. Find the electron density in the p-region immediately adjacent to the junction (point x_p in Fig. 7.6). Find also the electron current across the junction if the junction is circular with a radius of 0.15 mm. What is the hole current across the junction? The electron and hole mobilities are 0.36 and 0.17 m^2 $volt^{-1}$ sec^{-1} respectively; assume a lifetime of 100 microseconds for both electrons and holes.

7.8 For a p-n germanium junction, let the conductivity of the bulk p-region be 10^4 ohm^{-1} m^{-1} and that of the bulk n-region 100 ohm^{-1} m^{-1}. In thermal equilibrium, the voltage drop across the barrier layer is 0.5 volt. Calculate the capacitance of the junction if its circular cross section has a diameter of 0.15 mm. Calculate also the capacitance for a reverse voltage of 3 volts. You may use Table 7.1 and assume that the junction is abrupt. The relative dielectric constant of the material is 16.

Answers to Problems

1.1 2.69×10^{25} m^{-3}

1.2 appr. 3×10^{12} m^{-3}

1.3 0.039 ev; appr. 2000 m sec^{-1}

1.4 5.93×10^{5} m sec^{-1}; 1.38×10^{6} m sec^{-1}

1.5 $W_{kin} = 13.6$ ev; $W_{pot} = -27.2$ ev; $W_{total} = -13.6$ ev

1.6 $W_1 = -13.6$; $W_2 = -3.4$; $W_3 = -1.51$; $W_4 = -0.85$ ev
$r_1 = 0.529$; $r_2 = 2.12$; $r_3 = 4.76$; $r_4 = 8.46$ angstroms

1.7 $W_2 - W_1 = 10.2$ ev; $f = 2.47 \times 10^{15}$ sec^{-1}; in ultraviolet

1.10 600 volts

1.15 $(5 - 4 - 2.9) = -1.9$ ev

1.16 2 for b-c-c.; 4 for f-c-c.; 1 for simple cubic lattice

1.23 7500 volts; 54.6 volts

1.24 8.5×10^{28} m^{-3}

2.1 For $r < R$, $D = 0$, $E = 0$, $V = Q/4\pi\epsilon_0 R$ volts
For $r > R$, $D = Q/4\pi r^2$ coulomb m^{-2}, $E = Q/4\pi\epsilon_0 r^2$ volt m^{-1}, $V = Q/4\pi\epsilon_0 r$ volts

2.2 For $r < R$, $D = Qr/4\pi R^3$ coulomb m^{-2}, $E = Qr/4\pi\epsilon_0 R^3$ volt m^{-1}, $V = (Q/8\pi\epsilon_0 R)(3 - r^2/R^2)$ volts
For $r > R$, $D = Q/4\pi r^2$ coulomb m^{-2}, $E = Q/4\pi\epsilon_0 r^2$ volt m^{-1}, $V = Q/4\pi\epsilon_0 r$ volts

2.3 $C = 4\pi\epsilon_0 R$ farads; stored energy $= Q^2/8\pi\epsilon_0 R$ joules

2.4 $Q = 2\pi\epsilon_0\epsilon_r V/[\ln(R_2/R_1)]$ coulomb m^{-1};
$C = 2\pi\epsilon_0\epsilon_r/[\ln(R_2R_1)]$ farad m^{-1}

2.5 $C = 5.7$ microfarads

2.7 For TiO_2 dielectric, 0.50 and 0.495 joule; for mica, 0.50 and 0.407 joule

2.9 Q coulomb m, along positive x-axis

2.10 $E = 2.82 \times 10^{7}$ volt m^{-1}; $\mu = 2.5 \times 10^{-11}$ coulomb m

2.11 $\alpha = Q^2/f$ farad m^2

2.12 $\epsilon_r = 1.000435$

2.13 $\alpha = 3\pi\epsilon_0 r_1^3$ farad m^2

2.15 $\mu = 1.44 \times 10^{-31}$ coulomb m = 0.043 Debye units; attractive force = 4.2×10^{-13} newton
2.16 2.44×10^{25} m^{-3}; $\mu = 0.95$ Debye units; $\alpha = 1.6 \times 10^{-39}$ farad m^2
2.17 $\mu_{\text{ind}} = \alpha\mu/2\pi\epsilon_0 a^3$
2.18 $\mu_{\text{ind}} = -\alpha\mu/4\pi\epsilon_0 a^3$
2.19 $E_i = E/(1 - \alpha/2\pi\epsilon_0 a^3)$ volt m^{-1} for each atom; $E_i/E = 1.029$
2.20 $E_i = E(1 + \alpha/4\pi\epsilon_0 a^3)$ volt m^{-1} for each atom; $E_i/E = 0.986$
2.22 $E_i/E = 1.6$
2.23 The ferroelectric configuration

3.4 $C = 200$ micromicrofarads; $R = 2 \times 10^6$ ohms; $W = 10^{-6}$ joule sec^{-1}

4.1 $F_y = -5$ newtons
4.2 5×10^{-9} weber m^{-2}
4.4 10^{-4} newton (attr.)
4.5 $\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
4.9 $B = 1.257$ weber m^{-2}; $M = 5$ ampere m^{-1}
4.10 $B = 1.257$ weber m^{-2}; $M = 1400$ and 5000 ampere m^{-1}
4.12 2.2 Bohr magnetons per atom
4.13 2.8×10^{-3} and ~ 0.2 Bohr magnetons

5.1 $\rho = 1.72 \times 10^{-6}$ ohm m
5.2 $v_{\text{drift}} = 0.7$ m sec^{-1}; $\mu = 7 \times 10^{-3}$ m^2 volt^{-1} sec^{-1}; $\tau = 4 \times 10^{-14}$ sec
5.3 $v_F = 1.39 \times 10^6$ m sec^{-1}; $\lambda = 556$ angstroms
5.4 $\rho = 5.8 \times 10^{-8}$ ohm m; increase 222 percent
5.5 $\rho_i = 0.97 \times 10^{-6}$ ohm m
5.7 $F(+0.1 \text{ ev}) = 0.018$; $F(-0.1 \text{ ev}) = 0.982$
5.9 $\rho(300°) = 1.866 \times 10^{-6}$ ohm m; $\rho(4°) = 0.306 \times 10^{-6}$ ohm m; 0.28 percent for the alloy and 0.33 percent for pure Cu
5.10 $\Delta v_x = 0.476$ m sec^{-1}; 2.05×10^{-31} joule
5.11 1.5×10^8 joules
5.12 $K = 432$ and 5.25 watt m^{-1} degree^{-1}

6.1 8
6.2 $N = 5.00 \times 10^{28}$ m^{-3} for Si and 4.52×10^{28} m^{-3} for Ge
6.4 $n_i = 2.38 \times 10^{19}$ m^{-3}
6.5 1.1×10^{16} m^{-3}
6.6 $n_h = 4.52 \times 10^{24}$ m^{-3}; $n_e = 1.25 \times 10^{14}$ m^{-3}; $\rho = 0.36 \times 10^{-6}$ ohm m
6.7 $n_h = 2 \times 10^{22}$ m^{-3}; $\mu_h = 0.035$ m^2 volt^{-1} sec^{-1} if formula (6.24) is used
6.8 1.83 millivolt
6.9 1.1 degree

7.1 $n_i = 0.5 \times 10^{22}$ m^{-3}
7.2 $n_i = 0.78 \times 10^{22}$

7.3 $D_e = 0.0044$ m^2 sec^{-1}; $D_A = 0.00091$ m^2 sec^{-1}

7.4 $R_H = 3.7 \times 10^{-5}$ coulomb^{-1} m^{-3} λ if formula (6.24) is used

7.5 $L = 0.85$ and 1.77 mm

7.6 $V_0 = 0.5$ volt

7.7 $n_e(x_p) = 2.7 \times 10^{21}$ m^{-3}; $I_e = 0.3$ milliamperes; $I_A = 0.004$ milliamperes

7.8 3.5 micromicrofarads; 1.3 micromicrofarads